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A
COURSE
OF
MATHEMATICS.

IN TWO VOLUMES.

COMPOSED FOR THE USE OF

THE ROYAL MILITARY ACADEMY.

BY

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CONTINUED AND AMENDED BY

OLINTHUS GREGORY, LL.D. F.R.A.S.

VOL. II.

TWELFTH EDITION,

WITH CONSIDERABLE ALTERATIONS AND ADDITIONS,

BY

THOMAS STEPHENS DAVIES, F.R.S.L.&E. F.S.A.

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THE

EDITOR'S PREFACE.

THE public approbation of my efforts to render the first volume of *Dr. Hutton's Course of Mathematics* such as the present state of Mathematics requires, has been too undisguisedly manifested by the extensive sale of that volume, to be other than very gratifying to my feelings: and I hope in the next edition of it, to be able to fully carry out those contemplated improvements, which, from various circumstances, I was only able to partially develope in the last.

The second volume, here laid before the public, will, I trust, be considered more completely in accordance with the views under which I have officiated as editor, than was the case in the preceding one: and I anticipate with confidence, that the patronage with which the former volume was honoured, will be extended to the present one. This, at least, I can assure my readers, that no exertion or labour has been spared to render it worthy of the continuance of general approbation: and I cannot offer stronger evidence of this, than the fact, that there is not a single line of the original work which has not been recomposed. It may, likewise, be added, that I undertook only the ordinary duties of an editor—those of correcting the work for press, and reading the proofs—on corresponding pecuniary conditions. I may, hence, fairly disclaim every mercenary motive in the labours here brought to a close. Hutton's Course was, in fact, the *first* mathematical work I studied. I may, hence, be allowed to entertain a desire to render it as complete as possible, and, perhaps, to feel a deeper interest in its improvement, than in any work exclusively of my own production.

I now proceed to give a short account of the contents of the present volume. Prefatory to the Spherical Trigonometry there is given a chapter on the Geometry of the Sphere, comprising, besides the propositions usually to be found in other treatises, several curious and important ones used in the course of this work for the completion of discussions of trigonometrical properties, which have either not been noticed, or but very imperfectly developed.

In the Trigonometry itself, it will be seen that I have taken a view of Napier's rules as a scientific method, altogether the reverse of that usually adopted.

The deduction of the general properties of spherical triangles from the right-angled triangle is probably new: and the entire series of formulæ of solution is obtained by successive transformations without the aid of the polar triangle; as I consider the exercise in trigonometrical reduction thus obtained, to be infinitely more valuable than the time saved by the use of the polar triangle, elegant as that method is admitted to be. For the convenience, however, of those teachers who think otherwise, the method of deduction by the polar triangle has been added in the note on *page* 242. I have given numerous properties of the system of associated triangles and their polar ones, which are interesting for their remarkable symmetry and manifold applications in research; and likewise all the known formulæ (some of them, indeed, *but little* known) for the areas of spherical triangles.

In the solution of spherical triangles, I have introduced a new classification of the cases, which, from their analogy to the three cases of plane triangles, will render the rules of spherical trigonometry equally simple and easy to recollect with those in plane trigonometry. In all the examples it has been my aim to give the best working formularies, as well as the most elegant algebraical formulæ.

The doctrine of solid angles is curious in itself, and useful in its applications; and it is interesting to the readers of Hutton's Course from its being the re-invention of my late venerated colleague, Dr. Gregory,—the little that Stevin had written on the subject never having excited the least attention. I have, however, remodelled the chapter to adapt it to the general view which I entertain of the units of geometrical magnitude.

The series of properties of spherical figures contains a considerable number of theorems of much elegance in themselves, and still more for their analogy to, and indeed almost identity with, the most admired series of properties of plane figures investigated in the modern geometry. I had published some of them previously, though in a less completed state, in Leybourn's Repository, and in the Ladies' and Gentleman's Diaries.

The chapter on Spherical Astronomy, brief as it is, contains all that is required for the purpose. If practical astronomers, instead of tabulating an immense number of formulæ, would only take the trouble to familiarise themselves with the best solutions of spherical triangles, and use the two astronomical triangles, they would find the claims upon their memory and attention greatly abridged. The idea of the *two* astronomical triangles was suggested by a remark of Dr. Pearson, who, in his invaluable work on *Practical Astronomy*, speaks of *the* astronomical triangle which is our second one. The investigation of the Lunar problem is from the Trigonometry of Professor Young, who is undoubtedly amongst the most able, and by far the most useful, elementary mathematical writer of our age.

The treatise on the Conic Sections in the preceding edition was drawn up (as far as it was altered) by me; and, except a few changes in language and arrangement, it remains as it was then given. Those of greatest moment in this edition are in the parabola; and I have introduced, amongst other theorems, Lambert's very remarkable property, which has been attributed to several more recent authors. (*Vid. prop. xviii. p.* 198, and *Phil. Mag. Jan.* 1843.)

The chapter entitled *general properties of the conic sections* is new in this work; and being drawn up on the principles of the great authors of antiquity, embodying their most general properties and modes of investigation, it cannot but be a useful contribution to the young geometer's means of study. It also includes the doctrine of *poles and polars* originating in the earliest researches of the great Pascal, and terminating with the *Mystic Hexagram* of that extraordinary man, and the correlative property discovered by the very able French geometer, Brianchon. There are some theorems interspersed from the writings of Simson and Maclaurin—men not less gifted than the greatest of their predecessors of antiquity. In many respects I am much indebted to the *Geometry of Curve Lines*, of the late, and very imperfectly estimated, Sir John Leslie *, in the composition of this chapter: but well-read mathematicians will see that I have gone to his *original sources* of information, and perhaps in some respects with advantage to the student of this work.

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* The scientific character of Leslie has been greatly under-rated. As a geometer, he was, not only in learning, but invention, equalled by none of his cotemporaries, except Lowry and Swale; whilst in the literature of science he had no English competitor. His language, indeed, sometimes has too much eloquent hyperbole, and in, perhaps, a few of his demonstrations he has committed oversights: yet with these trivial objections, I can mention no work so improving to the taste of the young student as Leslie's *Geometrical Analysis and Doctrine of Curve Lines.* My present opinion, too, was that of a no less eminent judge of geometrical merit than the author of the *Life of Simson;* and Dr. Traill urged with much earnestness my close attention to this work at an early period of my own studies.

No man who reads a life of Leslie in *vol. vi.* of *Leybourn's Repository*, can help regretting that so distinguished a patron of mathematical science should have admitted into his work such a coarse, ignorant, and illiterate piece of biography; and no one who is familiar with Leslie's mathematical writings can rise from the reading of that paper without feelings of the deepest disgust.

In giving a few of the more obvious and elegant properties of the surfaces of revolution (conoïds), I have chosen only those which either from the elegance of their enunciation or demonstration, or their application to other researches, have the first claim upon our notice. Many of these are original; and I believe there is no work extant which contains, in a collected and condensed form, any series of propositions analogous to them. One important value which they possess is, the exercise they furnish to the student in the discussion of figures of three dimensions. The chapter on the cone, sphere, and cylinder, will be useful under the same view, if under no other.

The term *transversal*, even as a term, is scarcely known to English geometers. As a body of doctrine, it owes its existence to the celebrated Carnot; though the germs of the method, as well as the fundamental theorems, were familiar to the geometers of antiquity. The methods of investigation employed in the chapter on this subject, as well as many of the theorems developed, are new; whether considered in reference to lines, planes, or the conic sections. I believe that the most general and important properties delivered by Carnot and his successors will be found comprised in a very small space in this work; and I need not say, that to effect this, the general method and spirit of the investigations must be original. It will be seen, that the theorems of Pascal and Brianchon are demonstrated also under this aspect.

The doctrine of anharmonic ratio, developed by the greatest French geometer of the present day, *Chasles*, has been but slightly touched upon. This was contrary to my wish; and want of room was an imperative reason for my not entering further into the subject. I have, however, given my view of its great value in a long note at *page* 242; and I would refer the inquiring reader to the eleventh volume of the Memoirs of the Brussels Academy, or (which is a separate publication of the same work) Chasles's *Aperçu Historique des Méthodes en Géométrie*, for full details of this author's views. He may also consult some of the recent numbers of Quetelet's *Correspondance Mathématique et Physique* for applications of the doctrine.

I was uncertain at the commencement of writing the *geometry of co-ordinates of two dimensions*, whether it would be possible in the present volume to enter upon that of *three dimensions*. The result has proved that it could not be done: important as that subject is, it must be deferred to a supplementary volume. I judged, however, that in any case, the most general form of all the conclusions at which we arrived was preferable to the more confined ones; and have, therefore, given all the formulæ adapted to the most general cases, and as corollaries, those which relate to rectangular co-ordinates. I have long observed, that in actual instruction this is the most safe and beneficial course of proceeding; and experience becomes an undoubted authority on such questions.

Contrary to general usage, I have given the *transformation of co-ordinates* in the outset of the chapter; as it is a purely trigonometrical problem, and a valuable exercise in the use of the appropriate formulæ: it, moreover, enables us to make the transformations which are required as we proceed in our subsequent investigations. The general properties are deducible with much facility by this method of procedure; and I have had experimental proof, that in reference to the entire system of investigations, this is, in the end, much the shorter. A much greater number of examples is given than usual of the employment of the equations of the straight line, in reference to the demonstration of important properties of rectilinear figures; and possibly the results here obtained by such means may bring back a degree of respect for these elegant methods which the *transversals* had nearly destroyed. For a portion of these investigations I am indebted to two of the most able English geometers of our time, my friends and colleagues, Messrs. Rutherford and Fenwick.

The polar equations of the straight line (and subsequently in reference to the tangent and normal to any curve) are here given, for the first time in an *elementary* work, and, it is believed, were originally published by myself in the twelfth volume of the Edinburgh Transactions, in a note on my paper on *spherical co-ordinates.* A fuller and more expanded use of these, will, however, be given hereafter. The equation of *the radical axis* of two circles is also here given in a form fitted for following out a more extended series of investigations than those given by Gaultier and Steiner; but I could not enter into details, or even suggestions, upon looking at the paging of the work, and at what had yet to be given.

In even an algebraical treatise on the *conic sections*, it appears necessary, for the sake of consistency, to deduce the fundamental equation of the curves from *geometrical considerations.* After showing that every conic section is represented by an equation of the second degree between two variables, it is proved that every equation of the second degree is that of a conic section. The most general properties of the conic sections are then deduced by the method of co-ordinates; and some methods and discussions, not to be elsewhere found, will be recognised in this chapter. I may particularly refer, as either very useful or very elegant, to the conjugate equation, the determination of the foci, and the theorems of Pascal and Brianchon, the last two attributed in the text to their respective authors.

The special properties of the several curves are deduced in the form that, for some time, I had been in the habit of giving them in my professional duties; and in comparing the method here employed with those used in our ordinary class books, the preference given to this series of processes will be justified. The polar equations of the curves, their tangents and normals, are by this method rendered very effective; and in their applications to the conic sections, the general system (though but slightly developed in the present work) appears to great advantage. The tables of formulæ and values interspersed through the chapter will be found useful, and great care has been taken to secure accuracy both in the manuscript and in the printing. At the close of this chapter is another demonstration of Lambert's remarkable property of the parabola.

The *genesis and equations of various curves* are given in the manner of a series of exercises, and, to a student who is familiar with the preceding parts of the volume, is sufficient for the intended purpose. He should go through these with care and attention, as several of the curves here described will occur in a future part of the volume.

The *doctrine of Fluxions* is here replaced by a tolerably ample treatise on the *Differential and Integral Calculus.* In the developement of the first principles, the principle of *limiting ratios*, or, as I have here ventured to call it, of *tendent ratios*, is employed. Into the arguments in favour of this, or the rival method of series, the student would be incapable of entering; and amongst men of science, great names are so equally divided as advocates of both systems, that any quotation of authorities will be of little force. Though my own view has always been that the method here employed is the more logical, I have not been so much influenced in the composition of this work by this consideration, as by that of the greater simplicity and ease of comprehension, which characterises this method. At any rate, it has been well observed by a very judicious writer, (Dr. Jephson,) that all the methods meet in one important point, namely, *Taylor's theorem.* After this stage is attained, the whole inquiry resolves itself into mere algebraical transformation.

Many works of great elegance, but generally written with circumscribed objects, have appeared of late years in this country; and many of a much more expanded character have been published by continental writers. It has been remarked, that there is no work extant from which the calculus (especially the integral) can be learnt. Perhaps this, in a limited sense, is true; for in that department so much depends upon what, in the absence of a better term, we may call algebraical tact, that the want of any thing like general rules may sanction the opinion. However, every elementary work ought to contain the *general principles* of the Differential and of the Integral Calculus, as far as they have been reduced to any systematic form, whether in themselves, or in their application to geometry. To the extent of the geometry of two dimensions, I hope my task has been accomplished in a *useful* manner. Beyond this step, my limited space presented a barrier to my proceeding. No attempt has been made to introduce discussions which are merely curious, however interesting: and if nothing of real importance has been omitted, my purpose will be fully accomplished, and the contemplated objects of this series of chapters will be attained.

Some subjects of the second volume of the preceding edition have been omitted in the present one. As they were there given, they would be useless in the present state of science; and their republication in the shape they formerly assumed would neither be considered creditable to an editor, nor prove advantageous to a publisher.

The proof sheets have been read with great care by Messrs. Rutherford and Fenwick, as well as myself; and I hope that there will be few, if any, errata discovered. Should my readers, however, find any in either volume, they will confer a favour by giving me the information.

Grateful for the distinguished patronage of my labours, and for many kind attentions, which, in my connexion with Hutton's Course, I have received from various eminent mathematicians, I now take my leave of the public in the character of editor,—possibly for ever.

T. S. Davies.

Royal Military Academy,
May 5, 1843.

ERRATA.

PAGE
31, line 11, for $A + B' = \pi$, read $b + B' = \pi$.

39, line 4 from bottom, *for* $\frac{n^2}{\sin^2 s}$ *read* $\frac{n}{\sin^2 s}$.

41, *table, for* $s'_2 - a'_2 = \frac{1}{2}\Pi + (S - B)$ *read* $s'_2 - a'_2 = \frac{1}{2}\Pi + (S - C)$.

44, line 12 from bottom, *for* (54) *read* (53).

45, — 14, *for* tan r *read* sin s.

—, in equation (63) *for* tan r, tan r_2, and tan r_1 tan r_1, *read* tan r tan r_2 and tan r_3 tan r_1.

144, line 4 from bottom, *for* Ct *read* Ca.

247, — 11, *for* $f(x, y = 0)$ *read* $f(x, y) = 0$.

273, in equation (9) *for* a^3 *read* a_3.

280, in Example (31) *for* $\frac{x}{1}$ *read* $\frac{x}{a}$.

368, line 5 from bottom, *for* $\frac{1}{3} \cdot \frac{dA}{dx}$ *read* $\frac{1}{2} \cdot \frac{dA}{dx}$.

376, — 14 (denominator) *for* $f_3(a)\, h^3$ *read* $f_3(a)\, h^2$.

396, — 11, 12, from bottom, *for* ω *read* ω_1.

411, — 1 from bottom, and page 412, lines 8, 9: the statement is incorrect. The cusps of the curves are not at the foci. In the parabola, the focus is midway between the vertices of the curve and its evolute; and in the ellipse and hyperbola the cusp of the evolute is at the distance ae^2 from the centre.

CONTENTS.

SPHERICAL TRIGONOMETRY.

PAGE

Spherical Geometry :

- Definitions 2
- Elementary properties 3
- Polar triangles 6
- Right-angled triangles 8
- Properties of the sides and angles of triangles 10
- Duplicate solutions 13
- Associated triangles 16
- Spherical excess 18
- Platonic bodies 21
- Exercises 22

Spherical Triangles :

- Properties of right-angled triangles 24
- Napier's circular parts 25
- Exercises 28
- Quadrantal triangles 30
- Oblique-angled triangles 31
- The six formulæ of relation ib.
- Transformation to logarithmic forms 34
- Symmetrical functions of the sides and angles 38
- Polar triangles 40
- Table of values 41
- Circles inscribed and circumscribed to the associated triangles 42
- The areas of spherical triangles 47
- Spherical excess of the associated triangles 49
- Solution of right-angled triangles 52
- ——— quadrantal triangles 53
- ——— oblique triangles 55
  - First case ib.
  - Second case 57
  - Third case 60
  - Areas of spherical triangles 62
  - Solid angles 64
- Analogies, *etc.* 67
- Harmonical sections of an arc 76
- Spherical transversals 78
- Exercises in spherical geometry 83

Spherical Astronomy :

- Definitions 84
- Problems relating to the first astronomical triangle 87
- ——— second 91
- Lunar problem 94
- Exercises 96
- Miscellaneous exercises in spherics generally 98

THE CONIC SECTIONS, *etc.*

PAGE

- Definitions 104
- The parabola :
  - Properties related to the axes 107
  - Oblique ordinates 115
  - Lambert's theorem 118
  - Area 122
- The ellipse :
  - Lines related to the principal diameters 123
  - Oblique conjugate diameters 136
- The hyperbola :
  - Lines related to the principal diameters 143
  - Asymptotes 156
  - Oblique conjugate diameters 163
- Elementary Exercises 171
- General properties of the conic sections :
  - Harmonical sections, and the *pole* and *polar* 174
  - Inscribed polygons 183
  - Newton's theorem 187
  - Braikenridge's theorem 188
  - Pascal's theorem 190
  - Brianchon's theorem 191
- Exercises, general properties 192
- Conoids :
  - Properties demonstrated 194
  - ——— for demonstration 204
- Cone, sphere, and cylinder 205
  - Exercises 213

TRANSVERSALS.

- Definitions 214
- Theorem of Menelaus 215
- ——— Ceva 217
- Applications of Ceva's theorem 218
- Stewart's theorem 219
- Co-polar and co-axial triangles 221
- Quadrilateral 223
- Polygons, rectilinear and polyhedral 227
- Theorems of Pappus and Desargues 229
- Conical transversals 232
- Pascal's theorem 233
- Carnot's theorem 234
- Brianchon's theorem 238
- Theorem of Pappus on the anharmonic ratio 239
- Deductions by Chasles 241
- Exercises on transversals 243

GEOMETRY OF CO-ORDINATES.

- Definitions and notation 247
- Transformation of co-ordinates 249

PAGE
Rectilinear equations of the straight line.. 253
Polar equations of the straight line......... 264
Problems on the straight line 269
Exercises 278
The circle, rectilinear equations 281
—— ——— polar equations 285
Radical axes, *etc.* of circles 288
Exercises on the circle 290

LINES OF THE SECOND ORDER.

General properties and equations 293
Tangent 300
Equation of ordination 303
Two straight lines................................ 304
Pole and polar 305
Foci and directrices 308
Bret's method of finding the focus 310
Pascal's theorem (two solutions) 311
Brianchon's theorem 313
Synopsis of equations 315
Particular cases and properties............... 317
Tables of formulæ................................. 326
Properties of the curves by polar equations 329
Table of formulæ 330
Applications of polar equations 331
Lambert's theorem 336
Exercises 337

GENESIS AND EQUATIONS OF CURVES 341

DIFFERENTIAL CALCULUS.

Introduction 349
Differentiation of elementary functions... 356
Successive Differentiation 366
Taylor's theorem 368
Maclaurin's theorem............................. 369
Successive functions of one variable 371
Vanishing fractions 375
Maxima and minima............................. 377
Tangents, normals, and asymptotes 390
Do. in polar co-ordinates 396
Differentials of lengths and areas 402
Radius of curvature and evolutes 406
Do. in polar co-ordinates 414
Consecutive lines and envelopes 418
Singular points of curves 422
Points of inflexion...................... ...... ib.
Multiple points, 1st kind 424
—————— 2nd 427
Cusps 428
Conjugate points .... ib.
Tracing and discussion of curves........... 430

INTEGRAL CALCULUS.

PAGE
Definitions and principles..................... 434
Elementary Integrals 436
Rational fractions 443
Rationalisation 461
Correction of integrals 468
Integration by parts 471
    Binomial surds.......................... 472
    Trinomials 481
    Logarithmic functions 483
    Exponentials 485
    Trigonometrical functions 486
    Mixed functions 493
Integration by infinite series 499
Successive integrations 504
Formulæ for the lengths and areas of curves 508
The circle 510
Rutherford's ϖ 511
The parabola 512
The ellipse 513
    Fagnani's theorem 514
The hyperbola 516
The cycloïd 518
The tractory and catenary .... ib.
The logarithmic curve 522
The conchoïd 523
The Lemniscate of Bernoulli 525
The trigonometrical curves .... ib.
The spiral of Archimedes..................... 526
The hyperbolic spiral 527
The exponential spiral 528
The involute of the circle..................... 529
The epicycloïd .... ib.
Evolutes of curves 530
Differential curves 533
Examples for exercise 534

Surfaces and solids of revolution :
    The right cylinder .... ib.
    Lemma 536
    Area of a surface of revolution ... ib.
    Volume of a solid of revolution... 537
    The right cone.......................... ib.
    The sphere 538
    Spheroïds 539
    Paraboloïd .... ib.
    Surfaces produced from cycloïd... 540
    Exercises 541
    Pyramid or cone on plane base ... 543

Surfaces generated by a line or plane :
    The right, or common groin 544
    General theorem .... ib.
    The hyperbolic paraboloïd 545
    Oblique groin .... ib.
    Conical ungula 546
    Section of a cone and cylinder ... ib.
    Exercises 547

A
COURSE
OF
MATHEMATICS,
&c.

SPHERICAL TRIGONOMETRY.

SPHERICAL Trigonometry was primarily devoted to an investigation of the relations which subsist amongst the sides and angles of spherical triangles; that is, such triangles as are formed of arcs of circles upon the surface of the sphere, the planes containing which, all pass through the centre of the sphere. The term is now often extended to all investigations respecting every kind of figure traced upon the surface of the sphere, and bounded by arcs of such circles, as well as to the properties of circles made by planes not passing through the centre:—to all, in fact, which, in reference to the spherical surface, correspond to the subjects of inquiry comprehended under plane geometry.

The number of properties which can be stated in arcs of circles, in precisely the same form that corresponding properties of straight lines in plano can be stated, is extremely small: and yet the analogies between plane and spherical properties, as expressed by trigonometrical functions, is altogether unlimited. This, however, renders the fundamental theorems of spherical trigonometry, properly so called, indispensable in the investigation of geometrical properties, since trigonometrical functions enter into the enunciation of the great majority of the truths of spherical geometry.

A few of the leading properties of the spherical triangle must, however, be obtained by direct geometrical considerations: and such as are requisite for our purpose are so deduced in this treatise, under the title of *Spherical Geometry*. These are followed by such properties of spherical triangles, and of magnitudes connected with them, as are subsidiary to the actual solution of these triangles *numerically*, including quadrantal, as well as right and oblique-angled triangles. Then follows the application of the results so obtained to the numerical solution of triangles: and, lastly, an application of the principles of spherical trigonometry to the investigations of spherical geometrical theorems and astronomical problems; and a praxis for the student's exercise.

SPHERICAL GEOMETRY.

DEFINITIONS.

1. A *sphere* is that figure, every point of the surface of which is equally distant from a certain point within it; and that point is the *centre* of the sphere.

2. The distance of any point on the surface from the centre is called a *radius* of the sphere.

3. The *diameter* of the sphere is any line drawn through the centre, and terminated both ways by the surface.

4. A *diametral plane* is any plane drawn through the centre; and its intersection with the surface of the sphere is a *great circle of the sphere.*

5. An *excentric plane* is any plane which cuts the sphere, but does not pass through its centre; and the intersection of the surface of the sphere by an excentric plane is called a *less circle of the sphere.*

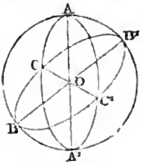
6. A *tangent-plane* to a sphere is a plane which meets the surface of the sphere in one point, and which being produced in all directions does not cut it.

7. If a perpendicular be drawn through the centre of the sphere, to a diametral plane, it intersects the surface in two points, which are called the *poles of the great circle* made by the diametral plane.

8. If a perpendicular be drawn through the centre to an excentric plane, it cuts the sphere in two points, which are called the *poles of the less circle.*

9. The poles of a less circle are called the *contiguous* and the *remote pole,* respectively, according as the pole is in the less, or in the greater, segment of the spherical surface, made by the plane.

10. When two great circles of the sphere intersect each other, they divide the surface into four portions, called *spherical lunes;* and the enclosed volumes made by the cutting planes and the intercepted surfaces are called *spherical wedges.*

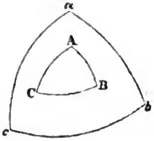


11. When three great circles intersect each other two and two, the portion of the surface enclosed by three contiguous segments of them is called a *spherical triangle,* as ABC in the figure.

12. As the three great circles intersect each other two and two a second time on the opposite sides of the sphere, at A′, B′, C′, there will be eight spherical triangles formed by the arcs of the three circles: viz., ABC, A′BC, AB′C, ABC′, and A′B′C′, AB′C′, A′BC′, A′B′C. The first four and the second four sets are called respectively, the *primary* and the *symmetrical, associated systems* of spherical triangles. The *primary system* is such as to have either AB, BC, or CA in each triangle; and the *symmetrical,* as to have either A′B′, B′C′, or C′A′ in each of the triangles which compose the system.

13. Whichever of these triangles be first traced, the entire and completed system will be the same; and hence, any one of them may be taken as the original triangle of the system. It has been usual to consider the central one as the original; but we may readily change the system, as may be easily seen, so that any other of the three shall be the original one. We shall in our investigations, however, consider, except otherwise specially stated, the central one ABC to be the *fundamental triangle of the associated system.* The sides of the other three are either equal to these or supplementary to them: and hence

the remaining three triangles will be spoken of as the *supplementary triangles of the associated system*. If we wish to specify which of the supplemental triangles we speak of, it will be conveniently done by mentioning the *side* of the fundamental triangle which is common to the two: thus, BA′C is supplemental with respect to *a*, AB′C with respect to *b*, and BC′A with respect to *c*.



14. If with the angular points A, B, C, as poles, three other great circles be described, their intersections, *a*, *b*, *c*, will be the angular points of another spherical triangle, *abc*, which is called the *polar triangle* with respect to ABC, the original triangle being called with respect to the polar one, the *primary triangle*.

15. If both the primary and polar system of associated triangles be completed, the systems are called the *primary associated* and the *polar associated* systems.

16. The arcs AB, BC, CA are called the *sides* of the triangle ABC.

17. By a *spherical angle* is meant the dihedral angle contained by the planes of the circles whose arcs form the boundaries of the lune on the surface of the sphere.

18. Great circles passing through the poles of a great circle are called *secondaries* to that circle.

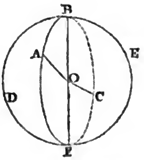
19. The *spherical excess* of a triangle is the excess of the three angles of that triangle above two right angles.

20. When two sides or two angles of a spherical triangle are both greater than $\frac{1}{2}\pi$, or both less than $\frac{1}{2}\pi$, they are said to be *of the same affection;* and when one (either side or angle) is greater and the other less than $\frac{1}{2}\pi$, they are said to be *of different affections*, or of *unlike affection*.

21. A *quadrantal triangle* has one of its sides a quadrant. It is conveniently called a *bi-quadrantal* and a *tri-quadrantal triangle*, when two sides or three sides, respectively, are quadrants.

PROP. I. THEOREM.

All diametral sections of the sphere are equal circles; of any two, one bisects the other.



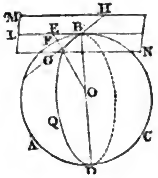
1. Let ABCF be a diametral section of the sphere, of which O is the centre. Then since O is the centre of the sphere, all the lines OA, OB, OC, *etc.* drawn to the surface are equal; and since A, B, C, *etc.* are in one plane, and equidistant from a point O in that plane, they are in the circumference of a circle. Hence the diametral section of the sphere is a circle, whose radius is equal to that of the sphere, and its centre coincident with that of the sphere.

In the same manner it will follow that another diametral section, BDFE, of the sphere, is a circle whose radius is equal to that of the sphere, and its centre coincident with that of the sphere. The two diametral sections, BAFC, BDFE, therefore, having equal radii, are equal circles.

2. The two diametral sections, ABCF, BDFE, passing through O, the centre of the sphere, their common section BF also passes through their common centre O. It is, hence, a diameter of each circle. The arcs BDF, BAF, BCF, BEF are, therefore, semi-circles; or the two circles mutually bisect each other at B and F, their intersections on the surface of the sphere.

PROP. II. THEOREM.

If a plane be drawn through any point on the surface of the sphere at right angles to a radius drawn to the same point, it will be a tangent plane to the sphere in that point.



Let the plane MN meet the sphere at B, and O be the centre of the sphere; then if MN be perpendicular to the radius BO, it will touch the sphere.

For, let E be any point in the plane, and join EO, and draw the plane EOB, cutting MN in the line LB, and the sphere in the great circle BFD.

Then, since OB is perpendicular to the plane MN, the angle EBO is a right angle, and hence greater than the angle BEO. The side EO of the triangle EBO is, therefore, greater than the side BO: and hence E is more remote from O than B is, and hence, again, than any point of the sphere is. Wherefore E is without the sphere. Making a similar construction, and pursuing a similar course of reasoning for any other point G of the plane MN, the point G is also proved to be without the sphere. Hence the plane MN is a tangent plane (*def.* 6).

Corollary 1.

Planes perpendicular to a diameter of the sphere at its extremities are parallel tangent planes.

Corollary 2.

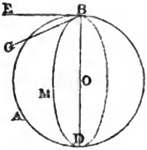
If tangent planes be drawn to a sphere at the opposite extremities of a diameter, they will be parallel.

Corollary 3.

Parallel tangent planes touch the sphere at the opposite extremities of a diameter.

PROP. III. THEOREM.

If tangents be drawn to two great circles of the sphere at either point of intersection, the angle formed by them is equal to the spherical angle formed by the circles themselves.



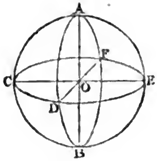
Let EB, BG be tangents to the great circles BAD, BMD at B their point of intersection; then EBG is equal to the spherical angle formed by the circles BAD, BMD.

For the tangents BE, BG to the circles are both perpendicular to their common diameter: and hence (*vol. i. p.* 347, *def.* 4,) the angle EBG is the measure of the dihedral angle formed by the planes of the circles, or to the spherical angle MBA, (*def.* 16.)

PROP. IV. THEOREM.

The plane of a great circle is cut by any two of its secondaries in lines which contain an angle equal to the spherical angle formed by those secondaries.

Let CDE be the primary great circle, and ACB, ADB any two of its secondaries, cutting the plane of CDE in CE, DF: then the angle DOC is equal to the spherical angle DAC.



For, since ACB, ADB are secondaries, their line of section AB is perpendicular to the plane CDE (*def.* 7 and 17); and hence the planes AOD, AOC cut CDE in lines CO, DO, which are perpendicular to OA (*vol. i. p.* 351, *th.* 7). Wherefore, the angle DOC is the measure of the inclination of the planes of the secondaries; and hence, again, equal to the spherical angle CAD.

Corollary 1.

If two great semicircles, ACB, ADB, be bisected in C and D, the arc CD of a great circle joining them is a measure of the spherical angle CAD.

Corollary 2.

The pole of a great circle is at the distance of a quadrant from every point of that circle: for since AC, AD subtend the right angles AOC, AOD, they are quadrants.

Corollary 3.

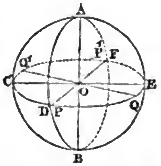
All the secondaries of a great circle are perpendicular to it: for their planes pass through OA, which is perpendicular to its plane; and hence the measure of the angle formed by the circle and its secondary is a quadrant.

Corollary 4.

Every great circle perpendicular to another is a secondary to it: for it passes through the poles of the former.

PROP. V. THEOREM.

The poles of all the secondaries are situated in the primary; and the distance of the poles of any two circles is the measure of the spherical angle contained by them.



1. Let BCA, BDA be any two secondaries to the primary CDE: then the poles, P, Q, of these secondaries will be in the primary CDE.

For, since AC, AD are secondaries to CE, the angles ACD, ADE are right angles (*iv. Cor.* 3); and since CE is perpendicular to CA, it passes through the pole P of CA; and similarly it passes through the pole Q of AD. Wherefore, the primary CD passes through the poles P, Q, of any two of its secondaries; and therefore, also, of the poles of all its secondaries.

2. Let ACB, ADB be any two circles intersecting in A, B; and let P and Q be their poles respectively: then the arc PQ is the measure of the angle CAD.

For let the circle PQ be drawn to meet the circles ADB, ACB in D and C: then, since P is the pole of ACB, PC is at right angles to ACB, and equal to a quadrant (*iv. Cor.* 2, 3); and for the same reason QD is at right angles to ADB, and equal to a quadrant. Hence to the secondaries ACB, ADB, CDPQ is the primary: and since CP is equal to DQ, each being a quadrant, we have PQ equal to CD. But since AC, AD are secondaries to CD, the arc CD is the measure of the spherical angle CAD (*iv. Cor.* 1); and hence, also, PQ is a measure of that angle.

Scholium.

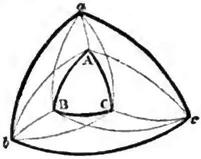
It is essential to remark, that in this demonstration the poles are taken in continuation of the circle CD in the *same direction*, viz., both in the direction from C towards or beyond D, as P and Q of the figure, or from D towards or beyond C, as Q′ and P′ of the figure. When the poles are taken *contrarily*, as P, Q′, or P′, Q, their distances measure the *supplementary angles* DAE and FAC.

We shall distinguish the two cases, the first as *the same direction*, and the second as *the cross direction.*

PROP. VI. THEOREM.

If with the angular points of a spherical triangle as poles three circles be described in cross direction to mutually intersect, a new triangle will be formed, such that :—

1. *The two triangles shall be mutually polar to each other ; and*
2. *The sides of each triangle shall be supplementary to the angles of the other.*



1. With poles A, B, C, describe great circles in cross direction with respect to the sides AB, BC, CA of the triangle ABC, mutually intersecting in *a*, *b*, *c* : then ABC, *abc* will be mutually polar to each other.

Since A is the pole of *bc*, the distances A*b*, A*c* are each a quadrant; and similarly B*a*, B*c*, C*a*, C*b*, are also, each of them a quadrant. Hence, arranging the order of them differently, *a*B and *a*C are each a quadrant; and therefore *a* is the pole of BC. Similarly *b* is the pole of AC, and *c* that of BA.

2. Since B and C are the poles of *ac*, *ab*, taken *in cross direction*, BC is the measure of the supplement of the angle *bac*, formed by *ab*, *ac* (*v. Schol.*), or the side BC is supplementary to the angle *bac*. In the same manner CA and AB are supplementary to the angles *abc* and *acb*.

Again, since the triangle ABC is polar to *abc*, the same reasoning will show that *bc* is supplementary to BAC, *ca* to ABC, and *ab* to ACB.

Scholium.

This theorem was discovered by Philip Lansberg, and has been since his time extensively employed in trigonometrical research. It is to be recollected, in the enunciation of this theorem, that by a spherical angle is meant its measure, viz., the circular arc intercepted by its sides upon the circle to which they are both primaries (*see iv. Cor.* 1). It must also be carefully remembered that this supplementary property only holds in the two triangles when they are taken in *cross directions*. For the remaining triangles formed by AB, BC, CA, and *ab*, *bc*, *ca*, the relation is investigated in the next proposition.

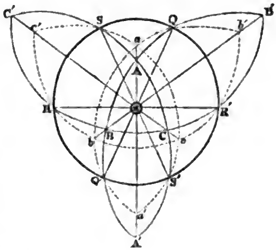
The distinction upon which the supplementary relation is founded, will render it convenient to denote the system as the *cross-polar triangles.*

PROP. VII. THEOREM.

If there be two cross-polar triangles, as in the last proposition, and taking these as the fundamental triangles of two associated systems (def. 14), *then these two associated systems will be, triangle for triangle, mutually cross-polar triangles.*

Let ABC, *abc*, be two mutually cross-polar triangles, and let the associated systems be completed by producing AB, AC to meet in A′, BA, BC to meet in B′, CA, CB to meet in C′, *ab*, *ac* to meet in *a*′, *ba*, *bc* to meet in *b*′, and *ca*, *cb* to meet in c': then the two systems will be, triangle for triangle, mutually cross-polars; such that

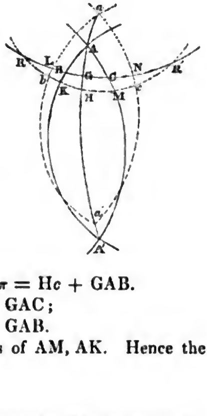
ABC, abc, are the first pair;
A′BC, $a'bc$, are the second;
AB′C, $ab'c$, are the third;
and ABC′, abc', are the fourth.



For, by the hypothesis, the first pair ABC, abc, are mutual polars; and hence B is the pole of ac, that is of ac'; A is the pole of bc, that is of bc'; and since C is the pole of ab, C′ is the other pole of ab. The points A, B, C′ are, therefore, the poles of the sides of the triangle abc'; and by the reciprocal nature of cross-polar triangles (which from the structure of the figure these triangles are), a, b, c' are the poles of the sides of the triangle ABC′.

In the same manner it is shown that A′BC, $a'bc$ are, mutually cross-polar triangles, and that AB′C and $ab'c$ are also cross-polars.

Corollary 1.



Draw the great circle aA to meet BC, bc in G and H. Then, because a is the pole of BC, and A that of bc, the arcs aG, AH are quadrants. Hence, $a\mathrm{G} + \mathrm{AH} = a\mathrm{H} + \mathrm{AG} = \pi$; and the angles at G and H are right angles.

Let R, R′ be the intersections of BC, bc: then because the angles at G and H are right angles, R, R′ are the poles of aA.

Corollary 2.

Let AB and bc meet in K, BC and ba meet in L, AC and bc meet in M, and ac, BC in N: then
$b\mathrm{H} \sim \mathrm{H}c = \mathrm{GAC} \sim \mathrm{GAB}$*, and $b\mathrm{H} + \mathrm{GAC} = \frac{1}{2}\pi = \mathrm{H}c + \mathrm{GAB}$.

For, $b\mathrm{M} = b\mathrm{H} + \mathrm{HM} = b\mathrm{H} + \mathrm{HAM} = b\mathrm{H} + \mathrm{GAC}$;

and, $c\mathrm{K} = c\mathrm{H} + \mathrm{HK} = c\mathrm{H} + \mathrm{HAK} = c\mathrm{H} + \mathrm{GAB}$.

Also $b\mathrm{M} = \frac{1}{2}\pi = c\mathrm{K}$, since b and c are the poles of AM, AK. Hence the conclusion follows.

Corollary 3.

Returning now to our original figure, let the three arcs Aa, Bb, Cc, be drawn; these will pass through the same point O, because, as has just been shown, they are perpendicular to the three sides BC, AC, AB. Also, because aA is perpendicular to bc, it will pass through the opposite pole a'. In like manner it will pass through A′. Or the points aAOa'A′ are in the same great circle, whose poles are R and R′, the intersections of BC, bc.

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* It is to be understood that angles are considered equivalent to the arcs which subtend them in equations such as these. The fact is, that both plane and dihedral angles are measured by their subtending arcs, and the *principle* has been laid down at p. 421, vol. i., upon which this comparison is founded.

In like manner *b*BO*b*′B′ are in one great circle, whose poles are S, S′, the intersections of AC, *ac*; *c*CO*c*′C′ are in one great circle, whose poles are Q, Q′, the intersections of AB, *ab*.

Again, because R, R′, are the poles of *aa*′, the arcs RO, OR′, are quadrants. In like manner SO, OS′, and QO, OQ′, are respectively quadrants; and as the quadrants are drawn from the poles of *aa*′, *bb*′, *cc*′, respectively, R, O, R′; S, O, S′, and Q, O, Q′, are respectively in the same great circles.

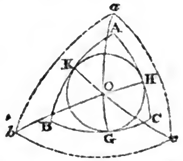
Corollary 4.

Also, since $OR = OR' = OS = OS' = OQ = OQ' = \frac{1}{2}\pi$, the six points R, R′, S, S′, Q, Q′, are in the same great circle, and O is its pole.

The complexity of the figure which would result from a further detail of these interesting researches, compels us to leave them to the industry and ingenuity of the student.

PROP. VIII. THEOREM.

The centre of the circle inscribed in either the primary or the polar triangle, coincides with the centre of that circumscribed respectively about the polar or the primary; and the radii of either pair of concentric circles are complementary to each other.



Let O be the centre of the circle GHK, inscribed in the primary triangle ABC; it will also be the centre of that described about the polar triangle *abc*; and the radii of the two circles will together be equal to a quadrant.

For, through *a*, *b*, *c*, draw *a*O, *b*O, *c*O, to meet the sides of ABC in G, H, K. Then since *a* is the pole of BC, the angles OGB, OGC, are right angles; and since O is the centre of inscribed circle, G is the point of contact, and OG the spherical radius or polar distance. In like manner *b*OH, *c*OK, are drawn to the points of contact, and OH, OK, are radii of the inscribed circle. The three arcs OG, OH, OK, are therefore equal.

Again, since *a*, *b*, *c*, are the poles of BC, CA, AB, the three arcs *a*G, *b*H, *c*K, are all quadrants, and therefore equal; and the parts OG, OH, OK, having been shown to be equal, the remainders *a*O, *b*O, *c*O, are equal; and O is, therefore, the centre of the circle through *a*, *b*, *c*, and *a*O, *b*O, *c*O, are each of them the radius.

It has been shown, likewise, that *a*O is complementary to OG.

In the same way it may be proved that the centre of the circle inscribed in *abc* is coincident with that of the circle described about ABC, and that the radii are complementary[*].

PROP. IX. THEOREM.

If from the oblique angles A, B, *of a triangle right-angled at* C, *as poles, two circles*, RQS, MKH, *be described to cut the three sides (produced where necessary) as in the figure; then,*

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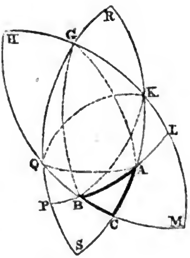
[*] Several curious and interesting investigations connected with this subject, by Mr. Lowry, may be seen in the Math. Repos. O.S., vols. i. ii. iii., and N.S., vol. i.



1. *The triangles,* ALK, KRG, GHQ, QPB, *will be right-angled at* L, R, H, P.
2. *The intersections* K, G, Q, *will be the poles of the sides* CB, BA, AC, *of the triangle* ABC*; and the opposite poles of* AQ *and* BK *will be* R, S, *and* H, M, *respectively.*

1. SINCE A is the pole of RS, the circles AR, AG, AQ, AP, AS are all secondaries to RS, and hence at right angles to it (IV. *Cor.* 3). In the same manner BH, BG, BK, BL, BM are secondaries to HM, and at right angles to it. Hence the angles at L, R, H, P, are all right angles; which establishes the first part of the theorem.

2. Since MK, by the preceding part of the proposition, is perpendicular to BC, and by hypothesis, CA is also perpendicular to it, they both pass through the pole of BC (IV. *Cor.* 4). Hence their intersection K is the pole of BC; and in the same way it may be proved that Q is the pole of AC.

Again, since B is the pole of HGM, BG is a quadrant; and for the same reason AG is a quadrant: hence G is the pole of AB.

Lastly, since BM, MK are quadrants, M is the pole of BK; and H, the other intersection of BM, MK is the opposite pole. In the same manner it may be proved that R and S are the opposite poles of AQ. The second part of the proposition is, therefore, proved.

PROP. X. PROBLEM[*].

Given the parts of the right-angled triangle ABC (last figure), *to find the sides and angles of the triangles* ALK, KRG, GHQ, QPB.

1. THE angles L, R, H, P were found in the last proposition to be right angles.

2. The angle AKL is measured by CM, since K is the pole of BCM. Hence $AKL = BM - BC = \frac{1}{2}\pi - a$; and similarly $PQB = \frac{1}{2}\pi - b$. Also, the angles KAL, QBP being vertical to BAC, ABC, are equal to A and B, respectively: and RKG, HQG being opposite to AKL, BQP, are, also, respectively equal to $\frac{1}{2}\pi - a$, and $\frac{1}{2}\pi - b$.

Again, B being the pole of HGK, and A that of RGQ, the angles RGK, HGQ are each formed by the polar circles taken in the *same direction:* and hence also they are each measured by AB, the distance of the poles, or $RGK = HGQ = c$.

All the angles of the four triangles are, therefore, determined.

3. Since BL, BM are quadrants, LM is the measure of the angle B, and $KL = \frac{1}{2}\pi - B$; and similarly $PQ = \frac{1}{2}\pi - A$. Also, KC being a quadrant, $AK = \frac{1}{2}\pi - b$; and similarly $BQ = \frac{1}{2}\pi - a$. Finally, since BL and AP are quadrants, $BP = AL = \frac{1}{2}\pi - c$. The sides of the triangles AKL, PBQ, are, therefore, determined.

Moreover, since B is the pole of HKM, BH is a quadrant; and $HQ = HB - BQ = \frac{1}{2}\pi - (\frac{1}{2}\pi - a) = a$: and, similarly, $RK = b$. Also, G being the pole of AB, and Q that of AC, the side QG of the triangle is the measure of the angle BAC, or $QG = A$; and, similarly, $GK = B$. Lastly, since Q is the pole of RCS, QR is a quadrant, and $GR = RQ - QG = \frac{1}{2}\pi - A$; and, similarly, $HG = \frac{1}{2}\pi - B$. The sides of all the four triangles, ALK, KRG, GHQ, and QPB, are hence found, and the problem entirely solved.

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[*] To avoid circumlocution, algebraic symbols are introduced into this investigation, viz., the angles of the triangle ABC are denoted by A, B, C, and their opposite sides by a, b, c: also the quadrant is represented by its usual algebraic value to radius 1, viz., by $\frac{1}{2}\pi$.

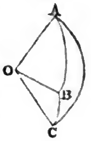
Scholium.

As these values will form the foundation of the entire system of *spherical trigonometry*, it will be convenient to tabulate them as follows:—

Parts.	ΔABC	ΔKAL	ΔGKR	ΔQGH	ΔBQP
Base	$BC=a$	$AL=\frac{1}{2}\pi-c$	$KR=b$	$GH=\frac{1}{2}\pi-B$	$PQ=\frac{1}{2}\pi-A$
Perpend.	$CA=b$	$LK=\frac{1}{2}\pi-B$	$RG=\frac{1}{2}\pi-A$	$HQ=a$	$PB=\frac{1}{2}\pi-c$
Hypoth.	$AB=c$	$KA=\frac{1}{2}\pi-b$	$GK=B$	$QG=A$	$BQ=\frac{1}{2}\pi-a$
∠ at base	$ABC=B$	$KAL=A$	$GKR=\frac{1}{2}\pi-a$	$QGH=c$	$BQP=\frac{1}{2}\pi-b$
∠ at vertex	$CAB=A$	$LKA=\frac{1}{2}\pi-a$	$RGK=c$	$HQG=\frac{1}{2}\pi-b$	$QBP=B$

PROP. XI. THEOREM.

Any two sides of a spherical triangle are together greater than the third, and the difference of any two is less than the third.



For, let O be the centre of the sphere upon which the spherical triangle ABC is described, and join OA, OB, OC.

Then AOB, BOC, COA, are the plane angles which contain the solid angle at O. Then of these angles, the sum of any two is greater than the third, and the difference of any two is less than the third (*vol.* i. *p.* 360). Also, these angles are measured by the arcs of the circles which form the spherical triangle ABC; and hence the proposition is true of these arcs, that is of the sides of the triangle.

PROP. XII. THEOREM.

In a spherical triangle, the three sides are together less than an entire circumference, and each side is less than a semicircle.

(*See figure, pr. xi.*)

1. For the three angles at O are together less than four right angles (*vol.* i. *p.* 361), and hence their measures AB, BC, CA, are together less than four quadrants: that is, than an entire circumference.

2. Since any two sides, AB, BC, are together greater than the third AC, it is obvious that one side, AC, must be less than half the sum of all the three sides AB, BC, CA; that is, by the preceding case, less than a semicircle*.

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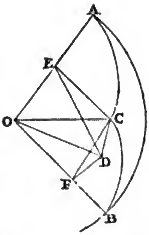
* The theorems given in this part of the work being limited to those which are essential to numerical solution, the student should exercise himself upon those enunciated as exercises at pages 22, 23.

PROP. XIII. THEOREM.

In a spherical triangle, if two of the angles be equal, the sides opposite to them are equal; and conversely, if two sides be equal, the angles opposite to them are equal.



1. Let the angles CAB, CBA, be equal; then shall BC be equal to CA.

For, draw CD, perpendicular to the plane AOB; and in the same plane draw DE, DF, perpendicular to OA, OB, and join EC, FC.

Then (*vol. i. p.* 351), CE and CF are perpendicular to OA and OB; and hence the angles CED, CFD, are equal to the spherical angles at A and B. But (*by hypothesis*) these are equal; and hence in the right-angled plane triangles CDE, CDF, we have the angles CED, CFD, equal, and the side CD common; wherefore also EC is equal to CF.

Again, in the right-angled triangles CEO, CFO, we have EC equal to CF and CO common; hence the angles COE, COF, are also equal, and their measures AC, CB, also equal.

2. Let the sides AC, CB, be equal, then the angles at A and B will also be equal.

For make the same construction: then, since the sides AC, CB, are equal, the angles AOC, COB, are equal; and since OC is common to the two right-angled triangles CEO, CFO, we have EC equal to FC. Wherefore, in the right-angled triangles CED, CFD, we have EC, CD, equal to FC, CD, each to each; and therefore the angles CED, CFD, also equal: and these are equal to the spherical angles at A and B.

Corollary.

In every spherical triangle, the greater side and greater angle are opposite.

For, let AC be greater than CB; then the angle COA is greater than COB, and therefore EC greater than CF; wherefore, pursuing a train of reasoning analogous to that in the proposition, we have CD common to the triangles CDE, CDF, but EC greater than CF; and hence the angle CFD greater than CED, that is, B greater than A.

PROP. XIV. THEOREM.

Every angle of a spherical triangle is less than two right angles, and all three together less than six right angles.

1. For every dihedral angle is less than two right angles, and hence every spherical angle which has the same measure, is less than two right angles.

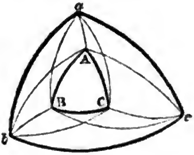
2. Hence the three together are less than three times two right angles; that is, less than six right angles.

Corollary.

Hence, contrary to what takes place in plane triangles, there may be two, or even three, of the angles of a spherical triangle, either right angles or obtuse angles.

PROP. XV. THEOREM.

The three angles of any spherical triangle are together greater than two right angles.



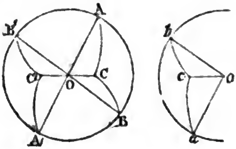
For, let ABC be any spherical triangle, and *abc* its cross-polar; and draw the arcs A*b*, A*c*, B*a*, B*c*, C*a*, C*b*.

Then, since A is the pole of *bc*, the angle *b*A*c* is measured by *bc*; and since BAC and *bc* are supplementary, the angles BAC and *b*A*c* are together equal to two right angles. In like manner, ABC and *a*B*c* are together equal to two right angles, and likewise BCA and *b*C*a* are together equal to two right angles. Hence the six angles ABC, BCA, CAB, *a*B*c*, *b*C*a*, and *c*A*b*, are together equal to six right angles.

But the angles *b*A*c*, *c*B*a*, *a*C*b*, are measured by the sides of the triangle *abc*, and these are together less than an entire circumference, and the angles themselves, therefore, less than four right angles. Wherefore the remaining three angles ABC, BCA, CAB, of the spherical triangle ABC, are greater than two right angles.

PROP. XVI. THEOREM.

If two triangles have three sides of the one equal to three sides of the other, each to each; the angles will be equal which are opposite to the equal sides: and conversely, if the three angles of one be equal to the three angles of the other, the sides opposite to the equal angles will be equal.



First. Suppose the sides to follow in the same order, and let the triangles be *abc*, A′B′C′. Then, as in plano, the sides *ab*, *bc*, *ca*, will be capable of coincidence respectively with A′B′, B′C′, C′A′, and the conclusion will follow.

Second. Suppose them not in the same order, as *abc*, ABC. Then produce the edges OA, OB, OC, to A′, B′, C′, and let the planes of each two cut the spherical surface in B′C′, C′A′, A′B′. Then these are equal to BC, CA, AB, respectively, and hence also to *bc*, *ca*, *ab*, respectively.

But the dihedral angles at OA′, OB′, OC′, are equal to those at OA, OB, OC, since they are vertical to them; and the dihedral angles at *oa*, *ob*, *oc*, are also equal to those at OA′, OB′, OC′, (*vol. i. th.* 24, *p.* 361,) since they are contained by equal plane angles, or those measured by equal arcs. Whence the dihedral angles, that is, the spherical angles of the triangles ABC, *abc*, are equal each to each.

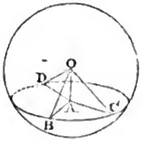
In like manner, by a mere contra-position of the terms of the argument, the converse case may be proved.

Scholium.

By methods correspondent to these, the equalities analogous to those where different combinations of the parts of plane triangles are concerned, may be established. It is not, however, necessary to pursue the subject further here; as it will be resumed in a future page, by means of the trigonometrical equations amongst the several parts.

PROP. XVII. THEOREM.

Every section of a sphere, made by an excentric plane, is a circle.

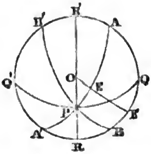


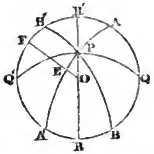
Let O be the centre of the sphere, and BCD any excentric plane section. From O draw OA perpendicular to the excentric plane BCD, and draw any lines in that plane, as AB, AC, AD, *etc.* and join OB, OC, OD, *etc.*

Then, (*vol. i. p.* 349,) OAB, OAC, OAD, *etc.* are right-angled triangles, having OA common, and the sides OB, OC, OD, *etc.* all equal (*def. i.*) Hence the sides AB, AC, AD, *etc.* are all equal. The points B, C, D, *etc.* therefore, being situated in the plane BCD, and their distances from A all equal, the figure BCD is a circle (*vol. i. def.* 15, *p.* 291).

PROP. XVIII. THEOREM.

Let two great semicircles cut at right angles: then from any point in one of them, innumerable pairs of circles may be drawn, so that each pair shall make equal angles with the other semicircle.





Let RPR′, RQR′, be two semicircles intersecting at right angles; then through any point P in the former, innumerable circles may be drawn in pairs, as PA, PB, which shall make, with the latter, angles PAR, PBR, equal to one another.

Bisect the semicircle RQR′ in Q; take any two equal arcs AQ, BQ, in RQR′, and draw the circles AP, PB. These will make the angles RBP, RAP, equal to one another.

For, produce the circles AP, BP, and draw QP to meet the completion of the circle RQR′ in A′, B′, Q′. Then, we have

$$BR+RA=(RQ-QB)+(RQ+QA)=2RQ=RQR'=\pi=ARA'=AR+R'A.$$

Wherefore, $A'R=RB$, and consequently also, $RBP=RA'P=RAP$. Whence, taking AQ, QB, *of any magnitude*, but equal to each other, the angles formed by PA, PB, with RQR′ will be equal; which establishes the proposition.

Corollary 1.

If the point P lie (as in the first figure) between R and the pole O of the circle RQQ′, the angles PAR, PBR, will be *acute*, and have as their greatest limit the right angles PRQ′, PR′Q: but if it lie beyond O estimated from R, (as in the second figure,) the angles PAR, PBR, will be *obtuse*, and have for their least limit the same right angles PRQ′, PR′Q.

Corollary 2.

On the hypothesis of the last corollary, PQR will be the least in the former figure, and the greatest in the latter, of all the angles that can be made by circles passing through P and cutting the circle RQR′.

For draw OEF perpendicular to PA, the one of the circles that passes between PR′ and PQ, meeting it in E, and RQR′ in F. Then since the angle R′PA is acute, it is less than OEP, (*fig.* 1,) and therefore also, OE is less than OP; and as OR = OF = $\frac{1}{2}\pi$, we have EF greater than PR. But AA′ is bisected in E, or AE is a quadrant, and at right angles to EF; wherefore, A is the pole of EF, and the angle EAF or PAR is measured by EF. Also PQR is measured by PR; wherefore, PQR is less than PAR.

In the second figure we may reason similarly: but it will be simpler to deduce the conclusion from the supplements of the angles. Thus, since R′QP (*fig.* 2) is less than R′AP, it follows that PQR is greater than PAR.

Corollary 3.

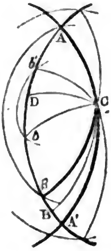
The circles PA, PB, make equal angles with the circle OP or RR′, as is at once obvious.

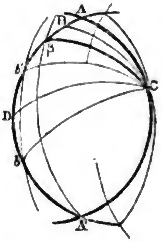
Corollary 4.

If from any point P on the surface of a hemisphere bounded by the circle R′QRQ′, circles PR, PB, PQ, PA, PR′, *etc.* be drawn to meet that circle: then the greatest is that which passes through the pole, and the least is that constituting the remainder of the semicircle; those which pass, one on each side of the greatest, at equal distances from the pole (or which make equal angles on opposite sides with the greatest) are equal, *etc.* to correspond with *Ex. iii.* 7, 8.

PROP. XIX. PROBLEM.

Two sides of a triangle and the angle opposite to one of them being given, it is required to investigate the number of solutions of which a triangle constructed with these data admits.





Let A be the given angle, BC the given side opposite to A, and AC the other given side: it is required to find when the point B can have two positions, and when only one, compatible with the given magnitudes A, a, b.

[In fig. 1. A is *acute*, and in fig. 2. A is *obtuse*.]

Produce AB, AC, to meet in A′; and from C draw CD perpendicular to ABA′. With pole C, and spherical radius equal to any given value of the side a, describe circles to cut ADA′, which each will do, as for instance, in two points, b, b'.

Now, whilst a is less than CD, there will be no intersection, and hence no solution at all; and when a = CD there will be one solution, the triangle being right-angled and isolated.

When a is less (in *fig.* 1) or greater (in *fig.* 2) than either CA or CA′ (but respectively greater or less than CD), there will be two intersections b and b' (*pr. xviii. cor.* 4); and hence two triangles AbC and Ab'C, which have all the data, and therefore two solutions.

When a is equal to, or greater than the less of the two CA, CA′, but less than the other, there will be only *one* solution as AβC, or ABC.

When a is greater both than CA and CA′, the triangle that will be formed will not have the given angle A, but its supplement in the one case; and though formed with A in the other case, one of its sides will be greater than a semicircle, contrary to *prop. xii.*

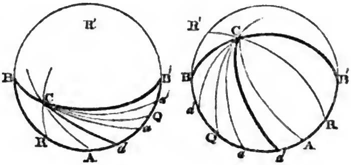
Whilst, therefore, the side a lies between b and $\pi - b$ in value, there will be only one solution: but in all other cases, either two or none.

Corollary.

The side b and its opposite angle B, in the case of the solution being unique, are of the same affection; as is evident from *prop. xviii. cor.* 2.

PROP. XX. PROBLEM.

Two angles of a triangle and the side opposite to one of them being given, it is required to investigate the number of solutions of which a triangle constructed with these data, admits.



Let B be the given angle to which the given side BC is adjacent: it is required to find when two solutions, and when only one, will be compatible with the data.

[In *fig.* 1. B is *acute*, and in *fig.* 2. B is *obtuse*].

Draw CR perpendicular to the side AB, and CQ perpendicular to CR: produce BC, BA, to meet again in B′, make Qd = QB′, and draw the great circle Cd.

Then, since CQ, RQ, are perpendicular to CR, Q is the pole of CR, and QR is a quadrant: wherefore (*pr. xviii.*) CdR = CB′R = CBR. There will hence for any value of the angle A between CdR and CQR be two solutions, as a'BC and aBC; but for any value which shall give to A a position between d and B in *fig.* 1, and between d and B′ in *fig.* 2, there is only one solution as ABC.

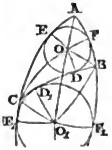
Now the limit to the value of A on the right being CdR, where A = B, and the limit on the left being CBR′ where A = Π − B: and the progress from the one to the other being continuous, the condition of the solution being single, is that A lies between B and π − B. When it is either less in *fig.* 1, or greater in *fig.* 2, than CQR, there is no solution; and when between the limits B and Q, there will be two solutions. Whilst, therefore, the value of the angle A is between B and Π − B, there will be but one solution: but in all other cases, either two or none.

Corollary.

The side b and its opposite angle B, in case of unique solution, will be of the same affection, as is evident from *prop. xviii. cor.* 4.

PROP. XXI.

If the three interior angles of a spherical triangle be bisected, the bisecting arcs will meet in one point, which is the centre of the inscribed circle: and if either two exterior angles be bisected, the bisecting arcs will meet in that which bisects the third interior one; and the point of section is the pole, or spherical centre, of the corresponding inscribed triangle.



Let O be the point of section of the arcs BO, CO, which bisect the interior angles B and C, and join OA: also draw OD, OE, OF, perpendicular to BC, CA, AB. Then the two triangles DOB, FOB, have the angles ODB, DBO, equal to OFB, FBO, and the side OB common; and they are symmetrically situated: whence OF = OD. Similarly OD = OE, and hence EO = OF.

Again, since EO, OA, are equal to FO, OA, and the angles at E and F equal, and the triangles symmetrically situated, the angles EAO, FAO, are equal, or A is bisected by the arc AO.

Also, since OD, OE, OF, are equal, and at right angles to the sides of the triangle ABC, it may be shown (as in plano) that O is the centre of the circle inscribed in ABC.

In a manner too obviously similar to the preceding to need specification in this place, the point O_1 is proved to be the centre of the circle escribed to A.

Scholium.

By a method perfectly analogous to that employed in plano, availing ourselves of *pr. xiii.* we may show that perpendiculars from the middles of the sides of a spherical triangle and its associates, will meet in the centres of their several circumscribing circles.

PROP. XXII.

To find the values of the several parts of the sides of the associated system of triangles made by the contacts of the inscribed and escribed circles; and of the angles of the system made by the arcs drawn from the angles of the several triangles to the centres of their circumscribing circles.

For the purpose of symmetrical notation, we shall use (*fig. next page*)

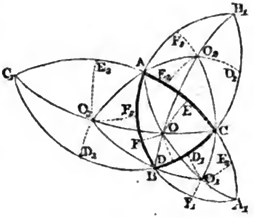
a, b, c, and A, B, C, for the sides and angles of ABC
a_1, b_1, c_1, and A_1, B_1, C_1, ... A′BC
a_2, b_2, c_2, and A_2, B_2, C_2, ... AB′C
a_3, b_3, c_3, and A_3, B_3, C_3, ... ABC′.

The subscribed numbers referring to the order a, b, c, of the common sides of the fundamental and supplemental triangles.

1. The values of the several sides and angles of the associated system, in terms of those of the fundamental triangle, are, evidently, as follow:—

$$\begin{array}{l|l|l|l|l|l}
a_1 = a & a_2 = \pi - a & a_3 = \pi - a & A_1 = A & A_2 = \Pi - A & A_3 = \Pi - A \\
b_1 = \pi - b & b_2 = b & b_3 = \pi - b & B_1 = \Pi - B & B_2 = B & B_3 = \Pi - B \\
c_1 = \pi - c & c_2 = \pi - c & c_3 = c & C_1 = \Pi - C & C_2 = \Pi - C & C_3 = C
\end{array}$$

2. Let O be the centre of the circle inscribed in the fundamental triangle ABC, O_1 of that in A′BC, O_2 of that in AB′C, and O_3 of that in ABC′: and draw the perpendiculars, as in the figure, from these centres upon the sides of the several triangles.



Then, by the properties of spherical triangles already developed, we shall have in the triangle ABC,
AF = AE, BF = BD, and CD = CE.
Put these arcs respectively equal to α, β, γ: that is, in fact, making

$$a = \beta + \gamma,\ b = \gamma + \alpha, \text{ and } c = \alpha + \beta.$$

Hence by the common methods of solution of these equations,

$$\left.\begin{aligned} \alpha + \beta + \gamma &= \tfrac{1}{2}(\quad a + b + c) = s \\ \alpha &= \tfrac{1}{2}(-a + b + c) = s - a \\ \beta &= \tfrac{1}{2}(\quad a - b + c) = s - b \\ \gamma &= \tfrac{1}{2}(\quad a + b - c) = s - c \end{aligned}\right\} \cdots\cdots\cdots\cdots \quad (1)$$

Again, in the triangle A′BC, pursuing a similar method, and denoting the several corresponding quantities by $\alpha_1, \beta_1, \gamma_1, a_1, b_1, c_1, s_1$, and then substituting the values of a_1, b_1, c_1, s_1, in terms of a, b, c, s, we shall have

$$a_1 = \beta_1 + \gamma_1,\ b_1 = \gamma_1 + \alpha_1, \text{ and } c_1 = \alpha_1 + \beta_1; \text{ and hence,}$$

$$\left.\begin{aligned} s_1 = \alpha_1+\beta_1+\gamma_1 &= \tfrac{1}{2}(\quad a_1+b_1+c_1) = \tfrac{1}{2}\{\quad a+(\pi-b)+(\pi-c)\} = \pi-(s-a) \\ s_1 - a_1 = \quad \alpha_1 &= \tfrac{1}{2}(-a_1+b_1+c_1) = \tfrac{1}{2}\{-a+(\pi-b)+(\pi-c)\} = \pi - s \\ s_1 - b_1 = \quad \beta_1 &= \tfrac{1}{2}(\quad a_1-b_1+c_1) = \tfrac{1}{2}\{\quad a-(\pi-b)+(\pi-c)\} = s - c \\ s_1 - c_1 = \quad \gamma_1 &= \tfrac{1}{2}(\quad a_1+b_1-c_1) = \tfrac{1}{2}\{\quad a+(\pi-b)-(\pi-c)\} = s - b \end{aligned}\right\} \quad (2)$$

In a precisely similar way, we shall obtain,

$$\left.\begin{aligned} s_2 = \alpha_2+\beta_2+\gamma_2 &= \tfrac{1}{2}(a_2 + b_2 + c_2) = \tfrac{1}{2}\{(\pi-a) + b + (\pi-c)\} = \pi-(s-b) \\ s_2 - a_2 = \quad \alpha_2 &= \tfrac{1}{2}(-a_2+b_2+c_2) = \tfrac{1}{2}\{-(\pi-a)+b+(\pi-c)\} = s - c \\ s_2 - b_2 = \quad \beta_2 &= \tfrac{1}{2}(a_2 - b_2 + c_2) = \tfrac{1}{2}\{(\pi-a) - b + (\pi-c)\} = \pi - s \\ s_2 - c_2 = \quad \gamma_2 &= \tfrac{1}{2}(a_2 + b_2 - c_2) = \tfrac{1}{2}\{(\pi-a) + b - (\pi-c)\} = s - a \end{aligned}\right\} . \quad (3)$$

And again, similarly,

$$\left.\begin{aligned} s_3 = \alpha_3+\beta_3+\gamma_3 &= \tfrac{1}{2}(a_3 + b_3 + c_3) = \tfrac{1}{2}\{(\pi-a) + (\pi-b) + c\} = \pi-(s-c) \\ s_3 - a_3 = \quad \alpha_3 &= \tfrac{1}{2}(-a_3+b_3+c_3) = \tfrac{1}{2}\{-(\pi-a)+(\pi-b)+c\} = s - b \\ s_3 - b_3 = \quad \beta_3 &= \tfrac{1}{2}(a_3 - b_3 + c_3) = \tfrac{1}{2}\{(\pi-a) - (\pi-b) + c\} = s - a \\ s_3 - c_3 = \quad \gamma_3 &= \tfrac{1}{2}(a_3 + b_3 - c_3) = \tfrac{1}{2}\{(\pi-a) + (\pi-b) - c\} = \pi - s \end{aligned}\right\} . \quad (4)$$

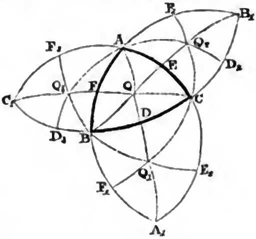
3. It may be of advantage for the sake of ready reference, to give the values of the several parts of the figure in terms of the sides, a, b, c, of the fundamental triangle.

$$\begin{aligned} AE_1 &= AF_1 = BD_2 = BF_2 = CD_3 = CE_3 = s \\ AE &= AF = BD_3 = BF_3 = CD_2 = CE_2 = s - a \\ BD &= BF = AE_3 = AF_3 = CE_1 = CD_1 = s - b \\ CE &= CD = AF_2 = AE_2 = BF_1 = BD_1 = s - c \end{aligned}$$

$$\begin{aligned} EE_1 &= FF_1 = BC = a \\ DD_2 &= FF_2 = CA = b \\ EE_3 &= DD_3 = AB = c \end{aligned}$$

$$\begin{aligned} DD_1 &= CD \sim DB = CA \sim AB = b \sim c \\ EE_2 &= AE \sim EC = AB \sim BC = c \sim a \\ FF_3 &= BF \sim FA = BC \sim CA = a \sim b. \end{aligned}$$

The parts thus valued are correspondent to the parts of the plane triangle, as divided by the contacts of the inscribed and escribed radii.



4. Let Q, Q_1, Q_2, Q_3, be the centres of the circles circumscribing the associate system of triangles, and draw the system of arcs as in the annexed figure. Let the angles QBC, QCB, made by the radii QB, QC, with BC be denoted by α'; QCA, QAC, made with AC by β', and QAB, QBA, made with AB by γ'.

Then we have $A = \beta' + \gamma'$, $B = \gamma' + \alpha'$, and $C = \alpha' + \beta'$; and hence

$$\left.\begin{aligned} \alpha' + \beta' + \gamma' &= \tfrac{1}{2}(\quad A + B + C) = S \\ \alpha' &= \tfrac{1}{2}(- A + B + C) = S - A \\ \beta' &= \tfrac{1}{2}(\quad A - B + C) = S - B \\ \gamma' &= \tfrac{1}{2}(\quad A + B - C) = S - C \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (5)$$

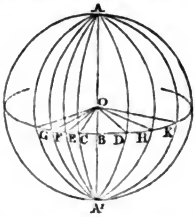
Proceeding in the same manner as in (2, 3, 4) we get

$$\left.\begin{array}{r|r|r} S_1 = \Pi - (S - A) & S_2 = \Pi - (S - B) & S_3 = \Pi - (S - C) \\ S_1 - A_1 = \Pi - S & S_2 - A_2 = S - C & S_3 - A_3 = S - B \\ S_1 - B_1 = S - C & S_2 - B_2 = \Pi - A & S_3 - B_3 = S - A \\ S_1 - C_1 = S - B & S_2 - C_2 = S - A & S_3 - C_3 = \Pi - S \end{array}\right\} \cdots \quad (6, 7, 8)$$

For the angles of the adjacent triangles are supplementary (being adjacent) as the sides were in the former case, and the *expressions* will be identical, A being substituted for *a*, A_1 for a_1, and so on.

PROP. XXIII.

Spherical lunes and spherical wedges (def. 10) are to one another as the spherical angles formed at the surface of the sphere by the enclosing arcs of great circles.



Let the planes meeting in the diameter AA′ of the sphere, cut the surface in ACA′, ABA′, ADA′. Then the lune contained by ACA′, ABA′, will be to that contained by ABA′, ADA′, as the angle CAB to the angle BAD.

For, let the plane CBD pass through the centre perpendicular to the diameter AA′, cutting the planes which form the lunes in C, B, D. Take in this the arcs CE, EF, FG, *etc.* each equal to BC, and the arcs DH, HK, *etc.* each equal to BD; and through the axis AA′, and the several points E, F, G, *etc.* and H, K, *etc.* draw planes cutting the surface of the sphere in the great circles AEA′, AFA′, AGA′, *etc.* AHA′, AKA′, *etc.*

Then it may be proved by super-position that the lunes BACA′, CAEA′, EAFA′, FAGA′, *etc.* are all equal; and likewise that BADA′, DAHA′, HAKA′, *etc.* are all equal. Moreover, that the angles CAB, CAE, EAF, FAG, *etc.* which are equal to COB, COE, EOF, FOG, *etc.* and which are subtended by the equal arcs BC, CE, EF, FG, *etc.* are all equal; as likewise the angles BAD, DAH, HAK, *etc.* are all equal.

Then, reasoning precisely as in *th.* 94, vol. i. p. 337, the asserted conclusion follows at once.

In the same manner it may be proved that the *wedges* are proportional to the angles at A.

Corollary 1.

The area of any lune is to the surface of the sphere as A to 2Π *, A being the angle of the lune.

Corollary 2.

Also, since the area of the spherical surface is represented by $4r^2\varpi$, we have the expression of the area of the lune to A,

$$\text{lune A} = \frac{\text{A}}{2\Pi} \cdot 4r^2\varpi = \frac{\text{A}}{\Pi} \cdot 2r^2\varpi.$$

Of particular cases notice should be taken of $\text{A} = \frac{1}{2}\Pi$ and $\text{A} = \Pi$; in which cases we have, severally,

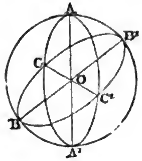
$$\text{lune } 90^\circ = r^2\varpi, \text{ and lune } 180^\circ = 2r^2\varpi.$$

It will be advantageous to write the mark) for the word lune; in which case, the expressions above will be briefly expressed as follows:—

$$\text{A}^\circ) = \frac{\text{A}^\circ}{90^\circ} \cdot r^2\varpi; \; 90^\circ) = r^2\varpi; \text{ and } 180^\circ) = 2r^2\varpi.$$

PROP. XXIV. THEOREM.

The area of a spherical triangle is to the surface of the hemisphere, as the excess of its three angles above two right angles is to four right angles.



Let ABC be the spherical triangle, and complete the great circles which constitute its containing sides. Then the triangle A′B′C′ is equal in all respects to ABC (*pr. xv.*), and the hemisphere is equal to the four triangles A′BC, AB′C, ABC′, and ABC or A′B′C′. But we have

$$\begin{aligned} \Delta\text{A}'\text{BC} &= \text{A}) - \Delta\text{ABC} \\ \Delta\text{AB}'\text{C} &= \text{B}) - \Delta\text{ABC} \\ \Delta\text{ABC}' &= \text{C}) - \Delta\text{ABC, or A}'\text{B}'\text{C}'. \\ \Delta\text{ABC} &= \qquad\;\; \Delta\text{ABC}. \end{aligned}$$

Whence by addition we have at once

$$\text{hemisphere} = \Pi) = \text{A}) + \text{B}) + \text{C}) - 2\Delta\text{ABC}; \text{ or}$$
$$2\Delta\text{ABC} = \text{A}) + \text{B}) + \text{C}) - \Pi).$$

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* In this work a notation is introduced that is intended to remove a source of perplexity which is found to arise from the use of a single symbol π to signify two different, though related, magnitudes;—the semicircle to radius unity, and two right angles. It is true, that in the great majority of cases, from the two magnitudes not coming into the same investigation, the inconvenience is not felt: yet, as in the present case, it does sometimes force itself upon our attention, and embarrass our reasonings, it is desirable to avoid it systematically. The symbol has, however, become so familiar in both uses, that it would be difficult to substitute another in its place, in either of the senses. Fortunately, the same letter having the two forms π and ϖ, we may, by constantly employing one form in one sense and the other in the other, remove the entire difficulty. I have used ϖ to signify the semicircle, solely from its curved form, and thus supplying a *memorial* ground for choice. I have, however, thought that the system would be still further improved by the use of the *capital letter*, Π, for designating two right angles in trigonometry proper, as the angles are usually denoted by the capital letters A, B, C. The first of the lower-case forms π, to designate the *arc* which measures two right angles considered in reference to trigonometrical purposes: whilst, as above, the second form ϖ signifies merely the number 3·14159 ......

But $A) = \frac{A}{\Pi} . 2\varpi r^2$, $B) = \frac{B}{\Pi} . 2\varpi r^2$, $C) = \frac{C}{\Pi} . 2\varpi r^2$, $\Pi) = \frac{\Pi}{\Pi} . 2\varpi r^2$; and hence, substituting and dividing by 2, we get

$$\Delta ABC = \frac{A + B + C - \Pi}{2\Pi} . 2\varpi r^2;$$

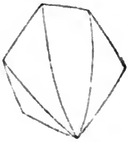
which is identical in signification with the verbal enunciation of the theorem.

Corollary 1.

Since the only variable quantity in the expression for the area of the spherical triangle is the *spherical excess*, $E = A + B + C - \Pi$, the excess is a correct representation of the area in terms of the unit $\frac{\varpi r^2}{\Pi}$.

Corollary 2.

A polygon of n sides being divisible into $n - 2$ triangles, and the area of each triangle being expressible by means of the spherical excess, we shall have, if Σ signify the sum of all the angles of the polygon,



$$\text{polygon} = \frac{E_1 + E_2 + . E_{n-2}}{2\Pi} . 2\varpi r^2 = \frac{\Sigma - (n-2)\Pi}{2\Pi} . 2\varpi r^2.$$

Corollary 3.

If on the surface of the sphere there be described p_2 polygons, one of which has n_1 sides, another has n sides, and so on to that which contains n_p sides (any two or more of these numbers of sides, however, may be equal); and if Σ_1, Σ_2, . . Σ_p, be the sums of the angles of the several polygons, P_1, P_2, . . . P_p:

$$\text{then } P_1 = \frac{\Sigma_1 - (n_1 - 2)\Pi}{2\Pi} . 2\varpi r^2, \quad P_2 = \frac{\Sigma_2 - (n_2 - 2)\Pi}{2\Pi} . 2\varpi r^2, \textit{ etc.}$$

And by addition, we have

$$P_1 + P_2 + \ldots P_p = \frac{\Sigma_1 + \Sigma_2 + \ldots \Sigma_p - (n_1 + n_2 + \ldots n_p - 2p)\Pi}{2\Pi} . 2\varpi r^2.$$

Corollary 4.

If the polygons P_1, P_2, . . . P_p, together entirely and exactly cover the sphere, we shall have

$$4\varpi r^2 = \frac{\Sigma_1 + \Sigma_2 + \ldots \Sigma_p - (n_1 + n_2 + \ldots n_p - 2p)\Pi}{2\Pi} . 2\varpi r^2;$$

$$\text{or } \Sigma_1 + \Sigma_2 + \ldots . \Sigma_p = \{n_1 + n_2 + \ldots n_p - 2(p-2)\}\Pi.$$

Corollary 5.

If the sums of the angles of any two spherical figures bounded by great circles be equal to one another, their areas will also be equal, whether the angles be equal each to each or not, or whether the figures have the same number of sides or not.

Corollary 6.

The areas of two similar * figures on *different spheres* whose radii are as r to

---

* *Similar spherical figures* are those which can be divided into triangles, such that all the angles of one set of triangles are equal to all the angles of the other set, each to each, and in the same order. These can only exist on spheres of different radii: for on the same or equal spheres such figures will be identical or equal. *See also note, p.* 61.

r_1, are to one another as r^2 to r_1^2; and where the *excesses* differ also, the areas are as r^2E to $r_1^2E_1$.

PROP. XXV.

If the surface of the sphere be divided into any number of polygons by arcs of great circles, and each having any number of sides: then the number of sides in all increased by two will be equal to the number of polygons together with the number of angular points.

Let there be E sides, F polygons, and S angular points: then universally

$$E + 2 = F + S.$$

For, let the F polygons have $n_1, n_2, \ldots . n_r$ sides respectively, and let the sums of their angles be severally denoted by $\Sigma_1, \Sigma_2, \ldots \Sigma_r$. Then

$$\Sigma_1 + \Sigma_2 + \ldots \Sigma_r = 2S\Pi \ \ldots\ldots\ldots. \ (1)$$

$$n_1 + n_2 + \ldots n_r = 2E \ldots\ldots\ldots\ldots \ (2)$$

the former because the sum of all the angles of all the polygons is equal to four times as many right angles as there are angular points: and the second because each side is common to two polygons.

Also, (*prop. xxiv. cor.* 4) we have, writing F instead of p,

$$\Sigma_1 + \Sigma_2 + \ldots . \Sigma_r = (n_1 + n_2 + \ldots n_r)\Pi - 2\,(F - 2)\Pi\,;$$

in which, inserting (1, 2), we get at once

$$2S\Pi = 2E\Pi - 2F\Pi + 4\Pi, \text{ or}$$

$$E + 2 = F + S \ \ldots\ldots \ (3)$$

Corollary 1.

The same relation will subsist in any polyhedron having F faces, E edges, and S solid angles: for drawing lines from all the angles to any point *within* the solid, and about that point describing a sphere, we shall have a spherical polygon with as many angular points as the solid has solid angles, as many polygons as the figure has faces, and each polygon having the same number of edges as the corresponding polygonal face of the solid; that is, as many sides as the solid has edges.

Corollary 2.

In such an assemblage of spherical polygons, the number of odd-sided ones is always even.

Let there be F_3 triangles, F_4 quadrilaterals, F_5 pentagons, and so on. Then

$$2F = 2F_3 + 2F_4 + 2F_5 + \ldots$$

$$2E = 3F_3 + 4F_4 + 5F_5 + \ \ldots.$$

Whence, these being substituted in $2E - 2F = 2(S - 2)$ and the result reduced to the appropriate form, we get

$$F_3 + F_5 + F_7 + \ldots = 2\left\{S - 2 - F_4 - F_5 - 2F_6 - 2F_7 - \ldots.\right\}$$

and as all terms, S, F_3, F_4, *etc.* are integer, the right-hand side is necessarily an even number; and the corollary is hence established.

PROP. XXVI.

To investigate the number and character of the equiangular polygons into which the surface of the sphere can be divided.

Let there be S points of junction; E arcs of junction, or sides of all the polygons, of which m meet in each of the S points; and let there be n sides to each polygon, and F polygons in all.

Then $\frac{2\Pi}{m}$ is the angle made by each two arcs of junction, or, in other words, the angle of the polygon; and hence $\Sigma = \frac{2n\Pi}{m}$ expresses the sum of the angles of one polygon; and the sum of the angles of the F equal polygons is, therefore,

$$n\Sigma = \frac{2nF\Pi}{m} \;\ldots\ldots\ldots\ldots\; (1).$$

But (*prop. xxiv. cor.* 4) supposing $\Sigma_1 = \Sigma_2 = \ldots \Sigma$, and $n_1 = n_2 = \ldots = n$, we have

$$\frac{2nF\Pi}{m} = \{Fn - 2(F-2)\}\,\Pi, \text{ or } F = \frac{4m}{2n-(n-2)m} \;\ldots\ldots\ldots\; (2).$$

Also, from *prop. xxiv. cors.* (2), (3), we have in the present case

$$E = \tfrac{1}{2}Fn \;\ldots\ldots\; (3) \qquad | \qquad S = E - F + 2 \;\ldots\ldots\; (4).$$

Now from the conditions of the problem, neither n nor m can be less than 3: subject to which conditions we must solve equation (2) in *positive integer values* of F, m, and n. This, it will be readily seen, will be effected *only* by the numbers given below. Moreover, if the chords joining the S points of junction be drawn, they will form plane regular polygons, the boundaries of the several figures, as named against them to the right in the following table.

n	F	m	E	S	Name of inscribed solid.
3	4	3	6	4	*Tetrahedron;*
3	8	4	12	6	*Octahedron;*
3	20	5	30	12	*Icosahedron;*
4	6	3	12	8	*Hexahedron*, or cube;
5	12	3	30	20	*Dodecahedron.*

Corollary 1.

Since $S = \frac{nF}{m}$ and $E = \frac{nF}{2}$, we shall have by substitution of (2), and also including (2) amongst the equations, the following elegant system of solutions.

$$S = \frac{4n}{2(m+n)-mn},\; E = \frac{2mn}{2(m+n)-mn},\; F = \frac{4m}{2(m+n)-mn}.$$

Corollary 2.

The sum of all the plane angles of a regular polyhedron is $2(S-2)\,\Pi$.

For the sum Σ of all the plane angles of one face is $(n-2)\,\Pi$, and hence that of the angles of F faces becomes

$$F\Sigma = F(n-2)\,\Pi = 2(E-F)\,\Pi = 2(S-2)\,\Pi.$$

Scholium.

These figures, called the *Platonic bodies*, are objects of real interest, from the circumstance of several of them being very familiar forms, assumed by crystallized substances: and moreover, from their being the only regular forms into which space can be divided without intervals.

EXERCISES IN PURE SPHERICAL GEOMETRY.

1. In the base AB of the triangle ABC, and in directions from each of the angular points towards the other, take segments AD, BE, equal to the sides AC, CB, and let O be the pole of the circle described through CDE: then the great circle OC will bisect the angle ACB.

2. From the angle B of the spherical triangle ABC draw the circle BD to bisect the angle B; from the middle E of the base AC draw ED perpendicular to the base meeting BD in D; and from D let DP be drawn perpendicular to the side AB: then

$$BP = \tfrac{1}{2}(AB + BC), \text{ and } PA = \tfrac{1}{2}(AB - BC).$$

3. Given two points A, B, on the same side of a great circle DE, to find in DE two points C, C′, such that AC + CB shall be the least possible, and that AC′ + C′B shall be the greatest possible.

4. Show that properties of the less circle analogous to those of *Euc. iii.* 7, 8, exist on the sphere, where great circles are substituted for straight lines.

5. If the perimeter and vertical angle of a spherical triangle be given, the base will touch a given circle.

6. If equal given arcs be taken in two parallel and equal less circles of the sphere, and these be joined in the manner of a parallelogram in plano, the figure enclosed will be of a given and constant magnitude.

7. If the base and area of a spherical triangle be given, the vertex of the triangle will always be in a less circle of the sphere, which may be determined.

8. If a great-circle quadrilateral be inscribed in a less circle of the sphere, the sum of one pair of opposite angles will be equal to the sum of the other pair.

9. If a great-circle quadrilateral be described about a less circle of the sphere, and from the angular points arcs be drawn to the pole of the less circle; then the sum of one pair of angles which are not adjacent to each other, will be equal to the sum of the other pair.

10. The following triads of great-circle arcs meet in a point:—

(1.) The arcs from the angles bisecting the sides.
(2.) The arcs bisecting the angles.
(3.) The perpendiculars from the angles to the sides.
(4.) The perpendiculars bisecting the sides.
(5.) The arcs bisecting one interior angle and the other two exteriorly.

11. Let ABC be a spherical triangle right-angled at C, and let another great circle cut AB in A′ and BC in C′, the latter at right angles: then if BD, BE, be the trigonometrical tangents of AB, BC, and BD′, BE′, those of A′B, BC′, it is to be proved that the straight lines DE, D′E′, will be parallel.

12. If P be the pole of a less circle circumscribing the spherical triangle ABC; prove that the spherical angle APB is double the plane angle ACB.

13. The arc of a great circle which joins the middle of the side of a spherical triangle with its pole, bisects the angle which joins the pole with the poles of the other two sides.

14. If in a spherical triangle, the angle C be equal to the sum of A and B, and an arc be drawn from C to bisect AB in D: then D is the spherical centre of the circle about ABC.

15. If arcs be drawn to bisect each pair of opposite sides of a spherical quadrilateral figure, and an arc to bisect its two diagonals: these three arcs will pass through the same point on the sphere.

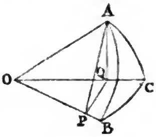
16. If from a fixed point on the surface of the sphere, arcs be drawn to cut a given less circle, each in two points; and if these four points be joined two and two by arcs, and produced (if necessary) to meet: then the points of intersection will always be in the same great circle, which is perpendicular to the arc drawn from the given point to the centre of the given circle.

RIGHT-ANGLED SPHERICAL TRIANGLES.

PROP. I.

In every right-angled triangle we shall have,

1. *Sin perpend.* = *sin hypothenuse* × *sin angle at base.*
2. *Sin base* = *cot angle at base* × *tan perpendicular.*



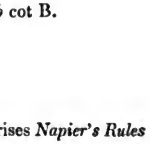
1. Let ABC be a spherical triangle right-angled at C, and BC be the base: then it is to be proved that $\sin b = \sin c \sin B$.

From A draw AQ perpendicular to OC, (O being the centre of the sphere,) and from Q draw QP perpendicular to OB, and join AP.

Then, since BCA is a right angle (*hypoth.*) the planes AOC, COB are perpendicular to each other; and AQ drawn perpendicular to their common section OC, is perpendicular to the plane BOC (*vol. i. p.* 355); and hence to any line QP in it (*vol. i. p.* 349): wherefore, also (*vol. i. p.* 351), since from the foot Q of the perpendicular AQ to the plane BOC, a line QP is drawn perpendicular to the line OB in that plane, and that PA is joined; the line AP is also perpendicular to OB. Wherefore, again, the angle APQ is the inclination of the planes AOB, COB; and therefore equal to the spherical angle ABC.

Now in the right-angled plane triangle APQ, of which Q is the right angle, we have

$$\sin B = \sin APQ = \frac{AQ}{AP} = \frac{\sin AC}{\sin AB} = \frac{\sin b}{\sin c}; \text{ or, } \sin b = \sin c \sin B.$$



2. Draw the tangent CQ to the side AC, to meet OA in Q, and the perpendicular CP from C to OB, and join PQ.

Then since the planes AOC, COB, are at right angles, and CQ is drawn perpendicular to the common section OC, it is perpendicular to the plane BOC: and hence, as before, CPQ is the inclination of the planes BOC, BOA, or is equal to the spherical angle ABC. Wherefore,

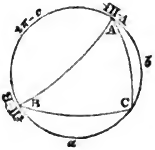
$$\tan B = \tan CPQ = \frac{CQ}{CP} = \frac{\tan b}{\sin a}; \text{ or, } \sin a = \tan b \cot B.$$

PROP. II.

Preliminary to enunciating this proposition, which comprises *Napier's Rules for the Circular parts*, one or two explanatory remarks will be requisite.

The preceding theorem relates to the right-angled triangle only, and the proofs of the properties depend upon the triangle having one right angle. In all the uses to be made of those properties, just as in right-angled plane triangles, the right angle may be excluded from the list of the data, and the properties of that particular triangle assumed instead of it. The parts of the triangle, then, are reduced to five, viz., the three sides and the two oblique angles. If we fix upon any one of these five parts, as a middle one, the remaining four will be ranged in pairs, two adjacent to the middle one, and the remaining two separated from the middle by the adjacent parts. The names given to these parts under such a classification would very naturally be, the *middle part*, the *adjacent parts*, and the *opposite parts:* but for giving unity to the enunciation of the properties of the figure, Napier selected the sides, the complements of the angles, and the complement of the hypothenuse; and these, collectively, he called the *circular parts*, viz., a, b, $\frac{1}{2}\Pi - A$, $\frac{1}{2}\Pi - B$, and $\frac{1}{2}\pi - c$.

The analogy from which he chose the expression will be apparent from the annexed figure, where a circle is described about the triangle, and the several parts marked upon the circumference.



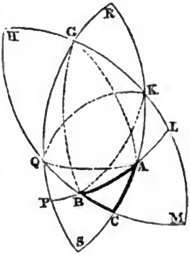
In the following table each part is taken successively as the middle, and the consequent adjustment of the adjacent parts and opposite parts as they are derived from the figure.

Opposite.	Adjacent.	Middle.	Adjacent.	Opposite.
$\frac{1}{2}\Pi - B$	a	b	$\frac{1}{2}\Pi - A$	$\frac{1}{2}\pi - c$
a	b	$\frac{1}{2}\Pi - A$	$\frac{1}{2}\pi - c$	$\frac{1}{2}\Pi - B$
b	$\frac{1}{2}\Pi - A$	$\frac{1}{2}\pi - c$	$\frac{1}{2}\Pi - B$	a
$\frac{1}{2}\Pi - A$	$\frac{1}{2}\pi - c$	$\frac{1}{2}\Pi - B$	a	b
$\frac{1}{2}\pi - c$	$\frac{1}{2}\Pi - B$	a	b	$\frac{1}{2}\Pi - A$

We are now prepared to enunciate and prove the proposition,

Sin middle part = tan one adjacent part × tan other adjacent part.
Sin middle part = cos one opposite part × cos other opposite part.

1. Let ABC be the right-angled triangle, and complete the figure as in *th. ix.* Then apply to each of the five triangles, in that theorem, the second part of the preceding. The results being written symmetrically, viz., the first sides of the equations all in sines, and the second all in tangents, we get



ΔKRG, $\sin b = \tan a \tan(\frac{1}{2}\Pi - A)$
ΔQPB, $\sin(\frac{1}{2}\Pi - A) = \tan b \tan(\frac{1}{2}\pi - c)$
ΔALK, $\sin(\frac{1}{2}\pi - c) = \tan(\frac{1}{2}\Pi - B) \tan(\frac{1}{2}\Pi - A)$
ΔGHQ, $\sin(\frac{1}{2}\Pi - B) = \tan a \tan(\frac{1}{2}\pi - c)$
ΔABC, $\sin a = \tan b \tan(\frac{1}{2}\Pi - B)$.

Now since the four triangles ALK, KRG, GHQ, QPB, have their sides and angles functions of those of ABC, whatever relations exist amongst the parts of the former triangles are also relations amongst those of ABC. The equations, therefore, just deduced by means of those four triangles, also express properties of the triangle ABC. But taken altogether, they are but one general property of ABC and its dependent triangles: whilst they express, in fact, all the cases of the first part of the proposition itself. The first of Napier's rules is hence established.

2. Apply to the same series of triangles the first part of the last proposition:

then we have, writing as before, the sides of the equation symmetrically, the left side in sines, and the right in cosines,

$$\begin{aligned}
\Delta\text{ABC}, \sin b &= \cos(\tfrac{1}{2}\pi - c)\cos(\tfrac{1}{2}\Pi - \text{B})\\
\Delta\text{KRG}, \sin(\tfrac{1}{2}\Pi - \text{A}) &= \cos a \cos(\tfrac{1}{2}\Pi - \text{B})\\
\Delta\text{QPB}, \sin(\tfrac{1}{2}\pi - c) &= \cos a \cos b\\
\Delta\text{ALK}, \sin(\tfrac{1}{2}\Pi - \text{B}) &= \cos b \cos(\tfrac{1}{2}\Pi - \text{A})\\
\Delta\text{GHQ}, \sin a &= \cos(\tfrac{1}{2}\pi - c)\cos(\tfrac{1}{2}\Pi - \text{A}),
\end{aligned}$$

which system of equations embrace all the cases of the second part of the proposition.

Scholium.

The method of establishing these remarkable theorems is an attempt to restore and develop the views of Napier himself. It is very obvious from his own statement[*] that he originally deduced his theorems from the five triangles that we have employed, and that he only put them in another form for the sake of simply enunciating the properties in reference to the triangle ABC proposed for actual solution. To exhibit them in reference to this triangle solely, he was also compelled to establish them by an induction from the several cases, or by showing that all the possible cases were comprehended in the theorems which he proposed: though, as appears from the preceding proposition, they only express *true* and *single* properties in each of an entire system of dependent triangles.

Attention was first called to these rules by Gillebrand; but as his object was to attract notice to their simplicity and generality, he employed also the method of establishing them by showing that they held good in all the cases. Perhaps, too, he did not clearly comprehend the original system of Napier. However, it has been the almost universal custom to look upon them as mere *technical rules*, or as a most ingenious system of artificial *memory*[†], rather than as the offspring of a subtle and profound intellect, directed not only to all the cases of the problem, but all the dependent parts of the figure under consideration. To view these theorems in the light of mere mental legerdemain, does great injustice to their author: and it is much to be desired that students should be led to contemplate them in their true character—an expression of *apparently different* relations, but still only *two single and isolated theorems* applied to the several parts of a mutually dependent system of triangles.

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[*] These theorems were first given in the *Mirifici Logarithmorum canonis descriptio*, 1614.

[†] The following are a few of the phrases employed in the works which are in most general use:—

"The whole by the help of an *ingenious contrivance*," *etc.*—*Young's Trigonometry*, p. 84.

"Some writers consider the employment of these rules as *unscientific*, and as having a tendency to render the knowledge of the subject both *confined* and *technical*, and suggest that every instance should be worked out when it occurs, as has been done above. This may be the case, when they are viewed in a *theoretical* light: but their practical importance in the hands of persons who cannot be expected to be very familiar with the transformation of trigonometrical formulæ, seems to show that an *elementary treatise* on the subject would be imperfect without them."—*Hind*, p. 205. 3rd ed.

"For common purposes, a *technical* memory has been invented, under the title of *Napier's* Rules for circular parts."—*Wilson*, p. 217.

"There is *a kind of artificial memory* by which," *etc.*—*Snowball*, p. 29.

"*An artificial memory* has been supplied," *etc.*—*Woodhouse*, p. 135. 2nd ed.

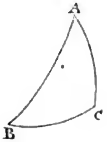
Foreign writers, as well as the earlier English, almost invariably employ corresponding language when speaking of these rules. Indeed, as far as I recollect, Mr. Ivory (*Phil. Mag.* 1821) is the *only* writer who has ventured to maintain a different view of the real nature of these theorems.

It may assist our recollection of these two theorems to keep in mind the fact that the first vowels of t*a*ngent of the *a*djacent, and of c*o*sine of the *o*pposite, are, in each theorem, the same.

PROP. III.

If C *be the right angle of a spherical triangle, then*

$$\sin(c - a) = \sin b \cos a \tan \tfrac{1}{2}\mathrm{B},$$
$$\sin(c - a) = \tan b \cos c \tan \tfrac{1}{2}\mathrm{B}.$$



By Napier's theorems, we have

$$\sin b = \cos(\tfrac{1}{2}\pi - c)\cos(\tfrac{1}{2}\pi - \mathrm{B}) = \sin c \sin \mathrm{B}, \text{ or } \sin c = \frac{\sin b}{\sin \mathrm{B}} \quad \dots\dots\dots\dots (1)$$

$$\cos \mathrm{B} = \tan a \tan(\tfrac{1}{2}\pi - c) = \frac{\sin a \cos c}{\cos a \sin c}, \text{ or, in another form,}$$

$$\sin a \cos c = \cos a \sin c \cos \mathrm{B} = \frac{\cos a \sin b \cos \mathrm{B}}{\sin \mathrm{B}} \quad \dots\dots (2).$$

Hence,
$$\sin(c - a) = \sin c \cos a - \cos c \sin a = \frac{\sin b \cos a}{\sin \mathrm{B}} - \frac{\cos a \sin b \cos \mathrm{B}}{\sin \mathrm{B}}$$
$$= \sin b \cos a \frac{1 - \cos \mathrm{B}}{\sin \mathrm{B}} = \sin b \cos a \tan \tfrac{1}{2}\mathrm{B},$$

which is the first part of the proposition, the transformations being effected by the equations (1, 2).

Again, for the second part, by Napier's theorem $\sin(\tfrac{1}{2}\pi - c) = \cos a \cos b$, and hence $\cos a = \dfrac{\cos c}{\cos b}$, which inserted in the preceding, gives,

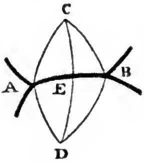
$$\sin(a - c) = \frac{\sin b \cos c \tan \tfrac{1}{2}\mathrm{B}}{\cos b} = \cos c \tan b \tan \tfrac{1}{2}\mathrm{B}.$$

Scholium.

These theorems were first given by Prony in the *Connaissance des Temps* for 1808, without demonstrations, he having used them in the construction of a chart of the River Po. They will often be found useful in Geodesic reductions, as well as their analogous ones in plane trigonometry: but a more suitable place will occur for speaking of them hereafter.

PROP. IV.

To find the inclination, I, *of two adjacent faces of a regular polyhedron.*



Let the points C, D be the poles of circles which on the surface of the sphere circumscribe two adjacent faces, and AB be the common side of two contiguous polygons. Draw the great circles CA, CB, CD, DA, DB.

Then, if diametral planes be drawn parallel to those of the faces, whose poles are C and D, these points will also be the poles of those circles, whilst their inclination to one another is the same with that of these two faces. The distance CD will, being cross-polar, measure the supplement of the inclination; and as it is evident that CD is bisected in E by its intersection with AB, we shall have $\mathrm{CE} = \tfrac{1}{2}\Pi - \tfrac{1}{2}\mathrm{I}$.

Again, since $\mathrm{ACB} = \frac{1}{2} \cdot \frac{2\Pi}{n} = \frac{\Pi}{n}$, and $\mathrm{CAE} = \frac{1}{2} \cdot \frac{2\Pi}{m} = \frac{\Pi}{m}$, we shall have by the right-angled triangle ACE, the relation

$$\cos \mathrm{CAE} = \cos \mathrm{CE} \sin \mathrm{ACE}, \text{ or } \cos \mathrm{CE} = \frac{\cos \mathrm{CAE}}{\sin \mathrm{ACE}}; \text{ that is}$$

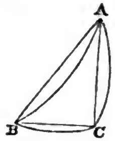
$$\sin \tfrac{1}{2}\mathrm{I} = \frac{\cos \frac{1}{m}\Pi}{\sin \frac{1}{n}\Pi}.$$

Corollary.

If $2a$ be the side of any one of the polygonal faces, and r, R the radii of the inscribed and circumscribed spheres, then

$$r = a \tan \tfrac{1}{2}\mathrm{I} \cot \frac{\Pi}{n}; \text{ and } \mathrm{R} = a \tan \tfrac{1}{2}\mathrm{I} \tan \frac{\Pi}{m}.$$

MISCELLANEOUS EXERCISES ON RIGHT-ANGLED TRIANGLES.



1. Let the triangle be subject to the usual notation: then,

$$\sin^2 \tfrac{1}{2}c = \sin^2 \tfrac{1}{2}a \cos^2 \tfrac{1}{2}b + \cos^2 \tfrac{1}{2}a \sin^2 \tfrac{1}{2}b.$$

For, $\cos c = \cos a \cos b$, and hence

$1 - \cos c = 1 - \cos a \cos b$, or in semi arcs.

$$\begin{aligned} 2 \sin^2 \tfrac{1}{2}c &= 1 - \left\{2 \cos^2 \tfrac{1}{2}a - 1\right\}\left\{2 \cos^2 \tfrac{1}{2}b - 1\right\} \\ &= 2 \cos^2 \tfrac{1}{2}a + 2 \cos^2 \tfrac{1}{2}b - 4 \cos^2 \tfrac{1}{2}a \cos^2 \tfrac{1}{2}b, \text{ or} \\ \sin^2 \tfrac{1}{2}c &= \cos^2 \tfrac{1}{2}a\,(1 - \cos^2 \tfrac{1}{2}b) + \cos^2 \tfrac{1}{2}b\,(1 - \cos^2 \tfrac{1}{2}a) \\ &= \cos^2 \tfrac{1}{2}a \sin^2 \tfrac{1}{2}b + \cos^2 \tfrac{1}{2}b \sin^2 \tfrac{1}{2}a, \end{aligned}$$

which is analogous to *th.* 34. *Pl. Geom. vol.* i. *p.* 310, and becomes that theorem when the sides are reduced to their limits.

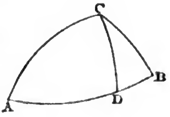
2. Draw the chords of the three sides: then denoting the *plane* angle ACB by θ, and recollecting that the chords have the following values,

$$\text{chord AB} = 2 \sin \tfrac{1}{2}c, \text{ chord AC} = 2 \sin \tfrac{1}{2}b, \text{ chord BC} = 2 \sin \tfrac{1}{2}a,$$
$$(2 \sin \tfrac{1}{2}c)^2 = (2 \sin \tfrac{1}{2}a)^2 - 2 \cos \theta\, (2 \sin \tfrac{1}{2}a.\ 2 \sin \tfrac{1}{2}b) + (2 \sin \tfrac{1}{2}b)^2,$$
$$\text{or } \sin^2 \tfrac{1}{2}c = \sin^2 \tfrac{1}{2}a - 2 \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \cos \theta + \sin^2 \tfrac{1}{2}b.$$

Equate this value of $\sin^2 \frac{1}{2}c$ with the preceding one; then

$$\begin{aligned} 2 \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \cos \theta &= \sin^2 \tfrac{1}{2}b\,(1 - \cos^2 \tfrac{1}{2}a) + \sin^2 \tfrac{1}{2}a\,(1 - \cos^2 \tfrac{1}{2}b) \\ &= 2 \sin^2 \tfrac{1}{2}b \sin^2 \tfrac{1}{2}a. \end{aligned}$$

Hence $\cos \theta = \sin \frac{1}{2}a \sin \frac{1}{2}b$.



3. From the right angle C draw the perpendicular CD (p), and let it divide the hypothenuse into the segments a_1, b_1, adjacent to the sides a and b; and the angle C into the angles C_1 and C_2. Then by Napier's theorems, $\sin p = \tan b_1 \cot \mathrm{C}_1$, and $\sin p = \tan a_1 \cot \mathrm{C}_2$. Multiply these, then, since $\mathrm{C}_1 + \mathrm{C}_2 = \frac{1}{2}\Pi$, $\cot \mathrm{C}_1 \cot \mathrm{C}_2 = 1$, and

$$\sin^2 p = \tan a_1 \tan b_1 \cot \mathrm{C}_1 \cot \mathrm{C}_2 = \tan a_1 \tan b_1,$$

which is analogous to *th.* 87, *Pl. Geom.* i. *p.* 333.

4. By the same triangles we have

$$\cos b = \cos b_1 \cos p, \text{ and } \cos a = \cos a_1 \cos p; \text{ from which}$$
$$\cos a_1 \cos b = \cos a \cos b_1, \text{ or } \cos a_1 \cos^2 b = \cos a \cos b \cos b_1 = \cos b_1 \cos c.$$

Wherefore, $1 - \cos^2 b = 1 - \dfrac{\cos b_1 \cos c}{\cos a_1} = \dfrac{\cos a_1 - \cos b_1 \cos c}{\cos a_1}$. But $\cos a_1 = \cos (c - b_1) = \cos b_1 \cos c + \sin b_1 \sin c$, which substituted in the preceding equation, gives

$$\sin^2 b = \frac{\cos c \cos b_1 + \sin b_1 \sin c - \cos b_1 \cos c}{\cos a_1} = \frac{\sin b_1 \sin c}{\cos a_1}.$$

$$\sin^2 a = \frac{\cos c \cos a_1 + \sin a_1 \sin c - \cos a_1 \cos c}{\cos b_1} = \frac{\sin a_1 \sin c}{\cos b_1}.$$

Multiply these; then $\sin^2 a \sin^2 b = \sin^2 c \tan a_1 \tan b_1$.

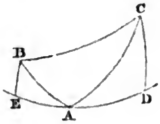
5. By the preceding, $\sin^2 p = \tan a_1 \tan b_1$, which inserted in the theorem just obtained, gives

$$\sin^2 a \sin^2 b = \sin^2 p \sin^2 c, \text{ or } \sin a \sin b = \sin p \sin c.$$

The plane analogy is obvious and well known.

6. Divide the results of (4): then we have the analogue to another well-known theorem in plano, viz.

$$\frac{\sin^2 a}{\sin^2 b} = \frac{\sin c \sin a_1 \cos a_1}{\sin c \sin b_1 \cos b_1} = \frac{\sin 2a_1}{\sin 2b_1}:$$



7. Let BAC be a right angle, and ED any great circle through A, and BE, CD, secondaries to it. Then $\tan^2 AB \tan^2 AC =$
$\tan^2 EA \tan^2 AC + \tan^2 AD \tan^2 AB$.
For, $\cos BAE = \tan EA \cot AB$, and $\cos CAD = \tan AD \cot AC$: also, $BAE + CAD = \frac{1}{2}\pi$, and hence $\cos CAD = \sin BAE$.

Add, therefore, the squares of the two equations, and we obtain

$$\tan^2 EA \cot^2 AB + \tan^2 AD \cot^2 AC = 1, \text{ or}$$
$$\tan^2 EA \tan^2 AC + \tan^2 AD \tan^2 AB = \tan^2 AB \tan^2 AC.$$

To give it a form analogous to the common expression in plano, we have

$$\frac{\tan^2 EA}{\tan^2 AB} + \frac{\tan^2 AD}{\tan^2 AC} = 1.$$

8. From the same triangles we have also the two equations
$\sin BE = \sin BA \sin BAE$, and $\sin CD = \sin CA \sin CAE$; and hence, as before,

$$\frac{\sin^2 BE}{\sin^2 BA} + \frac{\sin^2 CD}{\sin^2 CA} = 1.$$

EXERCISES FOR SOLUTION.

1. If C be the right angle, show that

(1.) $2 \cos c = \cos(a + b) + \cos(a - b)$,
(2.) $\tan^2 \frac{1}{2} b = \tan \frac{1}{2}(c + a) \tan \frac{1}{2}(c - a)$,
(3.) $\sin^2 b \cos^2 a = \sin(c + a) \sin(c - a)$,
(4.) $\cos^2 b \sin^2 a = \sin(c + b) \sin(c - b)$.

2. If the two right-angled triangles ABC, A′B′C, have their legs coincident in position: then,

$$\tan \tfrac{1}{2}(c + c') \tan \tfrac{1}{2}(a - a') = \tan \tfrac{1}{2}(c - c') \tan \tfrac{1}{2}(a + a'),$$
$$\tan \tfrac{1}{2}(c + c') \tan \tfrac{1}{2}(b - b') = \tan \tfrac{1}{2}(c - c') \tan \tfrac{1}{2}(b + b').$$

3. Given the hypothenuse and radius of the inscribed circle to find the sides and angles of the right-angled triangle.

4. Let the arc of a great circle AB be bisected in M; and at the extremities A, B, draw perpendicular great circles, AG, and HBK, and make on opposite sides of AB BH = BK; also draw the great circles GH, GK, cutting AB in C

and D: then, whatever be the lengths of the arcs AG, BH, or BK, we shall have $\tan \mathrm{MC} \tan \mathrm{MD} = \tan^2 \mathrm{MB}$.

5. If p be a perpendicular from C to c, and l an arc from C to bisect c: then

$$\cot^2 l = \frac{\cos a + \cos b}{\sin^2 a + \sin^2 b}\,;\ \sin \tfrac{1}{2}c = \frac{\sin^2 l}{1 + \sin^2 p}\,;\ \text{and } \cot^2 p = \cot^2 a + \cot^2 b.$$

6. Prove that in the triangle, right-angled at C, we have

$$(1.)\ \frac{\sin(a-b)}{\sin(a+b)} = \frac{\tan\frac{1}{2}(\mathrm{A}+\mathrm{B})}{\cot\frac{1}{2}(\mathrm{A}-\mathrm{B})} \qquad (3.)\ \tan^2\tfrac{1}{2}(c+90^\circ) = \frac{\tan\frac{1}{2}(\mathrm{B}+b)}{\tan\frac{1}{2}(\mathrm{B}-b)}$$

$$(2.)\ \tan^2\tfrac{1}{2}a = \frac{\tan\frac{1}{2}(c+b)}{\tan\frac{1}{2}(c-b)} \qquad (4.)\ \tan^2\tfrac{1}{2}(\mathrm{A}+90^\circ) = \frac{\cot\frac{1}{2}(\mathrm{B}+b)}{\tan\frac{1}{2}(\mathrm{B}-b)}$$

7. Also, that $\tan^2\frac{1}{2}\mathrm{A} = \dfrac{\sin(c-b)}{\sin(c+b)}$, and $\tan^2\frac{1}{2}a = \dfrac{\tan\frac{1}{2}\left\{\mathrm{B}+\mathrm{A}-90^\circ\right\}}{\tan\frac{1}{2}\left\{\mathrm{B}-\mathrm{A}+90^\circ\right\}}$.

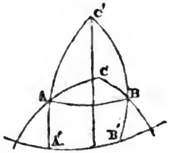
8. Let two arcs of great circles AC, BD, intersect a less circle of the sphere in the points A, C, and B, D, and each other in E: then, whether E be within or without the circle, $\tan\frac{1}{2}\mathrm{EB}\tan\frac{1}{2}\mathrm{ED} = \tan\frac{1}{2}\mathrm{EA}\tan\frac{1}{2}\mathrm{EC}$.

9. If E be *without* the circle, and EF be drawn to touch the circle; or if E be *within* the circle, and EF be drawn perpendicular to the diameter through E, we shall have $\tan^2\frac{1}{2}\mathrm{EF} = \tan\frac{1}{2}\mathrm{EB}\tan\frac{1}{2}\mathrm{ED}$.

10. If from the extremities of the diameter AB of a less circle arcs be drawn to any point F in the circumference, then, $\sin^2\frac{1}{2}\mathrm{BF} + \sin^2\frac{1}{2}\mathrm{FD} = \sin^2\frac{1}{2}\mathrm{BD}$.

11. If from any point P perpendiculars PD, PE, PF, be drawn to the sides BC, CA, AB: then we shall have
$\cos\mathrm{AF}\cos\mathrm{BD}\cos\mathrm{CE} = \cos\mathrm{FB}\cos\mathrm{DC}\cos\mathrm{EA}$;
$\sin^2\mathrm{PD}\sin^2\mathrm{PE}\sin^2\mathrm{PF} = \tan\mathrm{AE}\tan\mathrm{EC}\tan\mathrm{CD}\tan\mathrm{DB}\tan\mathrm{BF}\tan\mathrm{FA}$; and
$(\sin^2\mathrm{CD}-\sin^2\mathrm{DB})\cos\mathrm{DP}+(\sin^2\mathrm{AE}-\sin^2\mathrm{EC})\cos\mathrm{EP}+(\sin^2\mathrm{BF}-\sin^2\mathrm{FA})\cos\mathrm{FP}=0$.

QUADRANTAL TRIANGLES.



Let the side AB or c be a quadrant, and the other two sides AC, CB, or b, a, be of any magnitude whatever: then if we take as the circular parts of the system the magnitudes $a - \frac{1}{2}\pi$, $b - \frac{1}{2}\pi$, $\Pi - \mathrm{A}$, $\Pi - \mathrm{B}$, and $\mathrm{C} - \frac{1}{2}\Pi$: then the same relation amongst the five circular parts will be as in *prop. ii. Right-angled triangles, p.* 24 :—

sin middle part = tan one adjacent part × tan other adjacent part
sin middle part = cos one opposite part × cos other opposite part.

For describe the polar triangle A′B′C′ (*i. prop. vi.*); the angle C′ of which being the supplement of AB, is a right angle. Whence the proposition is true respecting the triangle A′B′C′, the circular parts being taken as in *ii. prop. ii.* But as equations, these are true when for each of the magnitudes we substitute their respective equals: and in the present case, the equals of the circular parts in ABC in terms of A′B′C′ are

$$\tfrac{1}{2}\Pi - \mathrm{A}' = a - \tfrac{1}{2}\pi\,;\ \tfrac{1}{2}\Pi - \mathrm{B}' = b - \tfrac{1}{2}\pi\,;\ \tfrac{1}{2}\pi - c' = \mathrm{C} - \tfrac{1}{2}\Pi;$$
$$a' = \Pi - \mathrm{A}\,;\ \text{and } b' = \Pi - \mathrm{B}.$$

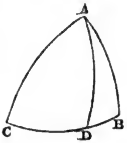
Whence every function of the magnitudes of the first sides of these equations, directed by the rule above, will be equivalent to the same functions of the second sides of the same equations: that is, the rule holds in the quadrantal triangle ABC by the substitution of these values of the circular parts.

This result might be obtained by an independent process analogous in its general principle to that by which the rule for the right-angled triangle was obtained. That process is, however, too long for insertion here; and the dependence of the quadrantal upon the rectangular triangle, renders it dispensable in an elementary work. It is to be remarked, however, that the *circular parts* in the quadrantal triangle as above given, differ from those usually found in books: but it will be found that they are correct, since in all the cases the conditions of polarity are fulfilled; viz. $a + A' = \pi$, $A + B' = \pi$, $c + C' = \pi$, $a' + A = \pi$, $b' + B = \pi$, and $c' + C = \pi$.

Some degree of ambiguity, however, in actual solution, arising out of having different sets of circular parts, will be removed by the adoption of a different plan of work; as will be seen in a future page. *See p.* 54.

PROPERTIES OF SPHERICAL TRIANGLES.

I. FORMULÆ NECESSARY FOR THE SOLUTION OF TRIANGLES.



Let ABC be a spherical triangle, and AD a perpendicular from the angle A to the opposite side BC. Denote the sides and angles by a, b, c, and A, B, C, as in plane trigonometry; and the perpendiculars from A, B, C, respectively by p_1, p_2, p_3. Let also $CD = a_1$, and $DB = a_2$.

Then, by Napier's theorems, we shall have, in CAD, BAD which are right-angled respectively at D.

$$\sin b \sin C = \sin p_1 = \sin c \sin B, \text{ or } \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Proceeding similarly for p_2 and p_3, we shall finally obtain

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \quad \ldots\ldots\ldots\ldots \quad (1.)$$

Again, by the same triangles we shall have

$$\cos a_1 = \cos b \sec p_1, \text{ and } \cos a_2 = \cos c \sec p_1.$$

Also, $\cos a = \cos (a_1 + a_2) = \cos a_1 \cos a_2 - \sin a_1 \sin a_2$,
or $\cos a - \cos a_1 \cos a_2 = - \sin a_1 \sin a_2$; and hence
$\cos^2 a - 2 \cos a \cos a_1 \cos a_2 + \cos^2 a_1 \cos^2 a_2 = (1 - \cos^2 a_1)\ (1 - \cos^2 a_2)$, or cancelling, and inserting the values of $\cos a_1$, $\cos a_2$, above,

$$1 - \cos^2 a = \sec^2 p_1 \left\{\cos^2 b - 2 \cos a \cos b \cos c + \cos^2 c\right\}, \text{ or}$$

$$\sin^2 p_1 = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a},$$

which is an expression for the perpendicular in terms of the three sides of the triangle. Performing similar operations for p_2 and p_3, and multiplying out, we get

$$\left.\begin{aligned} \sin^2 a \sin^2 p_1 &= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c \\ \sin^2 b \sin^2 p_2 &= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c \\ \sin^2 c \sin^2 p_3 &= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c \end{aligned}\right\} \ldots\ldots (2\,)$$

The quantity on the right side of the equation will be hereafter given in a different form, better adapted to the use of logarithms. *See p.* 34, *eq.* 7.

From the equality of the right sides of the last system of equations, we have

$$\sin a \sin p_1 = \sin b \sin p_2 = \sin c \sin p_3 \;\ldots\ldots\ldots\ldots\ldots\ldots \;(3.)$$

Since $\sin C = \frac{\sin p_1}{\sin b}$, we have $\cos^2 C = 1 - \frac{\sin^2 p_1}{\sin^2 b}$; and substituting for $\sin^2 p_1$ its value found in (2), we obtain

$$\cos^2 C = \frac{\sin^2 a \sin^2 b - 1 + \cos^2 a + \cos^2 b + \cos^2 c - 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b},$$

or, reducing the numerator, and extracting the square root, we finally obtain

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}.$$

Performing similar operations for cos B and cos A, we have, after reducing,

$$\left.\begin{array}{l} \cos a = \cos b \cos c + \sin b \sin c \cos A \\ \cos b = \cos c \cos a + \sin c \sin a \cos B \\ \cos c = \cos a \cos b + \sin a \sin b \cos C \end{array}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots \;(4.)$$

These formulæ were known to Albategnius[*], an Arabian astronomer of the ninth century, though they have frequently been attributed to so recent a period as that of Maupertuis, De Gua[†], and Lagrange[‡], who have been followed by all recent writers, in making them the foundation of the entire system of analysis of spherical triangles.

Resuming the triangles CDA, BDA, right-angled at D, we have

$$\sin B \sin C = \frac{\sin^2 p_1}{\sin b \sin c} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin b \sin c}.$$

Multiply both sides by cos a, subtract and add cos b cos c in the numerator of the second side, and separate the term 2 cos a cos b cos c into two parts. Then this equation is converted into

$$\cos a \sin B \sin C = \frac{\cos a - \cos b \cos c}{\sin b \sin c} + \frac{\cos b - \cos c \cos a}{\sin c \sin a} \cdot \frac{\cos c - \cos a \cos b}{\sin a \sin b}$$

$$= \cos A + \cos B \cos C, \text{ from eq. (4).}$$

Performing similar operations by means of p_2 and p_3, we shall readily get

$$\left.\begin{array}{l} \cos A = -\cos B \cos C + \sin B \sin C \cos a \\ \cos B = -\cos C \cos A + \sin C \sin A \cos b \\ \cos C = -\cos A \cos B + \sin A \sin B \cos c \end{array}\right\} \;\ldots\ldots\ldots\ldots \;(5.)\text{ §}$$

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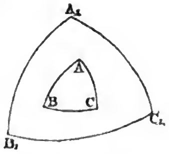
[*] Delambre, *Astronomie Moderne*, p. 353.

[†] *Mém. de l'Acad.* an 1783.

[‡] *Journal de l'Ec. Polyt.* tom. ii.; or *Leybourn's Repository*, vol. i. N.S.

[§] These formulæ are obtained from the preceding (4.) with great simplicity, by means of the *polar triangle*. Thus, for example,



Let A_1, B_1, C_1, and a_1, b_1, c_1, be the angles and sides of the triangle polar to ABC. Then we have (*th. vi. p.* 6).

$$a_1 = \Pi - A,\; b_1 = \Pi - B,\; c_1 = \Pi - C,$$
$$A_1 = \pi - a,\; B_1 = \pi - b,\; C_1 = \pi - c.$$

Also, in all triangles, and therefore in the polar one, we have

$$\cos a_1 = \cos b_1 \cos c_1 + \sin b_1 \sin c_1 \cos A_1 \text{ (by 4).}$$

Insert in this the corresponding values of the quantities concerned as functions of the parts of the primary triangle: then we get

$$\cos(\Pi - A) = \cos(\Pi - B)\cos(\Pi - C) + \sin(\Pi - B)\sin(\Pi - C)\cos(\pi - a)$$

by expanding which, and changing all the signs, we get

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

Multiply the third equation of system (4) by $\cos b$, and add the product to the first: then we get

$$\cos A \sin b \sin c + \cos C \sin a \sin b \cos b = \cos a\,(1 - \cos^2 b) = \cos a \sin^2 b.$$

But $\sin c = \dfrac{\sin a \sin C}{\sin A}$, which inserted, and the whole divided by $\sin a \sin b$, gives,

$$\cot a \sin b = \cos b \cos C + \cot A \sin C.$$

It is obvious that corresponding equations will result for any four continuously adjacent parts of the triangle. Whichever adjacent parts be taken, it is also clear that a side and its opposite angle must be the extremes of the four parts; and that for each pair of opposite parts there will be two different pairs of mean parts, and hence two resulting equations. There will, hence, be six equations in the entire system, there being three pairs of opposite parts, and two pairs of mean parts for each of these. The following is the entire system:—

$$\begin{array}{cl}
\textit{Extremes.} & \textit{Corresponding Equations}^{*}. \\
A,\ a. & \left\{\begin{array}{l}\cot a \sin b = \cos b \cos C + \cot A \sin C \\ \cot a \sin c = \cos c \cos B + \cot A \sin B\end{array}\right. \\
B,\ b. & \left\{\begin{array}{l}\cot b \sin a = \cos a \cos C + \cot B \sin C \\ \cot b \sin c = \cos c \cos A + \cot B \sin A\end{array}\right. \\
C,\ c. & \left\{\begin{array}{l}\cot c \sin a = \cos a \cos B + \cot C \sin B \\ \cot c \sin b = \cos b \cos A + \cot C \sin A\end{array}\right.
\end{array} \Bigg\} \quad \ldots\ldots\ldots\ldots (6).$$

The equations (1, 4, 5, 6) are generally denominated the fundamental ones of spherical trigonometry, since they contain the expressions of the value of any one part in terms of any three others. That they are adequate to the analytical solution, the following view (after Lagrange) will show: but in order to render them effective for numerical solution, by means of logarithms, considerable transformations will be requisite, as far as regards the systems (4, 5, 6).

Any three parts of a spherical triangle being given, the triangle is determinate, at least, abating the possible existence of double solutions, analogous to the first case of plane triangles (*vol. i. p.* 451). It follows, therefore, that any one part ought to be expressible in terms of any other three; and it is only requisite to show that the systems already given embrace every possible variety of combination of the six parts of the triangle, taken four and four together.

Taken in *fours*, six things admit of $\frac{6.\ 5.\ 4.\ 3}{1.\ 2.\ 3.\ 4}$, that is, 15, different combina-

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[*] These equations may be exhibited also under a different form, which are often useful in general investigations.

$$\begin{aligned}
\sin a \cos B &= \cos b \sin c - \sin b \cos c \cos A \\
\sin a \cos C &= \cos c \sin b - \sin c \cos b \cos A \\
\sin b \cos C &= \cos c \sin a - \cos c \sin a \cos B \\
\sin b \cos A &= \cos a \sin c - \sin a \cos c \cos B \\
\sin c \cos A &= \cos a \sin b - \sin a \cos b \cos C \\
\sin c \cos B &= \cos b \sin a - \sin b \cos a \cos C.
\end{aligned}$$

The investigation of the last of the set is given as a specimen of the method of obtaining this class of forms.

By (4) we have $\sin a \sin c \cos B = \cos b - \cos a \cos c$

$$\begin{aligned}
&= \cos b - \cos a\,(\cos a \cos b + \sin a \sin b \cos C) \\
&= \cos b\,(1 - \cos^2 a) - \cos a \sin a \sin b \cos C \\
&= \cos b \sin^2 a - \cos a \sin a \sin b \cos C, \text{ or}
\end{aligned}$$

$$\sin c \cos B = \cos b \sin a - \cos a \sin b \cos C.$$

tions, no regard being paid to the permutations of each set. Now these may be,

(1). Three combinations of two pairs of opposite parts, viz., Aa Bb, Bb Cc, Cc Aa: and the equations (1) are manifestly adequate to the solution of these cases.

(2). Three adjacent parts, and the opposite to the middle one. These divide themselves into two subordinate cases, according as the middle one is an angle or a side.

When the middle one is an angle, there are three cases, one for each angle, and the solution is contained in the system (4).

When, similarly, the middle one is a side, there are three cases, one for each side; and the solution is contained in the system (5).

(3). The four parts connected may be those of which a side and its opposite angle are the extreme parts, and the two parts which connect them in the figure, the other two. This will give for each pair of opposite parts two cases; and hence six in all. The equations adequate to their solution are those of system (6).

By due attention to the figure, the student will readily comprehend the foregoing classification, and perceive that these fifteen cases include every possible combination of four of them in a group. We, therefore, proceed to effect the successive transformations required for numerical solution.

The transformation of the expressions on the right side of the equations marked (2) is easily effected as follows:—

$$1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c$$
$$= (1 - \cos^2 a)(1 - \cos^2 b) - (\cos^2 c - 2\cos a \cos b \cos c + \cos^2 a \cos^2 b)$$
$$= \sin^2 a \sin^2 b - (\cos c - \cos a \cos b)^2$$
$$= (\sin a \sin b + \cos c - \cos a \cos b)(\sin a \sin b - \cos c + \cos a \cos b)$$
$$= \{\cos c - \cos(a + b)\}\{-\cos c + \cos(a - b)\}$$
$$= 2\sin \tfrac{1}{2}(a + b + c)\sin \tfrac{1}{2}(a + b - c) \,.\, 2\sin \tfrac{1}{2}(a - b + c)\sin \tfrac{1}{2}(-a + b + c).$$

Now if, as in plane trigonometry, (*p.* 448, *vol. i.,*) we put $2s = a + b + c$, this is converted at once into a form adapted to logarithms: viz. to

$$4\sin s \sin(s - a)\sin(s - b)\sin(s - c) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7).$$

As this expression, or its component parts enter into so many formulæ in trigonometry, it is important that the student should render himself perfectly familiar with it.

To transform the expressions for A, B, C, we have (*in eq.* 4),

$$\cos A \sin b \sin c = \cos a - \cos b \cos c, \text{ and hence}$$
$$(1 + \cos A)\sin b \sin c = \cos a - \cos b \cos c + \sin b \sin c$$
$$\text{or } 2\cos^2 \tfrac{1}{2}A \sin b \sin c = \cos a - \cos(b + c)$$
$$= 2\sin \tfrac{1}{2}(a + b + c)\sin \tfrac{1}{2}(-a + b + c)$$
$$\text{and finally, } \cos^2 \tfrac{1}{2}A = \frac{\sin s \sin(s - a)}{\sin b \sin c} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (a).$$

$$\text{Also, } (1 - \cos A)\sin b \sin c = -\cos a + \cos b \cos c + \sin b \sin c,$$
$$\text{or } 2\sin^2 \tfrac{1}{2}A \sin b \sin c = 2\sin \tfrac{1}{2}(a - b + c)\sin \tfrac{1}{2}(a + b - c),$$
$$\text{or again, } \sin^2 \tfrac{1}{2}A = \frac{\sin(s - b)\sin(s - c)}{\sin b \sin c} \quad \ldots\ldots\ldots\ldots \quad (b).$$

Dividing the result (*b*) by the result (*a*) we obtain

$$\tan^2 \tfrac{1}{2}A = \frac{\sin(s - b)\sin(s - c)}{\sin s \sin(s - a)}.$$

If we perform similar operations for the three angles, and collect the results together, we shall have

$$\begin{array}{l|l|l}
A = \dfrac{\sin(s-b)\sin(s-c)}{\sin b\sin c} & \cos^2\tfrac12 A = \dfrac{\sin s\sin(s-a)}{\sin b\sin c} & \tan^2\tfrac12 A = \dfrac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)} \\
B = \dfrac{\sin(s-c)\sin(s-a)}{\sin c\sin a} & \cos^2\tfrac12 B = \dfrac{\sin s\sin(s-b)}{\sin c\sin a} & \tan^2\tfrac12 B = \dfrac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)} \\
C = \dfrac{\sin(s-a)\sin(s-b)}{\sin a\sin b} & \cos^2\tfrac12 C = \dfrac{\sin s\sin(s-c)}{\sin a\sin b} & \tan^2\tfrac12 C = \dfrac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}
\end{array}$$

. . . .(8).

These formulæ are all adapted to logarithmic calculation; but, as in the corresponding case in plane trigonometry (*vol. i. p.* 455), the tangents are preferable to the sines or cosines, as requiring fewer logarithms to be taken out, when it is required to compute *all* the angles of the triangles, and therefore less labour to be bestowed on the computation. Even this, however, admits of improvement in the following manner:

$$\text{Put } \tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

$$\text{Then } \tan\tfrac12 A = \frac{\tan r}{\sin(s-a)},\ \tan\tfrac12 B = \frac{\tan r}{\sin(s-b)},\ \tan\tfrac12 C = \frac{\tan r}{\sin(s-c)} \ \ldots (9.)$$

This last method (*eq.* 9) is analogous to that in plane trigonometry (*vol. i. p.* 455): and it will be shown hereafter, that r is the *radius of the inscribed circle*, just as in plano.

Our next transformation will be that of the system of equations marked (5), for finding a side in terms of the three angles.

These equations, being like the others, symmetrical, the transformation of any one of them will suffice as a specimen of the whole. Take, then,

$$\cos a\sin B\sin C = \cos A + \cos B\cos C;\ \text{and hence}$$

$$(1+\cos a)\sin B\sin C = \cos A + \cos B\cos C + \sin B\sin C = \cos A + \cos(B-C)$$
$$(1-\cos a)\sin B\sin C = -\cos A - \cos B\cos C + \sin B\sin C = -\cos A - \cos(B+C)$$

Hence we have

$$\cos^2\tfrac12 a\sin B\sin C = \cos\tfrac12(A+B-C)\cos\tfrac12(A-B+C)$$
$$\sin^2\tfrac12 a\sin B\sin C = -\cos\tfrac12(A+B+C)\cos\tfrac12(-A+B+C)$$

Put $A + B + C = 2S$: then these become

$$\cos^2\tfrac12 a = \frac{\cos(S-B)\cos(S-C)}{\sin B\sin C},\ \text{and}\ \sin^2\tfrac12 a = -\frac{\cos S\cos(S-A)}{\sin B\sin C}.$$

$$\text{And from these, } \cot^2\tfrac12 a = -\frac{\cos(S-B)\cos(S-C)}{\cos S\cos(S-A)}.$$

Similar operations being performed on the other two equations, we have the collected results as follows:

$$\begin{array}{l|l|l}
{}^2\tfrac12 a = -\dfrac{\cos S\cos(S-A)}{\sin B\sin C} & \cos^2\tfrac12 a = \dfrac{\cos(S-B)\cos(S-C)}{\sin B\sin C} & \cot^2\tfrac12 a = -\dfrac{\cos(S-B)\cos(S-C)}{\cos S\cos(S-A)} \\
{}^2\tfrac12 b = -\dfrac{\cos S\cos(S-B)}{\sin C\sin A} & \cos^2\tfrac12 b = \dfrac{\cos(S-C)\cos(S-A)}{\sin C\sin A} & \cot^2\tfrac12 b = -\dfrac{\cos(S-C)\cos(S-A)}{\cos S\cos(S-B)} \\
{}^2\tfrac12 c = -\dfrac{\cos S\cos(S-C)}{\sin A\sin B} & \cos^2\tfrac12 c = \dfrac{\cos(S-A)\cos(S-B)}{\sin A\sin B} & \cot^2\tfrac12 c = -\dfrac{\cos(S-A)\cos(S-B)}{\cos S\cos(S-C)}
\end{array}$$

. . . . (10).

These formulæ, like those of (8), though well adapted to logarithms, yet admit of an improvement similar to that in (9).

$$\text{Put } \cot R = \sqrt{-\frac{\cos(S-A)\cos(S-B)\cos(S-C)}{\cos S}}:\ \text{then}$$

$$\cot\tfrac12 a = \frac{\cot R}{\cos(S-A)},\ \cot\tfrac12 b = \frac{\cot R}{\cos(S-B)},\ \cot\tfrac12 c = \frac{\cot R}{\cos(S-C)} \ \ldots (11).$$

In these equations, $-\cos S$ is essentially positive, since $2\,S$ being greater than Π, and less than $3\,\Pi$, the value of S is greater than $\tfrac{1}{2}\Pi$, and less than $\tfrac{3}{2}\Pi$, and $\cos S$ is essentially $-$, or $-\cos S$ essentially $+$.

A theorem analogous to (8) in plano was discovered by William Purser, of Dublin, more than 200 years ago. I have, therefore, called the corresponding spherical ones by his name, as the passage from one to the other is so obvious as to render these a necessary consequence of his discovery, even if he did not actually carry it so far himself. See Wallace's *Theorems* and *Formulæ*, p. 2.

We may also readily find expressions for $\sin A$ and $\sin a$ in terms of these functions: for

$$\left.\begin{aligned}\sin A &= 2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A = \frac{2\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}}{\sin b\sin c}\\ \sin a &= 2\sin\tfrac{1}{2}a\ \cos\tfrac{1}{2}a = \frac{2\sqrt{-\cos S\cos(S-A)\cos(S-B)\cos(S-C)}}{\sin B\sin C}\end{aligned}\right\}\text{..(12).}$$

and similar expressions for the other sides and angles.

We have supposed that in these two groups of equations, the angle and the side were, respectively, the unknowns: but in the former group, we may suppose the side opposite the angle, and in the latter, the angle opposite the side, to be the unknowns. The transformations for solution must in these cases be different, or rather derivative from those already given.

From (8) we have

$$\begin{aligned}\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B &= \frac{\sin(s-b)}{\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}} = \frac{\sin(s-b)\cos\tfrac{1}{2}C}{2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}c}\\ \cos\tfrac{1}{2}A\sin\tfrac{1}{2}B &= \frac{\sin(s-a)}{\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}} = \frac{\sin(s-a)\cos\tfrac{1}{2}C}{2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}c}\\ \sin\tfrac{1}{2}A\sin\tfrac{1}{2}B &= \frac{\sin(s-c)}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}} = \frac{\sin(s-c)\sin\tfrac{1}{2}C}{2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}c}\\ \cos\tfrac{1}{2}A\cos\tfrac{1}{2}B &= \frac{\sin s}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}} = \frac{\sin s\sin\tfrac{1}{2}C}{2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}c}.\end{aligned}$$

Hence by addition and subtraction, respectively, we get

$$\left.\begin{aligned}\sin\tfrac{1}{2}(A+B) &= \frac{\{\sin(s-b)+\sin(s-a)\}\cos\tfrac{1}{2}C}{2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}c} = \frac{\cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C}{\cos\tfrac{1}{2}c}\\ \sin\tfrac{1}{2}(A-B) &= \frac{\{\sin(s-b)-\sin(s-a)\}\cos\tfrac{1}{2}C}{2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}c} = \frac{\sin\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C}{\sin\tfrac{1}{2}c}\\ \cos\tfrac{1}{2}(A+B) &= \frac{\{\sin s-\sin(s-c)\}\sin\tfrac{1}{2}C}{2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}c} = \frac{\cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C}{\cos\tfrac{1}{2}c}\\ \cos\tfrac{1}{2}(A-B) &= \frac{\{\sin s+\sin(s-c)\}\sin\tfrac{1}{2}C}{2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}c} = \frac{\sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C}{\sin\tfrac{1}{2}c}\end{aligned}\right\}\text{(13).}$$

From these we can find $\tfrac{1}{2}(A+B)$ and $\tfrac{1}{2}(A-B)$ when a, b, and C are given, and hence A and B separately.

These expressions, however, may be slightly modified as to form; for divide the first by the third, and the second by the fourth: then

$$\tan\tfrac{1}{2}(A+B) = \frac{\cos\tfrac{1}{2}(a-b)}{\cos\tfrac{1}{2}(a+b)}\cot\tfrac{1}{2}C;\quad \tan\tfrac{1}{2}(A-B) = \frac{\sin\tfrac{1}{2}(a-b)}{\sin\tfrac{1}{2}(a+b)}\cot\tfrac{1}{2}C\text{..(14).}$$

We may now proceed to the case of two angles and the included side being given, to find the third angle.

By (10) we have

$$\sin \tfrac{1}{2}a \cos \tfrac{1}{2}b = \frac{\cos(\mathrm{S}-\mathrm{A})}{\sin \mathrm{C}} \sqrt{\frac{-\cos \mathrm{S} \cos(\mathrm{S}-\mathrm{C})}{\sin \mathrm{A} \sin \mathrm{B}}} = \frac{\cos(\mathrm{S}-\mathrm{A}) \sin \frac{1}{2}c}{2 \sin \frac{1}{2}\mathrm{C} \cos \frac{1}{2}\mathrm{C}}$$

$$\cos \tfrac{1}{2}a \sin \tfrac{1}{2}b = \frac{\cos(\mathrm{S}-\mathrm{B})}{\sin \mathrm{C}} \sqrt{\frac{-\cos \mathrm{S} \cos(\mathrm{S}-\mathrm{C})}{\sin \mathrm{A} \sin \mathrm{B}}} = \frac{\cos(\mathrm{S}-\mathrm{B}) \sin \frac{1}{2}c}{2 \sin \frac{1}{2}\mathrm{C} \cos \frac{1}{2}\mathrm{C}}$$

$$\sin \tfrac{1}{2}a \sin \tfrac{1}{2}b = \frac{-\cos \mathrm{S}}{\sin \mathrm{C}} \sqrt{\frac{\cos(\mathrm{S}-\mathrm{A}) \cos(\mathrm{S}-\mathrm{B})}{\sin \mathrm{A} \sin \mathrm{B}}} = \frac{-\cos \mathrm{S} \cos \frac{1}{2}c}{2 \sin \frac{1}{2}\mathrm{C} \cos \frac{1}{2}\mathrm{C}}$$

$$\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b = \frac{\cos(\mathrm{S}-\mathrm{C})}{\sin \mathrm{C}} \sqrt{\frac{\cos(\mathrm{S}-\mathrm{A}) \cos(\mathrm{S}-\mathrm{B})}{\sin \mathrm{A} \sin \mathrm{B}}} = \frac{\cos(\mathrm{S}-\mathrm{C}) \cos \frac{1}{2}c}{2 \sin \frac{1}{2}\mathrm{C} \cos \frac{1}{2}\mathrm{C}}$$

By addition and subtraction we obtain

$$\left.\begin{aligned}
\sin \tfrac{1}{2}(a+b) &= \frac{\{\cos(\mathrm{S}-\mathrm{A})+\cos(\mathrm{S}-\mathrm{B})\} \sin \frac{1}{2}c}{2 \sin \frac{1}{2}\mathrm{C} \cos \frac{1}{2}\mathrm{C}} = \frac{\cos \frac{1}{2}(\mathrm{A}-\mathrm{B}) \sin \frac{1}{2}c}{\sin \frac{1}{2}\mathrm{C}} \\
\sin \tfrac{1}{2}(a-b) &= \frac{\{\cos(\mathrm{S}-\mathrm{A})-\cos(\mathrm{S}-\mathrm{B})\} \sin \frac{1}{2}c}{2 \sin \frac{1}{2}\mathrm{C} \cos \frac{1}{2}\mathrm{C}} = \frac{\sin \frac{1}{2}(\mathrm{A}-\mathrm{B}) \sin \frac{1}{2}c}{\cos \frac{1}{2}\mathrm{C}} \\
\cos \tfrac{1}{2}(a+b) &= \frac{\{\cos(\mathrm{S}-\mathrm{C})+\cos \mathrm{S}\} \cos \frac{1}{2}c}{2 \sin \frac{1}{2}\mathrm{C} \cos \frac{1}{2}\mathrm{C}} = \frac{\cos \frac{1}{2}(\mathrm{A}+\mathrm{B}) \cos \frac{1}{2}c}{\sin \frac{1}{2}\mathrm{C}} \\
\cos \tfrac{1}{2}(a-b) &= \frac{\{\cos(\mathrm{S}-\mathrm{C})-\cos \mathrm{S}\} \cos \frac{1}{2}c}{2 \sin \frac{1}{2}\mathrm{C} \sin \frac{1}{2}\mathrm{C}} = \frac{\sin \frac{1}{2}(\mathrm{A}+\mathrm{B}) \cos \frac{1}{2}c}{\cos \frac{1}{2}\mathrm{C}}
\end{aligned}\right\} \ldots (15).$$

And by division, as in the preceding case,

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(\mathrm{A}-\mathrm{B})}{\cos \frac{1}{2}(\mathrm{A}+\mathrm{B})} \tan \tfrac{1}{2}c; \ \tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(\mathrm{A}-\mathrm{B})}{\sin \frac{1}{2}(\mathrm{A}+\mathrm{B})} \tan \tfrac{1}{2}c \ \ldots\ldots (16).$$

The six expressions here derived (15, 16) are, evidently, only *different in form* from the preceding (13, 14); and might have been at once deduced from them by mere trigonometrical division.

The four formulæ (13 or 15) are usually known as *Gauss's Analogies*[*], their demonstrations having been first given by that illustrious geometer, in his *Theoria motûs Corporum Cœlestium* (1809): but they had been published by *Delambre* some years previously in the *Connaissance des Temps* (for 1808); though, as was then usual in that work, without their investigations, probably with a view to attract attention to them. The two latter (14, 16) were discovered at a much earlier period by *Napier*, and are known as *Napier's Analogies.* For the purposes of *calculation*, Napier's Analogies are directly applicable, which is not the case with those of Gauss: but, in aid of *general investigations*, those of Gauss have a decided advantage over Napier's, from their being composed of more elementary trigonometrical functions. Delambre, however, did not appear (*Astron. tom. i. p.* 164) to attach considerable value to them; though they have ultimately proved to be amongst the most important and useful theorems we possess in elementary spherics.

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[*] Gauss did not deliver his theorems, or their investigation, in precisely the forms given in the text. The forms, indeed, are remarkable and elegant, and the investigations simple and easy. The forms are given here; but the investigations themselves are left as an exercise for the student. The theorem alleges, that if we put

$$\begin{aligned}
p &= \cos \tfrac{1}{2}c \sin \tfrac{1}{2}(\mathrm{A}+\mathrm{B}), &\quad \mathrm{P} &= \cos \tfrac{1}{2}\mathrm{C} \cos \tfrac{1}{2}(a-b), \\
q &= \cos \tfrac{1}{2}c \cos \tfrac{1}{2}(\mathrm{A}+\mathrm{B}), &\quad \mathrm{Q} &= \sin \tfrac{1}{2}\mathrm{C} \cos \tfrac{1}{2}(a+b), \\
r &= \sin \tfrac{1}{2}c \sin \tfrac{1}{2}(\mathrm{A}-\mathrm{B}), &\quad \mathrm{R} &= \cos \tfrac{1}{2}\mathrm{C} \sin \tfrac{1}{2}(a-b), \\
s &= \sin \tfrac{1}{2}c \cos \tfrac{1}{2}(\mathrm{A}-\mathrm{B}), \text{ and} &\quad \mathrm{S} &= \sin \tfrac{1}{2}\mathrm{C} \sin \tfrac{1}{2}(a+b);
\end{aligned}$$

then we shall have the following equalities of products, and thence of single factors;

$$\mathrm{PQ} = pq, \quad \mathrm{PR} = pr, \quad \mathrm{PS} = ps,$$
$$\mathrm{QR} = qr, \quad \mathrm{QS} = qs, \quad \mathrm{RS} = rs.$$

Whence finally by resolution of these equations,

$$\mathrm{P} = p, \ \mathrm{Q} = q, \ \mathrm{R} = r, \text{ and } \mathrm{S} = s;$$

which are the same results as were obtained in the text.

II. THE FUNCTIONS S, s, ETC. IN CONNECTION WITH EACH OTHER, AND THE SIDES AND ANGLES.

The formulæ already deduced are adequate to the solution by logarithms of all the cases of spherical trigonometry, where three of the parts are given to find the remaining parts. There are, however, many cases in which other formulæ become convenient; and others still, where the results are possessed of much analytical symmetry and elegance. A few such, therefore, are added here, after giving which we shall proceed at once to the actual resolution of triangles.

Before we proceed, however, we must lay down the following notation:—

Notation.

Let $N^2 = -\cos S \cos (S - A) \cos (S - B) \cos (S - C)$:
$n^2 = \sin s \sin (s - a) \sin (s - b) \sin (s - c)$:
Put Δ, Δ_1, Δ_2, Δ_3, for the associated triangles ABC, A′BC, AB′C, ABC′:
a, b, c, A, B, C, for the sides and angles of the triangle ABC;
a_1, b_1, c_1, A_1, B_1, C_1, for those of the triangle A′BC;
a_2, b_2, c_2, A_2, B_2, C_2, for those of the triangle AB′C;
a_3, b_3, c_3, A_3, B_3, C_3, for those of the triangle ABC′.
R, R_1, R_2, R_3, for the radii of circles circumscribing the associated triangles:
r, r_1, r_2, r_3, for those of the corresponding inscribed circles:
E, E_1, E_2, E_3, the spherical excesses of the associated system;
s, s_1, s_2, s_3, and S, S_1, S_2, S_3, the same parts of Δ, Δ_1, Δ_2, Δ_3, that s and S are of Δ; and all the other notation in correspondence with this.

The same letters with accents to represent the same parts of the polar triangles Δ', Δ'_1, Δ'_2, Δ'_3.

Employing *Purser's* theorems (8 and 10, *p.* 35) for the semi-sides and semi-angles, we have

$$\frac{\cos \frac{1}{2}B \cos \frac{1}{2}C}{\sin \frac{1}{2}A} = \frac{\sin s}{\sin a}; \text{ or } \sin s = \frac{\sin a \cos \frac{1}{2}B \cos \frac{1}{2}C}{\sin \frac{1}{2}A}.$$

And since s is symmetrical in respect of a, b, c, the same form of value holds good when b and c are put for a, the changes which symmetrically relate to it being also made, as given below in (17). Also

$$\frac{\sin \frac{1}{2}B \sin \frac{1}{2}C}{\sin \frac{1}{2}A} = \frac{\sin (s - a)}{\sin a}; \text{ or } \sin (s - a) = \frac{\sin a \sin \frac{1}{2}B \sin \frac{1}{2}C}{\sin \frac{1}{2}A}.$$

The entire collection of results are

$$\left.\begin{array}{l|l}
\sin s = \dfrac{\sin a \cos \frac{1}{2}B \cos \frac{1}{2}C}{\sin \frac{1}{2}A} & \sin (s - a) = \dfrac{\sin a \sin \frac{1}{2}B \sin \frac{1}{2}C}{\sin \frac{1}{2}A} \\
\sin s = \dfrac{\sin b \cos \frac{1}{2}C \cos \frac{1}{2}A}{\sin \frac{1}{2}B} & \sin (s - b) = \dfrac{\sin b \sin \frac{1}{2}C \sin \frac{1}{2}A}{\sin \frac{1}{2}B} \\
\sin s = \dfrac{\sin c \cos \frac{1}{2}A \cos \frac{1}{2}B}{\sin \frac{1}{2}C} & \sin (s - c) = \dfrac{\sin c \sin \frac{1}{2}A \sin \frac{1}{2}B}{\sin \frac{1}{2}C}
\end{array}\right\} \ldots\ldots (17).$$

In a corresponding manner we obtain the following:—

$$\left.\begin{array}{l|l}
\cos S = -\dfrac{\sin A \sin \frac{1}{2}b \sin \frac{1}{2}c}{\cos \frac{1}{2}a} & \cos (S - A) = \dfrac{\sin A \cos \frac{1}{2}b \cos \frac{1}{2}c}{\cos \frac{1}{2}a} \\
\cos S = -\dfrac{\sin B \sin \frac{1}{2}c \sin \frac{1}{2}a}{\cos \frac{1}{2}b} & \cos (S - B) = \dfrac{\sin B \cos \frac{1}{2}c \cos \frac{1}{2}a}{\cos \frac{1}{2}b} \\
\cos S = -\dfrac{\sin C \sin \frac{1}{2}a \sin \frac{1}{2}b}{\cos \frac{1}{2}c} & \cos (S - C) = \dfrac{\sin C \cos \frac{1}{2}a \cos \frac{1}{2}b}{\cos \frac{1}{2}c}
\end{array}\right\} \ldots. (18).$$

Again, the formula of Albategnius (*eq.* 4) gives

$$\sin A = \frac{2n}{\sin b \sin c} \quad \Big| \quad \sin B = \frac{2n}{\sin c \sin a} \quad \Big| \quad \sin C = \frac{2n}{\sin a \sin b}$$

$$\sin a = \frac{2N}{\sin B \sin C} \quad \Big| \quad \sin b = \frac{2N}{\sin C \sin A} \quad \Big| \quad \sin c = \frac{2N}{\sin A \sin B}.$$

These are identical with

$$\sin a \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C = \frac{N}{2\sin \tfrac{1}{2}B \sin \tfrac{1}{2}C} \quad \Big| \quad \sin A \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c = \frac{n}{2 \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c}$$

$$\sin a \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C = \frac{N}{2\cos \tfrac{1}{2}B \cos \tfrac{1}{2}C} \quad \Big| \quad \sin A \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c = \frac{n}{2 \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}.$$

And so on with the others. These values inserted in the sets of equations marked (16, 17) give

$$\left.\begin{array}{ll|ll}
\sin s & = \dfrac{N}{2 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C} & \cos S & = -\dfrac{n}{2 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c} \\
\sin (s-a) & = \dfrac{N}{2 \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C} & \cos (S-A) & = \dfrac{n}{2 \cos \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c} \\
\sin (s-b) & = \dfrac{N}{2 \cos \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C} & \cos (S-B) & = \dfrac{n}{2 \sin \frac{1}{2}a \cos \frac{1}{2}b \sin \frac{1}{2}c} \\
\sin (s-c) & = \dfrac{N}{2 \cos \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C} & \cos (S-C) & = \dfrac{n}{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \cos \frac{1}{2}c}
\end{array}\right\} (19)$$

Again, from the formulæ preceding the last, we have

$$\left.\begin{array}{l|l}
\sin b \sin c \sin A = 2n & \sin B \sin C \sin a = 2N \\
\sin c \sin a \sin B = 2n & \sin C \sin A \sin b = 2N \\
\sin a \sin b \sin C = 2n & \sin A \sin B \sin c = 2N
\end{array}\right\} \quad \ldots\ldots\ldots (20).$$

Multiply each triad of these, and divide the one product by the other: then

$$\frac{\sin A}{\sin a} \cdot \frac{\sin B}{\sin b} \cdot \frac{\sin C}{\sin c} = \frac{N^3}{n^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21).$$

And since all factors of the left side are equal, we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{N}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22).$$

Multiply all the six of the equations (20): then extracting the cube root, we have

$$\sin a \sin b \sin c \sin A \sin B \sin C = 4Nn \ldots\ldots\ldots\ldots\ldots\ldots\ldots (23).$$

Again, to obtain the values of N and n as symmetrical functions of the parts of the triangle, we have from each triad of (20)

$$\left.\begin{array}{l}
(2n)^3 = \sin^2 a \sin^2 b \sin^2 c \sin A \sin B \sin C \\
(2N)^3 = \sin^2 A \sin^2 B \sin^2 C \sin a \sin b \sin c
\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (24).$$

Recurring again to Purser's equations, we have

$$\left.\begin{array}{l}
\sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C = \dfrac{n^2}{\sin s \sin a \sin b \sin c} \\
\cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C = \dfrac{n \sin s}{\sin a \sin b \sin c} \\
\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C = \dfrac{n^2}{\sin^2 s}
\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (25).$$

$$\left.\begin{array}{l}
\sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c = \dfrac{-N \cos S}{\sin A \sin B \sin C} \\
\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c = \dfrac{N^2}{-\cos S \sin A \sin B \sin C} \\
\tan \tfrac{1}{2}a \tan \tfrac{1}{2}b \tan \tfrac{1}{2}c = \dfrac{\cos^2 S}{N}
\end{array}\right\} \quad \ldots\ldots\ldots (26).$$

Insert the values of sin a, sin A, *etc.* from (22) in (17, 18): then

$$\left.\begin{array}{ll|ll}
\sin s & = \dfrac{2n \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C}{N} & \cos S & = -\dfrac{2N \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{n} \\
\sin (s-a) & = \dfrac{2n \cos \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C}{N} & \cos (S-A) & = \dfrac{2N \sin \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}{n} \\
\sin (s-b) & = \dfrac{2n \sin \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C}{N} & \cos (S-B) & = \dfrac{2N \cos \frac{1}{2}a \sin \frac{1}{2}b \cos \frac{1}{2}c}{n} \\
\sin (s-c) & = \dfrac{2n \sin \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C}{N} & \cos (S-C) & = \dfrac{2N \cos \frac{1}{2}a \cos \frac{1}{2}b \sin \frac{1}{2}c}{n}
\end{array}\right\}(27).$$

Multiply together the values of sin s in (19, 27): then, those of sin $(s - a)$, *etc.*: and we get

$$\left.\begin{array}{ll|ll}
\sin^2 s & = n \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C & \cos^2 S & = N \tan \frac{1}{2}a \tan \frac{1}{2}b \tan \frac{1}{2}c \\
\sin^2 (s-a) & = n \cot \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C & \cos^2 (S-A) & = N \tan \frac{1}{2}a \cot \frac{1}{2}b \cot \frac{1}{2}c \\
\sin^2 (s-b) & = n \tan \frac{1}{2}A \cot \frac{1}{2}B \tan \frac{1}{2}C & \cos^2 (S-B) & = N \cot \frac{1}{2}a \tan \frac{1}{2}b \cot \frac{1}{2}c \\
\sin^2 (s-c) & = n \tan \frac{1}{2}A \tan \frac{1}{2}B \cot \frac{1}{2}C & \cos^2 (S-C) & = N \cot \frac{1}{2}a \cot \frac{1}{2}b \tan \frac{1}{2}c
\end{array}\right\} (28).$$

Multiply the first sides and the second sides of the series of equations (27): then

$$\left.\begin{array}{l}
4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c = -\sin s \cos S \\
4 \cos \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C \sin \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c = \sin (s-a) \cos (S-A) \\
4 \sin \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C \cos \frac{1}{2}a \sin \frac{1}{2}b \cos \frac{1}{2}c = \sin (s-b) \cos (S-B) \\
4 \sin \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C \cos \frac{1}{2}a \cos \frac{1}{2}b \sin \frac{1}{2}c = \sin (s-c) \cos (S-C)
\end{array}\right\} \cdot\cdot (29).$$

Innumerable other combinations, interesting for the symmetry and elegance of their forms might be easily deduced. They are, however, left as exercises for the student; and he is strongly recommended to pursue them, were it only for practice in transformation.

It has been seen at *p*. 31, that the perpendiculars from the angles to the opposite sides, are expressible as simple functions of n and the sides to which they are drawn. They may, however, be expressed in terms of N and the angles from which they are drawn.

For (*p*.31, *eq*. 2) and (*p*.39, *eq*. 22) $\sin p_1 = \dfrac{2n}{\sin a}$, and $n = \dfrac{N \sin a}{\sin A}$; whence

$$\sin p_1 = \frac{2N}{\sin A}; \text{ and so on with the others } \ldots\ldots\ldots\ldots\ldots\ldots \ (30).$$

Recurring to the formula of Albategnius, and its polar case, as given in the equations which precede (19), we have

$$\frac{\sin A}{\sin a} = \frac{2n}{\sin a \sin b \sin c}, \text{ and } \frac{\sin a}{\sin A} = \frac{2N}{\sin A \sin B \sin C};$$

from which we have by virtue of (21) the relations

$$\frac{N}{n} = \frac{2n}{\sin a \sin b \sin c}, \text{ and } \frac{n}{N} = \frac{2N}{\sin A \sin B \sin C} \ldots\ldots\ldots\ldots\ (31).$$

which are formulæ of great utility in general transformations.

III. THE POLAR TRIANGLES.

Some of the most obvious and important relations of the polar to the primary system have already been pointed out in *props. vi., vii., viii. p.* 6—8. A few others have reference to functions of the triangles which require trigonometrical expression. The majority of those, however, which have not yet been given, might have been brought in earlier, had there been any adequate reason for doing so.

In what follows, we shall express the sides and angles of the several associated triangles which are polar to the system whose sides are a, b, c, and angles A, B, C, in terms of these sides and angles. Any details beyond the mere results will be needless, as the formation of the several values is at once indicated by the figure itself.

TABLE.

The values of the sides and angles of the primary and polar associated triangles in terms of the sides and angles of the primary fundamental one.

PRIMARY ASSOCIATED TRIANGLES.			POLAR ASSOCIATED TRIANGLES.	
Sides.	*Angles.*		*Sides.*	*Angles.*
$a = a$	$A = A$		$a' = \Pi - A$	$A' = \pi - a$
$b = b$	$B = B$		$b' = \Pi - B$	$B' = \pi - b$
$c = c$	$C = C$		$c' = \Pi - C$	$C' = \pi - c$
$s = s$	$S = S$	Δ'	$s' = \frac{3}{2}\Pi - S$	$S' = \frac{3}{2}\pi - s$
$s - a = s - a$	$S - A = S - A$		$s' - a' = \frac{1}{2}\Pi - (S - A)$	$S' - A' = \frac{1}{2}\pi - (s - a)$
$s - b = s - b$	$S - B = S - B$		$s' - b' = \frac{1}{2}\Pi - (S - B)$	$S' - B' = \frac{1}{2}\pi - (s - b)$
$s - c = s - c$	$S - C = S - C$		$s' - c' = \frac{1}{2}\Pi - (S - C)$	$S' - C' = \frac{1}{2}\pi - (s - c)$
$a_1 = a$	$A_1 = A$		$a'_1 = \Pi - A$	$A'_1 = \pi - a$
$b_1 = \pi - b$	$B_1 = \Pi - B$		$b'_1 = B$	$B'_1 = b$
$c_1 = \pi - c$	$C_1 = \Pi - C$		$c'_1 = C$	$C'_1 = c$
$s_1 = \pi - (s - a)$	$S_1 = \Pi - (S - A)$	Δ'_1	$s'_1 = \frac{1}{2}\Pi + (S - A)$	$S'_1 = \frac{1}{2}\pi + (s - a)$
$s_1 - a_1 = \pi - s$	$S_1 - A_1 = \Pi - S$		$s'_1 - a'_1 = -\frac{1}{2}\Pi + S$	$S'_1 - A'_1 = -\frac{1}{2}\pi + s$
$s_1 - b_1 = s - c$	$S_1 - B_1 = S - C$		$s'_1 - b'_1 = \frac{1}{2}\Pi - (S - C)$	$S'_1 - B'_1 = \frac{1}{2}\pi - (s - c)$
$s_1 - c_1 = s - b$	$S_1 - C_1 = S - B$		$s'_1 - c'_1 = \frac{1}{2}\Pi - (S - B)$	$S_1 - C'_1 = \frac{1}{2}\pi - (s - b)$
$a_2 = \pi - a$	$A_2 = \Pi - A$		$a'_2 = A$	$A'_2 = a$
$b_2 = b$	$B_2 = B$		$b'_2 = \Pi - B.$	$B'_2 = \pi - b$
$c_2 = \pi - c$	$C_2 = \Pi - C$		$c'_2 = C$	$C'_2 = c$
$s_2 = \pi - (s - b)$	$S_2 = \Pi - (S - B)$	Δ'_2	$s'_2 = \frac{1}{2}\Pi + (S - B)$	$S'_2 = \frac{1}{2}\pi + (s - b)$
$s_2 - a_2 = s - c$	$S_2 - A_2 = S - C$		$s'_2 - a'_2 = \frac{1}{2}\Pi + (S - B)$	$S'_2 - A'_2 = \frac{1}{2}\pi - (s - c)$
$s_2 - b_2 = \pi - s$	$S_2 - B_2 = \Pi - S$		$s_2 - b'_2 = -\frac{1}{2}\Pi + S$	$S'_2 - B'_2 = -\frac{1}{2}\pi + s$
$s_2 - c_2 = s - a$	$S_2 - C_2 = S - A$		$s_2 - c'_2 = \frac{1}{2}\Pi - (S - A)$	$S'_2 - C'_2 = \frac{1}{2}\pi - (s - a)$
$a_3 = \pi - a$	$A_3 = \Pi - A$		$a'_3 = A$	$A'_3 = a$
$b_3 = \pi - b$	$B_3 = \Pi - B$		$b'_3 = B$	$B'_3 = b$
$c_3 = c$	$C_3 = C$		$c'_3 = \Pi - C$	$C'_3 = \pi - c$
$s_3 = \pi - (s - c)$	$S_3 = \Pi - (S - C)$	Δ'_3	$S'_3 = \frac{1}{2}\Pi + (S - C)$	$S'_3 = \frac{1}{2}\pi + (s - c)$
$s_3 - a_3 = s - b$	$S_3 - A_3 = S - B$		$s'_3 - a'_3 = \frac{1}{2}\Pi - (S - B)$	$S'_3 - A'_3 = \frac{1}{2}\pi - (s - b)$
$s_3 - b_3 = s - a$	$S_3 - B_3 = S - A$		$s'_3 - b'_3 = \frac{1}{2}\Pi - (S - A)$	$S'_3 - B'_3 = \frac{1}{2}\pi - (s - a)$
$s_3 - c_3 = \pi - s$	$S_3 - C_3 = \Pi - S$		$s'_3 - c'_3 = -\frac{1}{2}\Pi + S$	$S'_3 - C'_3 = -\frac{1}{2}\pi + s$

From this table we learn, very readily, several interesting particulars, one or two only of which we shall here instance.

$$\left.\begin{array}{l} n^2 = n_1^2 = n_2^2 = n_3^2 = N'^2 = N'^2_1 = N'^2_2 = N'^2_3, \\ \text{and } n'^2 = n'^2_1 = n'^2_2 = n'^2_3 = N^2 = N_1^2 = N_2^2 = N_3^2. \end{array}\right\} \cdots\cdots\cdots\cdots (32).$$

It has already been shown geometrically, (*prop. viii.,*) that the inscribed and circumscribed radii of each primary are the complements of the circumscribed and inscribed radii of the polar. This also may be elegantly proved by means of these equations, and the expressions for the several tangents, were it deemed necessary: but this, for the sake of room, is omitted, and the total collected results only put down as the consequences of that proposition.

It is, however, to be remarked, that we have in that proposition considered only the proper spherical centre or nearest pole. If we take the remote pole, the sum of the two radii instead of being $\frac{1}{2}\pi$ is $\frac{3}{2}\pi$. Both systems are put down in juxta-position.

$$\left.\begin{array}{l|l||l|l}
R' + r = \frac{1}{2}\pi & R'_2 + r_2 = \frac{1}{2}\pi & R' + r = \frac{3}{2}\pi & R'_2 + r_2 = \frac{3}{2}\pi \\
R + r' = \frac{1}{2}\pi & R_2 + r'_2 = \frac{1}{2}\pi & R + r' = \frac{3}{2}\pi & R_2 + r'_2 = \frac{3}{2}\pi \\
R'_1 + r_1 = \frac{1}{2}\pi & R_3 + r'_3 = \frac{1}{2}\pi & R'_1 + r_1 = \frac{3}{2}\pi & R'_3 + r_3 = \frac{3}{2}\pi \\
R_1 + r'_1 = \frac{1}{2}\pi & R'_3 + r_3 = \frac{1}{2}\pi & R_1 + r'_1 = \frac{3}{2}\pi & R_3 + r'_3 = \frac{3}{2}\pi
\end{array}\right\} \ldots\ldots \quad (33).$$

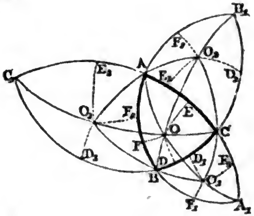
The sum of all the radii of the inscribed and circumscribed circles of the primary and polar triangles being taken, we have

$$\left.\begin{array}{l} R + R_1 + R_2 + R_3 + R' + R'_1 + R'_2 + R'_3 \\ + r + r_1 + r_2 + r_3 + r' + r'_1 + r'_2 + r'_3 \end{array}\right\} = 4\pi \text{ or } 12\pi \ldots\ldots \quad (34).$$

The continued products of all these sixteen tangents being taken, gives

$$\tan r \tan r_1 \ldots \tan r' \tan r'_1 \ldots\ldots \textit{etc.} = 1 \ldots\ldots\ldots\ldots\ldots\ldots \quad (35).$$

IV. THE CIRCLES INSCRIBED IN, AND CIRCUMSCRIBED ABOUT, THE ASSOCIATED TRIANGLES.



Let O be the centre of the circle inscribed in the fundamental triangle ABC, and D the point of contact with the base. Then BO, CO bisect the angles B and C, and OD is perpendicular to BC.

Put BD$=a_1$, DC$=a_2$, and OD$=r$. Then by the right-angled triangles BDO, DOC we have

$$\left.\begin{array}{l} \sin a_1 = \tan r \cot \frac{1}{2}B = \dfrac{\tan r \sin B}{1 - \cos B}, \\[2ex] \sin a_2 = \tan r \cot \frac{1}{2}C = \dfrac{\tan r \sin C}{1 - \cos C}. \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (36).$$

Also $\cos a = \cos (a_1 + a_2) = \cos a_1 \cos a_2 - \sin a_1 \sin a_2$: whence transposing

$$(\cos a + \sin a_1 \sin a_2)^2 = (1 - \sin^2 a_1)(1 - \sin^2 a_2), \text{ or}$$

$$\sin^2 a = \sin^2 a_1 + 2 \sin a_1 \sin a_2 \cos a + \sin^2 a_2 \ldots\ldots\ldots\ldots\ldots \quad (37).$$

Insert the values of $\sin a_1$ and $\sin a_2$ from (36): then

$$\frac{\sin^2 a}{\tan^2 r} = \frac{(1+\cos B)(1-\cos C)+(1-\cos B)(1+\cos C)+2\sin B \sin C \cos a}{(1-\cos B)(1-\cos C)}$$

$$= \frac{2\{1-\cos B \cos C+\sin B \sin C \cos a\}}{(1-\cos B)(1-\cos C)} = \frac{2(1+\cos A)}{(1-\cos B)(1-\cos C)} = \frac{\cos^2 \frac{1}{2}A}{\sin^2 \frac{1}{2}B \sin^2 \frac{1}{2}C};$$

or, by reduction and extraction, $\tan r = \dfrac{\sin a \sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A} \ldots\ldots\ldots\ldots\ldots \quad (38).$

But $\sin a = \dfrac{2N}{\sin B \sin C}$ which substituted in the preceding gives

$$\tan r = \frac{2N \sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A \sin B \sin C} = \frac{N}{2 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C} \ldots\ldots\ldots\ldots\ldots \quad (39).$$

Again, $2 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$

$$= \tfrac{1}{2}\{-(-\cos S) + \cos (S-A) + \cos (S-B) + \cos (S-C)\},$$

whence we have another value of $\tan r$.

$$\tan r = \frac{2N}{-(-\cos S) + \cos (S-A) + \cos (S-B) + \cos (S-C)} \ldots \quad (40).$$

Lastly, for $\cos \frac{1}{2}A$, *etc.* substitute their values from Purser's forms: then

$$\tan^2 r = \frac{\sin s \sin(s-a) \sin(s-b) \sin(s-c)}{\sin^2 s}; \text{ or } \tan r = \frac{n}{\sin s} \ldots\ldots (41).$$

Next, proceed in the same way with the triangle A'BC, employing the appropriate values of the sides and angles in terms of those of ABC from *prop. xxii., pp.* 16, 17, 18: and we get

$$\tan r_1 = \frac{N}{2\cos\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C} \ldots\ldots (42).$$

$$\tan r_1 = \frac{2N}{-\cos S - \cos(S-A) + \cos(S-B) + \cos(S-C)} \ldots\ldots (43).$$

$$\tan r_1 = \frac{n}{\sin(s-a)} \ldots\ldots (44).$$

Finding the others in the same way, we may tabulate the system thus:—

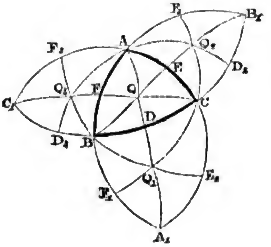
$$\begin{aligned}
\tan r &= \frac{N}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C} = \frac{2N}{+\cos S+\cos(S-A)+\cos(S-B)+\cos(S-C)} = \frac{n}{\sin s}\\
\tan r_1 &= \frac{N}{2\cos\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C} = \frac{2N}{-\cos S-\cos(S-A)+\cos(S-B)+\cos(S-C)} = \frac{n}{\sin(s-a)}\\
\tan r_2 &= \frac{N}{2\sin\frac{1}{2}A\cos\frac{1}{2}B\sin\frac{1}{2}C} = \frac{2N}{-\cos S+\cos(S-A)-\cos(S-B)+\cos(S-C)} = \frac{n}{\sin(s-b)}\\
\tan r_3 &= \frac{N}{2\sin\frac{1}{2}A\sin\frac{1}{2}B\cos\frac{1}{2}C} = \frac{2N}{-\cos S+\cos(S-A)+\cos(S-B)-\cos(S-C)} = \frac{n}{\sin(s-c)}
\end{aligned}$$

$$\ldots\ldots (45).$$

The first two forms are due to Mr. Rutherford, and the third to De Gua.

By reference to the mode of solution suggested in the equations marked (9), it will be seen that in $\tan r$, there employed, r is the radius of the inscribed circle. For, from the equation above, we have another value of $\tan r$.

$$\tan r = \frac{n}{\sin s} = \frac{\sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}}{\sin s} = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$



Let Q be the centre of the circumscribing circle, and draw the perpendiculars QD, QE, QF, and join QA, QB, QC. Then the sides are bisected in D, E, F.

Put CAQ $= A_1$ and QAB $= A_2$: then by right-angled triangles we have

$$\cos A_1 = \cot R \tan\tfrac{1}{2}b = \frac{\cot R\,(1-\cos b)}{\sin b}$$

$$\cos A_2 = \cot R \tan\tfrac{1}{2}c = \frac{\cot R\,(1-\cos c)}{\sin c}$$

$$\ldots\ldots (46).$$

Also, $\cos A = \cos(A_1 + A_2) = \cos A_1 \cos A_2 - \sin A_1 \sin A_2$; or by reduction, as before,

$$\sin^2 A = \cos^2 A_1 - 2\cos A_1 \cos A_2 \cos A + \cos^2 A_2 \ldots\ldots (47).$$

Insert the values of $\cos A_1$ and $\cos A_2$ from (A) in (B): then we find, after the manner of the preceding investigation,

$$\cot R = \frac{\sin A \cos\frac{1}{2}b \cos\frac{1}{2}c}{\sin\frac{1}{2}a} \ldots\ldots (48).$$

And since $\sin A = \dfrac{2n}{\sin b \sin c}$, we have again

$$\cot \mathrm{R} = \frac{n}{2 \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (49).$$

Transform the denominator: then this becomes

$$\cot \mathrm{R} = \frac{2n}{-\sin s + \sin (s - a) + \sin (s - b) + \sin (s - c)} \quad \ldots\ldots\ldots \quad (50).$$

And lastly, inserting the values of $\sin \tfrac{1}{2}a$, *etc.* from *Purser*,

$$\cot \mathrm{R} = \frac{\mathrm{N}}{-\cos \mathrm{S}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (51).$$

Corresponding operations being made upon the parts of A′BC, AB′C, and ABC′ in terms of the triangle ABC will give the results tabulated below.

$$\begin{aligned}
\cot \mathrm{R} &= \frac{n}{2 \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c} = \frac{2n}{-\sin s + \sin(s-a) + \sin(s-b) + \sin(s-c)} = \frac{\mathrm{N}}{-\cos \mathrm{S}} \\
\cot \mathrm{R}_1 &= \frac{n}{2 \sin \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c} = \frac{2n}{\sin s - \sin (s-a) + \sin (s-b) + \sin (s-c)} = \frac{\mathrm{N}}{\cos (\mathrm{S}-\mathrm{A})} \\
\cot \mathrm{R}_2 &= \frac{n}{2 \cos \tfrac{1}{2}a \sin \tfrac{1}{2}b \cos \tfrac{1}{2}c} = \frac{2n}{\sin s + \sin (s-a) - \sin (s-b) + \sin (s-c)} = \frac{\mathrm{N}}{\cos (\mathrm{S}-\mathrm{B})} \\
\cot \mathrm{R}_3 &= \frac{n}{2 \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \sin \tfrac{1}{2}c} = \frac{2n}{\sin s + \sin (s-a) + \sin (s-b) - \sin (s-c)} = \frac{\mathrm{N}}{\cos (\mathrm{S}-\mathrm{C})}
\end{aligned}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \quad (52).$$

The mode of solution suggested for one case of spherical triangles, given at *p*. 35, *eq*. 11, is founded on this principle. For

$$\begin{aligned}
\cot \mathrm{R} = \frac{\mathrm{N}}{-\cos \mathrm{S}} &= \sqrt{\frac{-\cos \mathrm{S} \cos (\mathrm{S}-\mathrm{A}) \cos (\mathrm{S}-\mathrm{B}) \cos (\mathrm{S}-\mathrm{C})}{(-\cos \mathrm{S})^2}} \\
&= \sqrt{\frac{\cos (\mathrm{S}-\mathrm{A}) \cos (\mathrm{S}-\mathrm{B}) \cos (\mathrm{S}-\mathrm{C})}{-\cos \mathrm{S}}}
\end{aligned}$$

The analogies of plane and spherical trigonometry are very numerous; but it is to be remarked, that these are confined to the *sides* of the triangles only: whilst the angles of the spherical triangle have a similar class of analogies to the sides themselves, but this is lost in comparing the angles of plane and spherical triangles.

From De Gua's values of $\tan r$, $\tan r_1$, *etc.* and Rutherford's, of $\cot \mathrm{R}$, $\cot \mathrm{R}_1$, *etc.* we have

$$\cot r + \cot r_1 + \cot r_2 + \cot r_3 = \frac{\sin s + \sin (s-a) + \sin (s-b) + \sin (s-c)}{n} \ldots (53),$$

$$\tan \mathrm{R} + \tan \mathrm{R}_1 + \tan \mathrm{R}_2 + \tan \mathrm{R}_3 = \frac{\sin s + \sin (s-a) + \sin (s-b) + \sin (s-c)}{n} \ldots (54),$$

whence $\cot r + \cot r_1 + \cot r_2 + \cot r_3 = \tan \mathrm{R} + \tan \mathrm{R}_1 + \tan \mathrm{R}_2 + \tan \mathrm{R}_3 \ldots\ldots (55).$

Again, from (54) subtract $2 \cot r = \dfrac{2 \sin s}{n}$: then we have

$$-\cot r + \cot r_1 + \cot r_2 + \cot r_3 = \frac{-\sin s + \sin (s-a) + \sin (s-b) + \sin (s-c)}{n} = 2 \tan \mathrm{R} \ \ldots (56)$$

In the same way, subtracting $2 \cot r_1$, $2 \cot r_2$, and $2 \cot r_3$, we get the system

$$\left.\begin{array}{rl}
2 \tan \mathrm{R} & = -\cot r + \cot r_1 + \cot r_2 + \cot r_3 \\
2 \tan \mathrm{R}_1 & = \cot r - \cot r_1 + \cot r_2 + \cot r_3 \\
2 \tan \mathrm{R}_2 & = \cot r + \cot r_1 - \cot r_2 + \cot r_3 \\
2 \tan \mathrm{R}_3 & = \cot r + \cot r_1 + \cot r_2 - \cot r_3
\end{array}\right\} \ldots\ldots\ldots\ldots \quad (57).$$

and proceeding in the same manner with respect to $2 \tan \mathrm{R}$, $2 \tan \mathrm{R}_1$, *etc.*

$$\left.\begin{array}{rl}
2 \cot r & = -\tan \mathrm{R} + \tan \mathrm{R}_1 + \tan \mathrm{R}_2 + \tan \mathrm{R}_3 \\
2 \cot r_1 & = \tan \mathrm{R} - \tan \mathrm{R}_1 + \tan \mathrm{R}_2 + \tan \mathrm{R}_3 \\
2 \cot r_2 & = \tan \mathrm{R} + \tan \mathrm{R}_1 - \tan \mathrm{R}_2 + \tan \mathrm{R}_3 \\
2 \cot r_3 & = \tan \mathrm{R} + \tan \mathrm{R}_1 + \tan \mathrm{R}_2 - \tan \mathrm{R}_3
\end{array}\right\} \ldots\ldots\ldots\ldots \quad (58).$$

These theorems, in a less distinct and definite form, were first given by the editor in *Leybourn's Repository, vol. vi.* N.S.: but in the above very elegant shape they are due to Mr. Rutherford, who gave them in the *Gentleman's Diary* for 1837.

Again, multiply De Gua's values of $\tan r$, $\tan r_1$, *etc.*: then we have

$$\tan r \tan r_1 \tan r_2 \tan r_3 = \frac{n^4}{n^2} = n^2 \text{.......................} \quad (59).$$

Divide this by the square of each of them in succession: this gives

$$\left.\begin{aligned} \sin^2 s &= \cot r \tan r_1 \tan r_2 \tan r_3 \\ \sin^2 (s-a) &= \tan r \cot r_1 \tan r_2 \tan r_3 \\ \sin^2 (s-b) &= \tan r \tan r_1 \cot r_2 \tan r_3 \\ \sin^2 (s-c) &= \tan r \tan r_1 \tan r_2 \cot r_3 \end{aligned}\right\} \text{.....................} \quad (60).$$

which results were given first by Mr. Lowry in the *Math. Repos. vol. v. p.* 3. The analogy of these to the original form of the equation between (45) and (46) is very remarkable. For in the case above, we have for the value of $\tan r$,

$$\sin s = \sqrt{\frac{\tan r_1 \tan r_2 \tan r_3}{\tan r}}, \text{ and so on.}$$

Take the products of De Gua's values of $\tan r$, $\tan r_1$, *etc.* two and two, when there results another system of equations, of which the first only is here worked out, and the others merely put down.

$$\begin{aligned} \tan r \tan r_1 &= \frac{n}{\sin s} \cdot \frac{n}{\sin (s-a)} = \sin (s-b) \sin (s-c) \\ &= \sin \tfrac{1}{2} \{a + (b-c)\} \sin \tfrac{1}{2} \{a - (b-c)\} = \sin^2 \tfrac{1}{2} a - \sin^2 \tfrac{1}{2} (b-c). \end{aligned}$$

The whole system of results deduced from the combinations indicated on the left sides of the equations will be

$$\left.\begin{aligned} \tan r \tan r_1 &= \sin^2 \tfrac{1}{2} a - \sin^2 \tfrac{1}{2} (b-c) \\ \tan r \tan r_2 &= \sin^2 \tfrac{1}{2} b - \sin^2 \tfrac{1}{2} (c-a) \\ \tan r \tan r_3 &= \sin^2 \tfrac{1}{2} c - \sin^2 \tfrac{1}{2} (a-b) \end{aligned} \;\middle|\; \begin{aligned} \tan r_1 \tan r_2 &= \sin^2 \tfrac{1}{2} (a+b) - \sin^2 \tfrac{1}{2} c \\ \tan r_1 \tan r_3 &= \sin^2 \tfrac{1}{2} (a+c) - \sin^2 \tfrac{1}{2} b \\ \tan r_2 \tan r_3 &= \sin^2 \tfrac{1}{2} (b+c) - \sin^2 \tfrac{1}{2} a \end{aligned}\right\} \text{..} \quad (61).$$

Whence by addition of these two and two as indicated,

$$\left.\begin{aligned} \tan r \tan r_1 + \tan r_2 \tan r_3 &= \sin^2 \tfrac{1}{2} (c+b) - \sin^2 \tfrac{1}{2} (c-b) = \sin b \sin c \\ \tan r \tan r_2 + \tan r_3 \tan r_1 &= \sin^2 \tfrac{1}{2} (c+a) - \sin^2 \tfrac{1}{2} (c-a) = \sin c \sin a \\ \tan r \tan r_3 + \tan r_1 \tan r_2 &= \sin^2 \tfrac{1}{2} (b+a) - \sin^2 \tfrac{1}{2} (b-a) = \sin a \sin b \end{aligned}\right\} \text{..} \quad (62).$$

And by addition of these three equations, there results the remarkable form,

$$\left.\begin{aligned} \tan r \tan r_1 + \tan r, \tan r_2 + \tan r \tan r_3 \\ + \tan r_2 \tan r_3 + \tan r, \tan r_1 + \tan r_1 \tan r_2 \end{aligned}\right\} = \sin a \sin b + \sin b \sin c + \sin c \sin a \text{..} (63).$$

an expression which was first given in my *Supplement to Young's Trigonometry*, and subsequently in the *Repos. vol. vi.*

$$\begin{aligned} \text{Since } \tan r \tan r_1 - \tan r_2 \tan r_3 &= \sin (s-b) \sin (s-c) - \sin s \sin (s-a) \\ &= - (\cos a - \cos b \cos c) \\ &= - \sin b \sin c \cos \mathrm{A}, \text{ we have} \end{aligned}$$

$$\cos \mathrm{A} = \frac{\tan r \tan r_1 - \tan r_2 \tan r_3}{- \sin b \sin c} = - \frac{\tan r \tan r_1 - \tan r_2 \tan r_3}{\tan r \tan r_1 + \tan r_2 \tan r_3}.$$

Whence, and operating in the same way for cos B, cos C, we have

$$\left.\begin{aligned} \tan^2 \tfrac{1}{2} \mathrm{A} &= \frac{1 - \cos \mathrm{A}}{1 + \cos \mathrm{A}} = \frac{\tan r \tan r_1}{\tan r_2 \tan r_3} \\ \tan^2 \tfrac{1}{2} \mathrm{B} &= \frac{1 - \cos \mathrm{B}}{1 + \cos \mathrm{B}} = \frac{\tan r \tan r_2}{\tan r_3 \tan r_1} \\ \tan^2 \tfrac{1}{2} \mathrm{C} &= \frac{1 - \cos \mathrm{C}}{1 + \cos \mathrm{C}} = \frac{\tan r \tan r_3}{\tan r \tan r_2} \end{aligned}\right\} \text{.....................} \quad (64).$$

These first formulæ were given in the *Repository* (in the same paper), and lead to several others, which may be seen there and in other places.

Processes too much like these to need the least detail here will give the following corresponding equations with respect to R, R_1, R_2, R_3.

$$\cot R \cot R_1 \cot R_2 \cot R_3 = \frac{N^4}{N^2} = N^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (65).$$

$$\left.\begin{aligned} (-\cos S)^2 &= \tan R \cot R_1 \cot R_2 \cot R_3 \\ \cos^2 (S - A) &= \cot R \tan R_1 \cot R_2 \cot R_3 \\ \cos^2 (S - B) &= \cot R \cot R_1 \tan R_2 \cot R_3 \\ \cos^2 (S - C) &= \cot R \cot R_1 \cot R_2 \tan R_3 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (66).$$

$$\left.\begin{aligned} \cot R \cot R_1 + \cot R_2 \cot R_3 &= \sin B \sin C \\ \cot R \cot R_2 + \cot R_3 \cot R_1 &= \sin C \sin A \\ \cot R \cot R_3 + \cot R_1 \cot R_2 &= \sin A \sin B \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (67).$$

$\cot R \cot R_1 + \ldots$(all the left side)$= \sin A \sin B + \sin B \sin C + \sin C \sin A$ .. (68).

$$\left.\begin{aligned} \tan^2 \tfrac{1}{2}a &= \frac{\cot R \ \cot R_1}{\cot R_2 \cot R_3} \\ \tan^2 \tfrac{1}{2}b &= \frac{\cot R \ \cot R_2}{\cot R_3 \cot R_1} \\ \tan^2 \tfrac{1}{2}c &= \frac{\cot R \ \cot R_1}{\cot R_1 \cot R_2} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (69).$$

The corresponding equations for the polar triangles are obtained at once from the equations in the tables at *p.* 41, and though interesting, need scarcely be more than referred to here, and left as an exercise for the student. One or two examples may, however, be added.

Since $\tan r = \frac{n}{\sin s}$, and $\tan r' = \frac{n'}{\sin s}$; and that we have also (*eq.* 42)

$$\tan r' = \frac{N'}{2 \cos \frac{1}{2}A' \cos \frac{1}{2}B' \cos \frac{1}{2}C'} = \frac{n}{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}; \text{ hence}$$

$$\frac{\tan r}{\tan r'} = \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{\sin \frac{1}{2}(a + b + c)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (70).$$

Again in the same way we have

$$\tan R = -\frac{\cos S}{N} = \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{n}, \text{ and } \tan R' = -\frac{\cos S'}{N'} = \frac{\sin s}{n},$$

and hence

$$\frac{\tan R}{\tan R'} = \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{\sin \frac{1}{2}(a + b + c)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (71).$$

From the two last, and similarly for the other triangles

$$\left.\begin{array}{l|l} \dfrac{\tan r}{\tan r'} = \dfrac{\tan R}{\tan R'} & \dfrac{\tan r_2}{\tan r'_2} = \dfrac{\tan R_2}{\tan R'_2} \\ \dfrac{\tan r_1}{\tan r'_1} = \dfrac{\tan R_1}{\tan R'_1} & \dfrac{\tan r_3}{\tan r'_3} = \dfrac{\tan R_3}{\tan R'_3} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (72).$$

Multiply the values of $\tan r$ and $\tan R'$, and likewise $\tan r'$ and $\tan R$: then there results

$$\tan r \tan R' = 1, \text{ and } \tan r' \tan R = 1:$$

which leads to the same conclusion as in (33); and corresponding results are found to flow from each pair of primary and polar triangles.

Expressions may also be readily found for the products $\tan r \tan R$, $\tan r_1 \tan R_1$, *etc.* and their polars. For example,

$$\tan r \tan \mathrm{R} = \frac{n}{\sin s} \cdot \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{n} = \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{\sin \frac{1}{2}(a+b+c)} \ldots\ldots (73).$$

which is the same result as (71), and again would have established the equations $\tan r = \cot \mathrm{R}'$, *etc.* given in (33).

V. THE AREA OF A SPHERICAL TRIANGLE.

1. *When the three angles are given.*

It has been shown in *pr. xxiv. Cor.* 2, *p.* 20, that the area, Δ, is expressed by

$$\Delta = \frac{\mathrm{A}+\mathrm{B}+\mathrm{C}-\Pi}{\Pi} \cdot r^2\varpi, \ldots\ldots\ldots\ldots\ldots\ldots (74).$$

which is computable when A, B, C, r are given [*].

Scholium.

As the only variable part of this expression in reference to a given sphere is that of the *spherical excess*, $\mathrm{A} + \mathrm{B} + \mathrm{C} - \Pi$, it is usual to speak of the excess and the area as convertible terms. Recollecting that to the unit $\frac{r^2\varpi}{\Pi}$, the *excess* does express the area, there is but little danger of actual mistake by such synonymous employment of the terms, yet as some confusion might arise in the mind of the student from such cause, it is better to avoid the indiscriminate use of them. Denoting the excess by E, we have the relation

$$\Delta = \frac{\mathrm{E}r^2\varpi}{\Pi}, \text{ or } \mathrm{E} = \frac{\Pi\Delta}{r^2\varpi}; \ldots\ldots\ldots\ldots\ldots\ldots (75).$$

which will always serve to convert an expression for the excess into one for the area, and *vice versâ.*

2. *Given two sides and the included angle.*

We have in this case

$$\sin \tfrac{1}{2}\mathrm{E} = \sin \tfrac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C}-\Pi) = -\cos \tfrac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})$$
$$= \sin \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) \sin \tfrac{1}{2}\mathrm{C} - \cos \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) \cos \tfrac{1}{2}\mathrm{C}.$$

Substitute for $\sin \frac{1}{2}(\mathrm{A}+\mathrm{B})$ and $\cos \frac{1}{2}(\mathrm{A}+\mathrm{B})$ from Gauss's analogies (13, *p.* 36): viz.

$$\sin \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}c} \cos \tfrac{1}{2}\mathrm{C}, \text{ and } \cos \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}c} \sin \tfrac{1}{2}\mathrm{C}:$$

then, after slight reductions, we have

$$\sin \tfrac{1}{2}\mathrm{E} = \frac{\sin \frac{1}{2}a \sin \frac{1}{2}b}{\cos \frac{1}{2}c} \sin \mathrm{C} \ldots\ldots\ldots\ldots\ldots\ldots (76).$$

This formula is very remarkable on account of its elegance and simplicity;

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[*] This expression was first given by Albert Girard more than two hundred years ago, and it is generally known as *Girard's theorem:* but the method of proof now usually employed was invented by Dr. Wallis.

The theorem itself was considered merely in the light of an elegant speculative truth, till Mr. Dalby, in 1787, was employed to correct the errors of spherical triangles in the great trigonometrical survey. See *Phil. Trans. vol.* 80. This employment of Girard's theorem has usually, but erroneously, been attributed to General Roy: for the whole of the mathematical department of that survey was solely conducted by Mr. Dalby, subsequently Professor of Mathematics in the Royal Military College. See Leybourn's *Repos. vol. v. N. S. p.* 201.

but as it involves also the side c in the denominator, it will require some modifications to fulfil the actual conditions of the problem.

Reasoning as above we have

$$\cos \tfrac{1}{2}E = \sin \tfrac{1}{2}(A + B + C) = \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} C + \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}C,$$

and inserting from Gauss as before, we get

$$\cos \tfrac{1}{2}E = \frac{\cos \frac{1}{2}(a - b) \cos^2 \frac{1}{2}C + \cos \frac{1}{2}(a + b) \sin^2 \frac{1}{2}C}{\cos \frac{1}{2}c}$$

$$= \frac{\cos \frac{1}{2}a \cos \frac{1}{2}b + \sin \frac{1}{2}a \sin \frac{1}{2}b \cos C}{\cos \frac{1}{2}c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (77).$$

Divide (77) by (76): then there results

$$\cot \tfrac{1}{2}E = \frac{\cot \frac{1}{2}a \cot \frac{1}{2}b + \cos C}{\sin C} = \left\{1 + \frac{\cot \frac{1}{2}a \cot \frac{1}{2}b}{\cos C}\right\} \cot C \ldots\ldots\ldots \quad (78).$$

This theorem is due to De Gua, *Mém. de l'Acad.* 1783. The practice of the solution by means of a subsidiary angle is very simple, in the following manner:—

(1.) *When cos C is positive, and of any consistent value whatever.*

$$\text{Take } \frac{\cot \frac{1}{2}a \cot \frac{1}{2}b}{\cos C} = \tan^2\theta: \text{ then } \cot \tfrac{1}{2}E = \cot C \sec^2\theta.$$

(2.) *When cos C is negative, and numerically greater than cot ½a cot ½b.*

$$\text{Take } \frac{\cot \frac{1}{2}a \cot \frac{1}{2}b}{\cos C} = -\sin^2\theta: \text{ then } \cot \tfrac{1}{2}E = -\cot C \cos^2\theta.$$

(3.) *When cos C is negative, and numerically less than cot ½a cot ½b.*

$$\text{Take } \frac{\cot \frac{1}{2}a \cot \frac{1}{2}b}{\cos C} = -\sec^2\theta: \text{ then } \cot \tfrac{1}{2}E = -\cot C \tan^2\theta.$$

4. *When the three sides are given.*

Employing Purser's theorems (8), *p.* 35, we have

$$\sin C = 2 \sin \tfrac{1}{2}C \cos \tfrac{1}{2}C = \frac{2\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}}{\sin a \sin b}.$$

Insert this in (76): then

$$\sin \tfrac{1}{2}E = \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}}{4 \sin a \sin b \cos \frac{1}{2} c}$$

$$= \frac{\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}}{2 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c} \ldots\ldots\ldots\ldots \quad (79).$$

This is Cagnoli's *theorem for the excess,* and is adapted to logarithms.

Squaring both sides of (79), and substituting vers E for $2 \sin^2 \frac{1}{2}E$, we have

$$\text{vers } E = \frac{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}{2 \cos^2 \frac{1}{2}a \cos^2 \frac{1}{2}b \cos^2 \frac{1}{2}c} \ldots\ldots\ldots\ldots\ldots \quad (80).$$

This form was, so far as I am aware, first given by myself in another place, and deduced in another manner.

Substituting the values of the numerator and denominator of (80) in terms of a, b, c, we get at once

$$\text{vers } E = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{(1 + \cos a)(1 + \cos b)(1 + \cos c)} \ldots\ldots\ldots (81),$$

which, as well as the *four next following,* was given by De Gua, in the memoir before referred to.

Again, $\cos^2 \frac{1}{2}E = \frac{1}{2}(1 + \cos E) = \frac{1}{2}(2 - \text{vers } E)$, in which, substituting (81), we get

$$\text{os}\,\tfrac{1}{2}\text{E} = \sqrt{\frac{2(1+\cos a)\,(1+\cos b)\,(1+\cos c) - (1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c)}{2\,(1+\cos a)\,(1+\cos b)\,(1+\cos c)}}$$

$$= \frac{1+\cos a+\cos b+\cos c}{\sqrt{2\,(1+\cos a)\,(1+\cos b)\,(1+\cos c)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (82).$$

Also, since $1 + \cos a = 2\cos^2 \frac{1}{2}a$, *etc.* we have

$$\cos\tfrac{1}{2}\text{E} = \frac{1+\cos a+\cos b+\cos c}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (83).$$

From (79, 82) we get by division

$$\cot\tfrac{1}{2}\text{E} = \frac{1+\cos a+\cos b+\cos c}{2\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}} \quad \ldots\ldots\ldots\ldots \quad (84.)$$

Again, to return to cos E, we have its value from (81): then

$$\cos\text{E} = \frac{(1+\cos a+\cos b+\cos c)^2-(1+\cos a)\,(1+\cos b)\,(1+\cos c)}{(1+\cos a)\,(1+\cos b)\,(1+\cos c)} \quad \ldots\ldots \quad (85).$$

The following is also an elegant form, though not adapted to logarithms:—

Multiply (80, 84), and put $\cos a = 2\cos^2\frac{1}{2}a - 1$, *etc.*: then

$$\cos\tfrac{1}{2}\text{E} = \frac{\cos^2\frac{1}{2}a+\cos^2\frac{1}{2}b+\cos^2\frac{1}{2}c-1}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (86).$$

The most remarkable formula, however, for the spherical excess in terms of the three sides, is that of Lhuillier; which may be thus investigated by recurring to Girard's fundamental theorem.

We have from the definition of the *spherical excess*,

$$\tfrac{1}{4}\text{E} = \tfrac{1}{4}(\text{A}+\text{B}+\text{C}-\Pi);\ \text{and hence}$$

$$\tan\tfrac{1}{4}\text{E} = \frac{2\sin\frac{1}{4}(\text{A}+\text{B}+\text{C}-\Pi)}{2\cos\frac{1}{4}(\text{A}+\text{B}+\text{C}-\Pi)}\cdot\frac{\cos\frac{1}{4}(\text{A}+\text{B}-\text{C}+\Pi)}{\cos\frac{1}{4}(\text{A}+\text{B}-\text{C}+\Pi)}$$

$$= \frac{\sin\frac{1}{2}(\text{A}+\text{B})-\sin\frac{1}{2}(\Pi-\text{C})}{\cos\frac{1}{2}(\text{A}+\text{B})+\cos\frac{1}{2}(\Pi-\text{C})} = \frac{\sin\frac{1}{2}(\text{A}+\text{B})-\cos\frac{1}{2}\text{C}}{\cos\frac{1}{2}(\text{A}+\text{B})+\sin\frac{1}{2}\text{C}}$$

Insert in the last expression for $\tan\frac{1}{4}\text{E}$ the values of $\sin\frac{1}{2}(\text{A}+\text{B})$ and $\cos\frac{1}{2}(\text{A}+\text{B})$ from *Gauss* (13): then we have

$$\tan\tfrac{1}{4}\text{E} = \frac{\cos\frac{1}{2}(a-b)-\cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b)+\cos\frac{1}{2}c}\cot\tfrac{1}{2}\text{C} = \frac{\sin\frac{1}{2}(s-a)\sin\frac{1}{2}(s-b)}{\cos\frac{1}{2}s\cos\frac{1}{2}(s-c)}\cot\tfrac{1}{2}\text{C}.$$

Insert the value of $\cot\frac{1}{2}\text{C}$ from *Thacker* (8): then

$$\tan\tfrac{1}{4}\text{E} = \frac{\sin\frac{1}{2}(s-a)\sin\frac{1}{2}(s-b)}{\cos\frac{1}{2}s\cos\frac{1}{2}(s-c)}\sqrt{\frac{\sin s\sin(s-c)}{\sin(s-a)\sin(s-b)}}$$

$$= \sqrt{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)} \quad \ldots\ldots\ldots\ldots \quad (87).$$

Scholium.

No expression for the spherical excess in a form adapted to calculation, and in other terms than already discussed, is known. The combinations thus excluded are A, B, c; a, b, A; and A, B, a. For the first of these, a theorem of considerable elegance in respect to its symmetry, is known, and appears amongst the *Exercises*. In reference, however, to actual calculation, it is useless. The most advantageous method in practice is, undoubtedly, either to complete from the data the three sides or the three angles of the triangle, and then to employ, according to the case, the theorems of Lhuillier or Girard.

5. *The spherical excesses of the associated triangles.*

Denote the parts as before defined, and we shall have

$$\left.\begin{array}{l|l} \text{E} = \text{A}+\text{B}+\text{C}-\Pi & \text{E}_2 = -\text{A}+\text{B}-\text{C}+\Pi \\ \text{E}_1 = \text{A}-\text{B}-\text{C}+\Pi & \text{E}_3 = -\text{A}-\text{B}+\text{C}+\Pi \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (88).$$

By addition

$$E + E_1 + E_2 + E_3 = 2\Pi$$

Add each of the last three equations to the first, successively, and proceed as indicated by the left side: then

$$A = \tfrac{1}{2}(E + E_1);\; B = \tfrac{1}{2}(E + E_2);\; C = \tfrac{1}{2}(E + E_3) \ldots\ldots\ldots\ldots\ldots \;(89).$$

$$\left.\begin{array}{rl||rl} S = & \tfrac{1}{2}(\Pi + E) & -\cos S = & \sin \tfrac{1}{2}E \\ S - A = & \tfrac{1}{2}(\Pi - E_1) & \cos (S - A) = & \sin \tfrac{1}{2}E_1 \\ S - B = & \tfrac{1}{2}(\Pi - E_2) & \cos (S - B) = & \sin \tfrac{1}{2}E_2 \\ S - C = & \tfrac{1}{2}(\Pi - E_3) & \cos (S - C) = & \sin \tfrac{1}{2}E_3 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots \;(90).$$

Inserting these values in Purser's expressions, we get

$$\left.\begin{aligned} \cot^2 \tfrac{1}{2}a &= \frac{\sin \tfrac{1}{2}E_2 \sin \tfrac{1}{2}E_3}{\sin \tfrac{1}{2}E \;\sin \tfrac{1}{2}E_1} \\ \cot^2 \tfrac{1}{2}b &= \frac{\sin \tfrac{1}{2}E_3 \sin \tfrac{1}{2}E_1}{\sin \tfrac{1}{2}E \;\sin \tfrac{1}{2}E_2} \\ \cot^2 \tfrac{1}{2}c &= \frac{\sin \tfrac{1}{2}E_1 \sin \tfrac{1}{2}E_2}{\sin \tfrac{1}{2}E \;\sin \tfrac{1}{2}E_3} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(91).$$

and any other functions may be deduced from them.

In the preceding, the given parts are A, B, C, and we may proceed to consider a, b, c, as given. Referring then to the table of values of s_1, $s_1 - a_1$, *etc.* *p.* 41, we have, by applying Lhuillier's theorem to each of the four triangles successively,

$$\left.\begin{aligned} \tan \tfrac{1}{4}E &= \sqrt{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b) \tan \tfrac{1}{2}(s - c)} \\ \tan \tfrac{1}{4}E_1 &= \sqrt{\cot \tfrac{1}{2}s \cot \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b) \tan \tfrac{1}{2}(s - c)} \\ \tan \tfrac{1}{4}E_2 &= \sqrt{\cot \tfrac{1}{2}s \tan \tfrac{1}{2}(s - a) \cot \tfrac{1}{2}(s - b) \tan \tfrac{1}{2}(s - c)} \\ \tan \tfrac{1}{4}E_3 &= \sqrt{\cot \tfrac{1}{2}s \tan \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b) \cot \tfrac{1}{2}(s - c)} \end{aligned}\right\} \ldots\ldots \;(92).$$

Multiply these: then

$$\tan \tfrac{1}{4}E \tan \tfrac{1}{4}E_1 \tan \tfrac{1}{4}E_2 \tan \tfrac{1}{4}E_3 = \cot \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c) \ldots (93).$$

Again,

$$\begin{aligned} \tfrac{1}{4}E &= \tfrac{1}{4}(-\Pi + A + B + C) \\ \tfrac{1}{4}E_1 &= \tfrac{1}{4}(\;\;\Pi + A - B - C) = \tfrac{1}{2}\Pi - \tfrac{1}{4}(\Pi - A + B + C) \\ \tfrac{1}{4}E_2 &= \tfrac{1}{4}(\;\;\Pi - A + B - C) = \tfrac{1}{2}\Pi - \tfrac{1}{4}(\Pi + A - B + C) \\ \tfrac{1}{4}E_3 &= \tfrac{1}{4}(\;\;\Pi - A - B + C) = \tfrac{1}{2}\Pi - \tfrac{1}{4}(\Pi + A + B - C), \end{aligned}$$

from which

$$\left.\begin{aligned} \tan \tfrac{1}{4}(-\Pi+A+B+C) &= \sqrt{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)} \\ \tan \tfrac{1}{4}(\;\;\Pi-A+B+C) &= \sqrt{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \cot \tfrac{1}{2}(s-b) \cot \tfrac{1}{2}(s-c)} \\ \tan \tfrac{1}{4}(\;\;\Pi+A-B+C) &= \sqrt{\tan \tfrac{1}{2}s \cot \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \cot \tfrac{1}{2}(s-c)} \\ \tan \tfrac{1}{4}(\;\;\Pi+A+B-C) &= \sqrt{\tan \tfrac{1}{2}s \cot \tfrac{1}{2}(s-a) \cot \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)} \end{aligned}\right\} \ldots (94).$$

Multiply these: then

$$\tan^2 \tfrac{1}{4}(a+b+c) = \tan \tfrac{1}{4}(-\Pi+A+B+C) \tan \tfrac{1}{4}(\Pi-A+B+C) \tan \tfrac{1}{4}(\Pi+A-B+C) \tan \tfrac{1}{4}(\Pi+A+B-C) \ldots (95).$$

By giving to the terms of Lhuillier's expression for the excess of the fundamental triangle their values in terms of a, b, c, we have

$$\tan^2 \tfrac{1}{4}(-\Pi + A + B + C) = \tan \tfrac{1}{4}(a + b + c) \tan \tfrac{1}{4}(-a + b + c) \tan \tfrac{1}{4}(a - b + c) \tan \tfrac{1}{4}(a + b - c)$$

the analogy of which to (95) is very remarkable; the only difference of general form being that Π enters into the composition of all the angular functions, but π not into those of the sides.

Again, the three last factors on the right side of (95) are $\cot \tfrac{1}{4}E_1$, $\cot \tfrac{1}{4}E_2$, and $\cot \tfrac{1}{4}E_3$, whilst the first is $\tan \tfrac{1}{4}E$; we have, therefore,

$$\tan^2 \tfrac{1}{4}(a+b+c) = \tan^2 \tfrac{1}{2}s = \tan \tfrac{1}{4}E \cot \tfrac{1}{4}E_1 \cot \tfrac{1}{4}E_2 \cot \tfrac{1}{4}E_3 \ldots\ldots\ldots (96).$$

But as the four triangles are symmetrical, their expressions are so likewise, and corresponding equations exist for s_1, s_2, s_3. Whence the entire results will be

$$\left.\begin{aligned}
\tan \tfrac{1}{4}E \cot \tfrac{1}{4}E_1 \cot \tfrac{1}{4}E_2 \cot \tfrac{1}{4}E_3 &= \tan^2 \tfrac{1}{2}s \;= \tan^2 \tfrac{1}{2}s \\
\cot \tfrac{1}{4}E \tan \tfrac{1}{4}E_1 \cot \tfrac{1}{4}E_2 \cot \tfrac{1}{4}E_3 &= \tan^2 \tfrac{1}{2}s_1 = \cot^2 \tfrac{1}{2}(s-a) \\
\cot \tfrac{1}{4}E \cot \tfrac{1}{4}E_1 \tan \tfrac{1}{4}E_2 \cot \tfrac{1}{4}E_3 &= \tan^2 \tfrac{1}{2}s_2 = \cot^2 \tfrac{1}{2}(s-b) \\
\cot \tfrac{1}{4}E \cot \tfrac{1}{4}E_1 \cot \tfrac{1}{4}E_2 \tan \tfrac{1}{4}E_3 &= \tan^2 \tfrac{1}{2}s_3 = \cot^2 \tfrac{1}{2}(s-c)
\end{aligned}\right\} \ldots\ldots\ldots (97).$$

Again, multiply the reciprocals of all these together: then we get

$$\tan \tfrac{1}{4}E \tan \tfrac{1}{4}E_1 \tan \tfrac{1}{4}E_2 \tan \tfrac{1}{4}E_3 = \cot \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c) \ldots\ldots\ldots\ldots\ldots\ldots (98).$$

By means of the known values of the parts of the polar system of associated triangles, tabulated at *p.* 41, we can find in a very simple manner the corresponding parts of that system in terms of A, B, C, or of a, b, c. We thus get

$$\left.\begin{aligned}
\tan \tfrac{1}{4}E' &= \cot \tfrac{1}{2}s & \tan \tfrac{1}{4}E'_2 &= \tan \tfrac{1}{2}(s-b) \\
\tan \tfrac{1}{4}E'_1 &= \tan \tfrac{1}{2}(s-a) & \tan \tfrac{1}{4}E'_3 &= \tan \tfrac{1}{2}(s-c)
\end{aligned}\right\} \ldots\ldots\ldots (99).$$

Multiply these four together: then

$$\tan \tfrac{1}{4}E' \tan \tfrac{1}{4}E'_1 \tan \tfrac{1}{4}E'_2 \tan \tfrac{1}{4}E'_3 = \cot \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c) \ldots\ldots\ldots\ldots\ldots\ldots (100).$$

Comparing (98) and (100), we find

$$\tan \tfrac{1}{4}E' \tan \tfrac{1}{4}E'_1 \tan \tfrac{1}{4}E'_2 \tan \tfrac{1}{4}E'_3 = \tan \tfrac{1}{4}E \tan \tfrac{1}{4}E_1 \tan \tfrac{1}{4}E_2 \tan \tfrac{1}{4}E_3 \ldots\ldots (101).$$

By comparing the values of tan s, cot $(s-a)$ *etc.* in (98) and (100), we have the value of each of the polar excesses in terms of those of the primary associated system; and by interchange of primary and polar in the investigation, we get those of the primary in terms of the polar excesses. We thus have,

$$\left.\begin{aligned}
\cot^2 \tfrac{1}{4}E' &= \tan \tfrac{1}{4}E \cot \tfrac{1}{4}E_1 \cot \tfrac{1}{4}E_2 \cot \tfrac{1}{4}E_3 \\
\cot^2 \tfrac{1}{4}E'_1 &= \cot \tfrac{1}{4}E \tan \tfrac{1}{4}E_1 \cot \tfrac{1}{4}E_2 \cot \tfrac{1}{4}E_3 \\
\cot^2 \tfrac{1}{4}E'_2 &= \cot \tfrac{1}{4}E \cot \tfrac{1}{4}E_1 \tan \tfrac{1}{4}E_2 \cot \tfrac{1}{4}E_3 \\
\cot^2 \tfrac{1}{4}E'_3 &= \cot \tfrac{1}{4}E \cot \tfrac{1}{4}E_1 \cot \tfrac{1}{4}E_2 \tan \tfrac{1}{4}E_3
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (102),$$

$$\left.\begin{aligned}
\cot^2 \tfrac{1}{4}E &= \tan \tfrac{1}{4}E' \cot \tfrac{1}{4}E'_1 \cot \tfrac{1}{4}E'_2 \cot \tfrac{1}{4}E'_3 \\
\cot^2 \tfrac{1}{4}E_1 &= \cot \tfrac{1}{4}E' \tan \tfrac{1}{4}E'_1 \cot \tfrac{1}{4}E'_2 \cot \tfrac{1}{4}E'_3 \\
\cot^2 \tfrac{1}{4}E_2 &= \cot \tfrac{1}{4}E' \cot \tfrac{1}{4}E'_1 \tan \tfrac{1}{4}E'_2 \cot \tfrac{1}{4}E'_3 \\
\cot^2 \tfrac{1}{4}E_3 &= \cot \tfrac{1}{4}E' \cot \tfrac{1}{4}E'_1 \cot \tfrac{1}{4}E'_2 \tan \tfrac{1}{4}E'_3
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (103).$$

Multiplying together either of these last sets of equations, we should obtain the reciprocals of (101).

Lastly, taking the values of $\tan \tfrac{1}{4}E$, $\tan \tfrac{1}{4}E'$, *etc.* from (99), we find

$$\left.\begin{aligned}
\tan^2 \tfrac{1}{4}E \tan^2 \tfrac{1}{4}E' &= \cot \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c) \\
\tan^2 \tfrac{1}{4}E_1 \tan^2 \tfrac{1}{4}E'_1 &= \cot \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c) \\
\tan^2 \tfrac{1}{4}E_2 \tan^2 \tfrac{1}{4}E'_2 &= \cot \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c) \\
\tan^2 \tfrac{1}{4}E_3 \tan^2 \tfrac{1}{4}E'_3 &= \cot \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)
\end{aligned}\right\} \ldots\ldots (104),$$

and as the right sides of these equations are equal, the left sides are also equal, and we have

$$\tan \tfrac{1}{4}E \tan \tfrac{1}{4}E' = \tan \tfrac{1}{4}E_1 \tan \tfrac{1}{4}E'_1 = \tan \tfrac{1}{4}E_2 \tan \tfrac{1}{4}E'_2 = \tan \tfrac{1}{4}E_3 \tan \tfrac{1}{4}E'_3 \ldots (105).$$

For other properties of this nature, the student may consult my supplement to Young's Trigonometry, the Mathematical Repository, vol. vi. N. S., and the Lady's and Gentleman's Diaries, 1831-43.

THE SOLUTION OF RIGHT-ANGLED TRIANGLES.

The general principle on which the solution of right-angled triangles depends has been laid down at *pp.* 24—26, as founded on Napier's *circular parts.* The method of proceeding will be as follows:—

Mark down in a circular position the five circular parts of the right-angled triangle, which are exclusive of the right angle; viz. $\frac{1}{2}\Pi - A$, $\frac{1}{2}\Pi - B$, $\frac{1}{2}\pi - c$, a, b, as in the first figure at p. 25. Notice in this the positions of the three parts immediately concerned, viz. the two data and the part sought.

If they all three lie in succession (except the interposition of the right angle, which is rejected entirely from consideration), the middle one is the *middle* in Napier's theorem, and the two others are the "*adjacent*" parts. Call these M and A_1, A_2: then

$$\sin M = \tan A_1 \tan A_2 \dots\dots \quad (1),$$

from which any one of the three magnitudes may be found, the other two being given.

Again, if one of the parts be separated by an interval from each of the other two; then this is the *middle*, and the other two are the opposite parts. Call them M and O_1, O_2: then

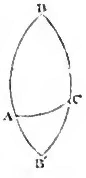
$$\sin M = \cos O_1 \cos O_2 \dots\dots \quad (2),$$

from which any one can be found when the other two are given.

Whatever two parts be given, and whatever part be required, the solution will fall under one or other of these cases.

Before we proceed to the actual exhibition of the work, we must advert to the fact, that in one case there are *two solutions*, or that the case is, as it is technically said, "*ambiguous.*" It is perfectly analogous in its general character to what was shown to take place in one case of plane triangles; viz. where the angle (or it may be, arc) is to be determined from its sine as the given result of the preceding calculation, *vol. i. p.* 451.

The only case, however, of right-angled triangles which is really *double* is *when the data are a side and its opposite angle.*



For if the perpendicular BC and hypothenuse BA be produced to meet again in A′, it is clear that BAC, B′AC have the angles at B and B′ equal, and the side AC opposite to them common: whence the data belong alike to both triangles, or *the solution is double.*

All the quæsitæ of one triangle are supplementary to those of the other, each to each.

The other cases in which a side or angle is found from the sine become definite from the two following considerations.

1. *The greater side is opposite the greater angle.*
2. *A side and its opposite angle are of the same affection* (*def.* 20).

The former of these is the *cor. to pr. x.*, and the latter may be readily established as below. It may, also, be readily proved geometrically.

In the circular arrangements the following is the order of the parts concerned, a, b, $\frac{1}{2}\Pi - A$. Whence the relation would be expressed by (1), or

$$\sin b = \tan(\tfrac{1}{2}\Pi - A) \tan a = \frac{\tan a}{\tan A}.$$

Now as b is essentially less than π, $\sin b$ is essentially $+$; and hence $\tan a$ and $\tan A$ must have the same sign. Now these are both limited between 0 and 180°; and (*vol. i. p.* 424) the signs are both $+$ when a and A are less than 90°, and both $-$ when they are between 90° and 180°. They must, therefore, to fulfil this condition, be of the same affection.

3. The hypothenuse is less than 90° when a, B, or A, b, or a, b, or A, B are of *like affection;* and it is greater than 90° when they are of *unlike affections.*

4. A leg is greater or less than 90° according as its adjacent angle and the hypothenuse, or the other leg and hypothenuse are *like* or *unlike in affection.*

5. An angle is acute or obtuse according as its adjacent leg and the hypothenuse, or the other angle and the hypothenuse are like or unlike.

EXAMPLE.

Given $a = 48° \; 24' \; 16''$ and $c = 70° \; 23' \; 42''$ to find the other parts of the triangle.

(1) To find the other side b, we have $\frac{1}{2}\pi - c$ for the *middle* between the opposites a and b; and hence

$$\sin(\tfrac{1}{2}\pi - c) = \cos a \cos b, \text{ or } \cos c = \cos a \cos b, \text{ or again, } \cos b = \cos c \sec a.$$

$$\begin{aligned} \text{Now } \cos c &= \cos 70° \; 23' \; 42'' &= 9{\cdot}5257360 \\ \text{and } \sec a &= \sec 48 \;\; 24 \;\; 16 &= \underline{10{\cdot}1779181} \\ \text{Hence } \cos b &= \cos 59 \;\; 38 \;\; 27 &= 9{\cdot}7036541 \end{aligned}$$

(2). To find the angles A, B, we may either suppose b found or not; that is, we may either avail ourselves of the previous work, or find them independently, and, as is often desirable, without any preliminary operations. It will be sufficient to show the forms taken by the equations of solution under these conditions.

In terms of a, b, c, all supposed known:—

$$\sin(\tfrac{1}{2}\Pi - A) = \tan b \tan(\tfrac{1}{2}\pi - c); \text{ that is, } \cos A = \tan b \cot c,$$

$$\sin(\tfrac{1}{2}\Pi - B) = \tan a \tan(\tfrac{1}{2}\pi - c); \text{ that is, } \cos B = \tan a \cot c.$$

In terms of a, c only, supposed to be known:—

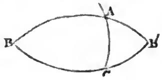
$$\sin a = \cos(\tfrac{1}{2}\Pi - A) \cos(\tfrac{1}{2}\pi - c); \text{ that is, } \sin A = \sin a \operatorname{cosec} c,$$

and, as before, $\cos B = \tan a \cot c.$

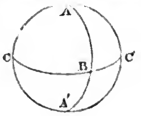
Whichever mode be employed, we find $A = 52° \; 32' \; 54''$ and $B = 66° \; 20' \; 40''$.

Scholium.

It will often happen that one or more of the data is greater than 90°; in which case some of the factors will have their signs changed. This can, however, create no real difficulty in computation, although it may give rise to oversight in the details of working; just as in Horner's method of solution of equations, the numerical developments of negative roots are more likely to give rise to mistakes than that of the positive ones. It should, therefore, in analogy to the course adopted in the resolution of equations (*vol. i. p.* 232-3), be recommended for actual calculation, the invariable employment of the supplementary triangle in the following manner.



Let ABC be given, in which either one (or both) the data b, c is greater than 90°, and complete the great circles BAB′ BCB′. Thus there is a new triangle AB′C formed, whose relation to the given one is at once obvious. Put a, b, c, A, B the sides and angles of ABC, and a', b', c', A′, B′ those of AB′C: then $b' = b$, $a' = \pi - a$, $c' = \pi - c$, $B' = B$, $A = \Pi - A$, and this triangle can be solved as above.



If the given angle be greater than 90°, the same course may be resorted to: for let ABC, having A greater than 90°, be proposed. Complete the semicircles ACA′ CBC′, ABA′ as in the figure. Then, as before, either the triangle ABC′ or A′BC′ will have its parts all less than 90°, and this triangle being resolved, the corresponding parts of the original or given one become known.

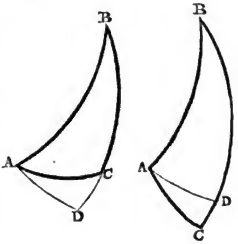
EXERCISES.

1. Given $c = 66^\circ\ 32'\ 12''$, and $a = 37^\circ\ 48'\ 12''$ to find A, B, b.
Ans. $A = 41^\circ\ 55'\ 42''$ $B = 70^\circ\ 19'\ 21''$, $b = 59^\circ\ 44'\ 24''$.
2. Given $a = 148^\circ\ 27'\ 10''$, and $c = 37^\circ\ 10'\ 20''$ to find the other parts.
Ans. $A = 59^\circ\ 59'\ 17''$, $B = 144^\circ\ 3'\ 40''$, $b = 159^\circ\ 13'\ 44''$.
3. Given $a = 98^\circ\ 20'\ 20''$, and $B = 57^\circ\ 43'\ 12''$ to find b, c, A.
Ans. $b = 57^\circ\ 26'\ 40''$, $c = 94^\circ\ 28'\ 33''$, $A = 97^\circ\ 2'\ 35''$.
4. Given $a = 104^\circ\ 12'\ 40''$, and $b = 97^\circ\ 29'\ 15''$ to find A, B, c.
Ans. $c = 88^\circ\ 10''\ 0''$, $A = 104^\circ\ 5'\ 46''$, $B = 97^\circ\ 15'\ 40''$.
5. Show that $a = 133^\circ\ 20'$, and $A = 137^\circ\ 35'$ is impossible.
6. In a right-angled triangle, $A = 60^\circ$, $B = 45^\circ$: what is a? *Ans.* 45°.
7. In a right-angled isosceles triangle the equal sides a, b are each 30°: show that the hypothenuse is $41^\circ\ 24'\ 35''$.
8. In an isosceles spherical triangle the equal sides a, b are each $95^\circ\ 30'$, and their included angle 100°: find the side c by right-angled triangles.
Ans. $99^\circ\ 22'\ 24''$.
9. If A, b be given, show that:
$\cot c = \cos A \cot b$, $\cos B = \cos b \sin A$, and $\tan a = \sin b \tan A$.
10. If b, c be given, shew that:—
$\cos A = \tan b \cot c$, $\sin B = \sin b \operatorname{cosec} c$, and $\cos a = \cos c \sec b$.

THE SOLUTION OF QUADRANTAL TRIANGLES.

The principle of the circular parts has been shown to be equally applicable in this case as in right-angled triangles, but as those parts are not identical in both cases, it is preferable to solve a right-angled triangle, the data of which can be assigned from those of the quadrantal, and the quæsita of which are also in specific and obvious relation to those of the quadrantal.

Let ABC be the quadrantal triangle, AB being the quadrant; also in BC take BD a quadrant, and join AD.



Then since BA, BD are quadrants, BDA or CDA is a right angle. The triangle ACD is, therefore, soluble by the preceding rules.

Now its parts are $CD = \frac{1}{2}\pi - a$, $AC = b$, $CAD = \frac{1}{2}\Pi - A$, and ACD, either supplemental to ACD or equal to it, according as the side BC is less or greater than a quadrant, and as represented in the two figures annexed.

The *double solution* will arise in corresponding circumstances to those of the preceding case.

EXAMPLE.

C being the angle opposite the quadrant AB, and equal to $122^\circ\ 10'\ 15''$, whilst b is $27^\circ\ 12'\ 12''$, what are the other parts of the triangle ABC?

Here by completing the quadrant BCD we have $ACD = 57^\circ\ 49'\ 45''$, and $AC = 27^\circ\ 12'\ 12''$. Whence by the right-angled triangle ACD we have

$AC = b = 43^\circ\ 59'\ 23''$; $AD = B = 36^\circ\ 0'\ 26''$; and $(\frac{1}{2}\Pi - CAD) = A = 48^\circ\ 48'\ 35''$.

EXERCISES.

1. Given $c = 90°$, $A = 54° 43'$, $B = 42° 12'$: what are the other parts?
Ans. $a = 64° 34' 40''$, $b = 48° 0' 9''$, $C = 115° 20' 5''$.

2. Given $c = 90°$, $A = 112° 2' 9''$, and $b = 67° 3' 14''$, to find the rest.
Ans. $B = 65° 27' 9''$, $a = 110° 12' 44'$, and $C = 81° 1' 58''$.

3. Given in the quadrantal triangle the two sides $22° 53' 30''$ and $51° 4' 35''$ to find the angle opposite the quadrant. *Ans.* $70° 3' 44''$.

4. In the quadrantal triangle ABC, $a = 132° 5' 40''$, and $b = 118° 9' 31''$: find the angles. *Ans.* $C = 118° 55' 4''$, $B = 129° 29' 30''$, $A = 139° 29' 40''$.

THE SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

The different classes of conditions for the complete determination of a spherical triangle, are very similar to the corresponding classes for plane triangles: and if we consider also the polar triangle in its relation to the primary one, the cases become precisely the same in number and character in spherical as in plane trigonometry, and the methods of solution have in both a striking similarity.

There are cases in spherical trigonometry, which at first sight seem to have no analogy to cases in plano; as, for instance, when the two angles and their included side, or the three angles of the triangle, are given, to find the remaining parts: but the employment of the polar triangle, transforms these into the converse; viz. given two sides and the included angle of the polar triangle,—or the three sides of the polar triangle,—to find the rest. For the sake, then, of expressing a distinction amongst the cases, which is dependent on general principles, we shall here divide the entire number of cases into three general ones, and each of these into two sub-cases. The former, viz. those where not more than one angle enters into the data, we shall call ***primary sub-cases:*** for as these have their analogies in plane trigonometry, they may be considered fundamentally analogous to cases of plane triangles: and the second, which do not contain more than one side, which become correspondingly so in the polar triangle, we shall call the ***polar sub-cases.*** They will then be arranged as below.

	Primary sub-cases.	*Polar sub-cases.*
CASE I.	Given a, b, A, to find B, C, c	Given A, B, a, to find b, c, C
CASE II.	Given a, b, C, to find A, B, c	Given A, B, c, to find a, b, C
CASE III.	Given a, b, c, to find A, B, C	Given A, B, C, to find a, b, c

CASE I.

1. *Primary sub-case.*

Given two sides, a, b, and the angle A opposite to one of them, to find the angle B opposite to the other.

First to find B: from *eq.* 1, *p.* 31, we have the requisite equations for solution,

$$\sin B = \frac{\sin b}{\sin a} \sin A = \sin b \operatorname{cosec} a \sin A.$$

But as this value of sin B may cause it to be either B or $\Pi - B$, the criterion deduced in *prop. xix.* and *Cor.* must be applied to the data. It may be stated thus :—

Cagnoli's *primary criterion of a unique solution.*

When two sides a, b, and an angle A opposite to one of them, are given to find B, then B will have only one value, whilst a is between b and $\pi - b$: and the side and its opposite angle will be of the same affection.

EXAMPLE.

Given $a = 80^\circ\ 5'\ 4''$, $b = 109^\circ\ 49'\ 30''$, and $A = 33^\circ\ 15'\ 7''$, to find B.

$$\begin{array}{rl} \sin 33^\circ\ 15'\ \ 7'' = & 9{\cdot}7390354 \\ \sin 109\ \ 49\ \ 30\ = & 9{\cdot}9734663 \\ \operatorname{cosec} 80\ \ \ 5\ \ \ 4\ = & 10{\cdot}0065361 \\ \hline \sin B = & 9{\cdot}7190378 \end{array}$$

Now a ($80^\circ\ 5'\ 4''$) lies between b ($109^\circ\ 49'\ 30''$) and $\pi - b$ ($70^\circ\ 10'\ 30''$), and therefore, by Cagnoli's *Criterion*, the solution is unique, and B is of the same affection as b, that is, greater than 90°. Whence, of the two angles expressed by means of the above value of sin B, viz. $31^\circ\ 34'\ 37{\cdot}7''$, and $148^\circ\ 25'\ 22{\cdot}3''$, the latter is the proper answer.

2. *Polar sub-case.*

Given two angles A, B, and the side a, opposite to one of them, to find the side b opposite to the other.

From the same equations, $\sin b = \dfrac{\sin B}{\sin A} \sin a = \sin a \sin B \operatorname{cosec} A.$

The *Criterion* of Cagnoli for this case also follows from *prop. xx.* and *Cor.* in the following form :—

Cagnoli's *polar criterion of a unique solution.*

When two angles A, B, and a side opposite to one of them, are given to find b; then b will have only one value, whilst A is contained between B and $\Pi - B$: and the side and its opposite angle will be of the same affection.

EXAMPLE.

Given $A = 47^\circ\ 19'\ 10''$, $B = 51^\circ\ 32'\ 15''$, and $a = 56^\circ\ 17'\ 12''$, to find b.

$$\begin{array}{rl} \sin 51^\circ\ 32'\ 15'' = & 9{\cdot}8937703 \\ \sin 56\ \ 17\ \ 12\ = & 9{\cdot}9200320 \\ \operatorname{cosec} 47\ \ 19\ \ 10\ = & 10{\cdot}1336272 \\ \hline \sin b = & 9{\cdot}9474295 \end{array}$$

Also, by Cagnoli's Criterion, the solution is not unique; and hence there are two triangles, the sides of which, b and b', are $62^\circ\ 22'\ 25{\cdot}6''$, and $117^\circ\ 37'\ 34{\cdot}4''$ respectively.

3. *To find C and c in both sub-cases.*

By Napier's Analogies, forming (14) (16) *pp.* 36, 37, we have

$$\cot \tfrac{1}{2}C = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A+B),$$

$$\tan \tfrac{1}{2}c = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b);$$

where, as we now know A, B, a, b, in both the sub-cases, we can find C and c.

Scholium.

In plane triangles the third angle C becomes known from the other two in virtue of $A + B + C = \Pi$; and the side c can, consequently, be found by the primary sub-case, as b was found. The circumstance of $3\Pi > A + B + C > \Pi$ interrupts the analogy in this respect, so far as mode of solution is concerned: and to a certain extent the same kind of interruption exists in all the *polar sub-cases*.

EXERCISES.

1. In a spherical triangle, $A = 34° 15' 2''$, $B = 42° 15' 13{\cdot}25''$, and $a = 40° 0' 10''$: find b. *Ans.* $b = 50° 10' 30''$.

2. If $a = 60°$, $b = 90°$, and $A = 60°$, what is B? *Ans.* 90°.

3. Given $a = 63° 50'$, $b = 80° 19'$, and $A = 51° 30'$, to find the rest.
Ans. $B = 59° 15' 47''$, $C = 131° 29' 46''$, $c = 120° 48' 0''$.

4. If $a = 40° 36' 37''$, $b = 91° 3' 25''$, and $A = 35° 57' 15''$, what is c?
Ans. $c = 58° 30' 56''$.

5. Given $A = 103° 59' 57{\cdot}5''$, $B = 46° 18' 7{\cdot}25''$, and $a = 42° 8' 48''$, to find C. *Ans.* $C = 36° 7' 52{\cdot}5''$.

6. Given $A = 17° 46' 16\tfrac{1}{2}''$, $B = 151° 43' 52''$, and $a = 37° 48'$, to find the other parts.

CASE II.

1. *Primary sub-case.*

Two sides a, b, and their included angle C, are given to find the remaining parts.

First method, solving the triangle completely by Napier's Analogies: which give

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}C$$

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2}C;$$

from which we find A and B: and then for c we have *either* of the other two analogies giving

$$\tan \tfrac{1}{2}c = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b), \text{ and}$$

$$\tan \tfrac{1}{2}c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b).$$

In this formula, to avoid negative quantities, we must take a for the greater angle.

EXAMPLE.

Let $a = 84° 14' 29''$, $b = 44° 13' 45''$, and $C = 36° 45' 28''$.

$$\begin{aligned} \text{Here we have } a + b &= 128\ 28\ 14 \\ \text{and } a - b &= 40\ \ 0\ 44, \\ \text{or } \tfrac{1}{2}(a + b) &= 64\ 14\ \ 7 \\ \tfrac{1}{2}(a - b) &= 20\ \ 0\ 22 \\ \tfrac{1}{2}\mathrm{C} &= 18\ 22\ 44 \end{aligned}$$

For tan ½ (A + B).	*For tan ½ (A − B)*
cot 18° 22′ 44″ = 10·4785396	cot 18° 22′ 44″ = 10·4785396
cos 20 0 22 = 9·9729689	sin 20 0 22 = 9·5341789
sec 64 14 7 = 10·3618336	cosec 64 14 7 = 10·0454745
10·87133421	10·0581930
or ½ (A + B) = 81° 15′ 44″	or ½ (A − B) = 48° 49′ 38″

For tan ½c using both the other analogies.

tan 64° 14′ 7″ = 10·3163591	tan 20° 0′ 22″ = 9·5612099
cos 81 15 44 = 9·1815947	sin 81 15 44 = 9·9949300
sec 48 49 38 = 10·1815550	cosec 48 49 38 = 10·1233620
9·6795088	9·6795019
or ½c = 25° 33′ 6·8″	or ½c = 25° 33′ 5·6″.

In these results there is a discrepancy of 1·2″: the cause is easily shown. The value of ½ (A + B) is more accurately 81° 15′ 44·4″, and the value of ½ (A − B) is taken slightly greater than the truth. Both these operate to ***increase*** tan ½c in the first, and to ***diminish*** it in the second operation: and upon performing the operations with the numbers so corrected, the discrepancy will be diminished to what will inevitably arise from the limited extent of the tables themselves.

The answers then required are very nearly

$$\mathrm{A} = 130^\circ\ 5'\ 22'',\ \mathrm{B} = 32^\circ\ 26'\ 6'', \text{ and } c = 51^\circ\ 6'\ 12''.$$

This problem, however, very often occurs in astronomical practice, where, with the same data, ***the third side only***, and not the angles A and B, is required. In this case the following method is generally considered preferable.

Second method: to find c by means of a subsidiary angle.

$$\begin{aligned} \text{By (4), } \cos c &= \cos a \cos b + \sin a \sin b \cos \mathrm{C}, \\ &= \cos a (\cos b + \tan a \sin b \cos \mathrm{C}) \end{aligned}$$

$$\text{Assume } \cot \omega \left(= \frac{\cos \omega}{\sin \omega}\right) = \tan a \cos \mathrm{C};$$

$$\begin{aligned} \text{then } \cos c &= \cos a \frac{\sin \omega \cos b + \cos \omega \sin b}{\sin \omega} \\ &= \frac{\cos a \sin (\omega + b)}{\sin \omega}. \end{aligned}$$

Let us, as an example, find c in the preceding example by this method.

tan 84° 14′ 29″ = 10·9963396	cos 84° 14′ 29″ = 9·0014632
cos 36 45 28 = 9·9037260	sin 51 24 12·5 = 9·8929404
	cosec 7 10 27·5 = 10·9034785
cot ω = 10·9000656	
or ω = 7° 10′ 27·5″	cos c = 9·7978821
b = 44 13 45	or c = 51° 6′ 20″.
ω + b = 51 24 12·5	

This differs from the other determination by 8″, probably arising from the imperfect interpolations for tan a and cosec ω; there being less dependence to be placed upon these functions for their values as they occur in this example, than upon those which occurred in the other solution *.

2. *Polar sub-case.*

Two angles A, B, and their interjacent side c are given, to find the other parts of the triangle a, b, C.

First method. This, like the primary sub-case, is soluble by Napier's Analogies, giving

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(\mathrm{A}-\mathrm{B})}{\cos \frac{1}{2}(\mathrm{A}+\mathrm{B})} \tan \tfrac{1}{2}c,$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(\mathrm{A}-\mathrm{B})}{\sin \frac{1}{2}(\mathrm{A}+\mathrm{B})} \tan \tfrac{1}{2}c;$$

from which a and b can be found separately.

Then, having A, B, a, b, the angle C and side c can be found by the formulæ given under the primary sub-case for the same purpose.

Second method. To find C the third angle without the aid of A and B.

By (*eq.* 5, *p.* 32) we have

$$\cos \mathrm{C} = \cos \mathrm{A} \,\{\tan \mathrm{A} \sin \mathrm{B} \cos c - \cos \mathrm{B}\}.$$

Put $\cot \omega \left(= \frac{\cos \omega}{\sin \omega}\right) = \tan \mathrm{A} \cos c$: then we have

$$\cos \mathrm{C} = \cos \mathrm{A} \,.\, \frac{\sin \mathrm{B} \cos \omega - \sin \omega \cos \mathrm{B}}{\sin \omega}$$

$$= \frac{\cos \mathrm{A} \sin (\mathrm{B}-\omega)}{\sin \omega} \dagger$$

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[*] The subsidiary angles may be taken in different ways from that employed above, though in none perhaps with equal simplicity and conciseness of operation. For instance, we may operate analogously to the process employed in the corresponding case of *plane triangles*, *vol. i. p.* 454 or 449. Thus:—

$$\text{Put } \tan \chi = \frac{\sin a \sin b \sin^2 \frac{1}{2}\mathrm{C}}{\sin^2 \frac{1}{2}(a-b)}; \text{ then } \sin \tfrac{1}{2}c = \sin \tfrac{1}{2}(a-b) \sec \chi;$$

$$\text{or, put } \cos^2\theta = \frac{\sin a \sin b \cos^2 \frac{1}{2}\mathrm{C}}{\sin^2 \frac{1}{2}(a+b)}; \text{ then } \sin \tfrac{1}{2}c = \sin \tfrac{1}{2}(a+b) \sin \theta;$$

$$\text{or, take } \tan \omega = \surd \tan a \tan b \cos \mathrm{C}; \text{ then } \cos c = \frac{\cos a \cos b}{\cos^2\omega};$$

$$\text{or again, take } \tan \psi = \tan a \cos \mathrm{C}; \text{ then } \cos c = \frac{\cos a \cos (b-\psi)}{\cos \psi}.$$

The two last are not, however, practicable in all cases, though they are very convenient when they admit of being used.

The investigations of these will form a useful exercise; and as such are left for the student to supply the proofs.

[†] Other methods of taking the subsidiary angle analogous to those employed in the primary case may be adopted.

$$\text{Put } \tan^2\mathrm{X} = \frac{\sin \mathrm{A} \sin \mathrm{B} \sin^2 \frac{1}{2}c}{\cos \frac{1}{2}(\mathrm{A}+\mathrm{B})}; \text{ then } \sin \tfrac{1}{2}\mathrm{C} = \cos \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) \sec \mathrm{X};$$

$$\text{or, } \cos^2\Theta = \frac{\sin \mathrm{A} \sin \mathrm{B} \cos^2 \frac{1}{2}c}{\cos \frac{1}{2}(\mathrm{A}-\mathrm{B})}; \text{ then } \sin \tfrac{1}{2}\mathrm{C} = \cos \tfrac{1}{2}(\mathrm{A}-\mathrm{B}) \sin \Theta;$$

$$\text{or, put } \cos \theta = \surd \cos a \cos b \sec a: \text{ then } \cos \mathrm{C} = \frac{\tan^2\theta}{\tan a \tan b}.$$

EXERCISES.

1. Given $A = 39^\circ 23'$, $B = 33^\circ 45' 3''$, $c = 68^\circ 46' 2''$, to find the rest.
Ans. $a = 43^\circ 37' 37''$, $b = 37^\circ 10' 1''$, $C = 120^\circ 59' 46''$.
2. Given $A = 31^\circ 34' 26''$, $B = 30^\circ 28' 12''$, $c = 70^\circ 2' 3''$, to find C.
Ans. $130^\circ 3' 11''$.
3. Given $A = 90^\circ$, $B = 60^\circ$, and $c = 90^\circ$, to find the rest.
Ans. $a = 90^\circ$, $b = 60^\circ$, $C = 90^\circ$.

CASE III.

Primary sub-case.

Given the three sides a, b, c, to find the three angles A, B, C, of the triangle.

The solution is obtained from any one of the three groups of formulæ marked (8): but the tangent is the most convenient, *when more than one angle* is required.

A still preferable method, where two or all the angles are required, is by the use of the group (9).

These methods are obviously analogous to those for the corresponding case of plane triangles, *pp.* 455, 456, of *vol. i.*: and actually become such when for the sines and tangents we substitute the arc in its ultimate state [*].

EXAMPLE.

The sides are $68^\circ 46' 2''$, $43^\circ 37' 38''$, $37^\circ 10'$: what are the angles?

$$\begin{aligned} a &= 68^\circ 46'\ 2'' \\ b &= 43\ \ 37\ \ 38 \\ c &= 37\ \ 10\ \ 0 \\ 2\,&|\,149\ \ 33\ \ 40 \\ s &= 74\ \ 46\ \ 50 \\ s-a &= 6\ \ 0\ \ 48 \\ s-b &= 31\ \ 9\ \ 12 \\ s-c &= 37\ \ 36\ \ 50 \end{aligned}$$

For $\tan \frac{1}{2}A = \sqrt{\dfrac{\tan(s-b)\tan(s-c)}{\sin s \sin(s-a)}}$

$$\begin{aligned} \sin 31^\circ\ 9'\ 12'' &= 9\cdot7137678\ \dagger \\ \sin 37\ 36\ 50 &= 9\cdot7855698 \\ \text{ac} \sin 74\ 46\ 50 &= 10\cdot0155053 \\ \text{ac} \sin 6\ \ 0\ 48 &= 10\cdot9800453 \\ 2\,&|\,20\cdot4948882 \\ \tan \tfrac{1}{2}A &= 10\cdot2474441 \end{aligned}$$

$\tan \frac{1}{2}B = \sqrt{\dfrac{\sin(s-c)\sin(s-a)}{\sin s \sin(s-b)}}$

$$\begin{aligned} \sin 37^\circ 36' 50'' &= 9\cdot7855698 \\ \sin 6\ \ 0\ 48 &= 9\cdot0199547 \\ \text{ac} \sin 74\ 46\ 50 &= 10\cdot0155053 \\ \text{ac} \sin 31\ \ 9\ 12 &= 10\cdot2862322 \\ 2\,&|\,19\cdot1072620 \\ \tan \tfrac{1}{2}B &= 9\cdot5536310 \end{aligned}$$

$\tan \frac{1}{2}C = \sqrt{\dfrac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}}$

$$\begin{aligned} \sin 6^\circ 0' 48'' &= 9\cdot0199547 \\ \sin 31\ \ 9\ 12 &= 9\cdot7137678 \\ \text{ac} \sin 74\ 46\ 50 &= 10\cdot0155053 \\ \text{ac} \sin 37\ 36\ 50 &= 10\cdot2144302 \\ 2\,&|\,18\cdot9636580 \\ \tan \tfrac{1}{2}C &= 9\cdot4818290 \end{aligned}$$

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[*] The analogy between plane and spherical trigonometry is very striking: and the simplest mode of exhibiting it appears to be, by considering the plane to be a sphere with a radius increased *ad infinitum.* In this case the sine and tangent become identical with the arc itself. See, however, the note at p. 67.

[†] In practice it will be better to take out, in a tabular form, the four sines involved in these expressions, and form their arithmetical complements as they are required—at least when they involve seconds: but if they go no further than minutes, it will perhaps be better to take the cosecants from the tables directly into the blank forms—these forms being supposed, of course, to be previously made out, as in plane trigonometry. *See vol. i. p.* 461.

Or thus, by the second theorem.

$$\tan r = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}; \tan \tfrac{1}{2}\mathrm{A} = \frac{\tan r}{\sin (s-a)}, \textit{etc.}$$

$$\begin{array}{rl}
\sin 6^\circ\ 0'\ 48'' = & 9{\cdot}0199547 \\
\sin 31\ \ 9\ \ 12 = & 9{\cdot}7137678 \\
\sin 37\ 36\ \ 50 = & 9{\cdot}7855698 \\
\text{ac} \sin 74\ 46\ \ 50 = & 10{\cdot}0155053 \\
\hline
2\,| & 38{\cdot}5347976 \\
\hline
\tan r = & 19{\cdot}2673988
\end{array}$$

$\tan \frac{1}{2}\mathrm{A} = 10{\cdot}2474441$, or $\frac{1}{2}\mathrm{A} = 60^\circ\ 29'\ 53''$, or $\mathrm{A} = 120^\circ\ 59'\ 46''$
$\tan \frac{1}{2}\mathrm{B} = \ \ 9{\cdot}5336310$, or $\frac{1}{2}\mathrm{B} = 18\ \ 51\ \ 53$, or $\mathrm{B} = 37^\circ\ 43'\ 46''$
$\tan \frac{1}{2}\mathrm{C} = \ \ 9{\cdot}4818290$, or $\frac{1}{2}\mathrm{C} = 16\ \ 52\ \ 16$, or $\mathrm{C} = 33^\circ\ 44'\ 32$.

In which the results are the same as before, and a comparison of the work in the two methods will convince the student of the greater facility, as well as the greater elegance, of the latter method.

EXERCISES.

1. Find the angles of an equilateral spherical triangle, each of whose sides is 60°. *Ans.* 70° 31′ 44″.

2. In a spherical triangle having $a = 120^\circ\ 28'\ 10''$, $b = 83^\circ\ 10'$, and $c = 96^\circ\ 50'$, find all the angles.
Ans. A = 120°, B = 86° 4′ 13′, and C = 93° 55′ 47″.

3. Given $a = 40^\circ\ 0'\ 10''$, $b = 50^\circ\ 10'\ 30''$, $c = 76^\circ\ 35'\ 36''$, to find A, B, C.
Ans. 34° 15′ 2″, 42° 15′ 13·26″, and 121° 36′ 20″.

4. The three sides of a triangle are 48° 24′ 16″, 59° 38′ 27′, and 70° 23′ 42″, what are the angles? *Ans.* 52° 32′ 54′, 66° 20′ 40″, and 90°.

5. The distances of three stars were measured by the quadrant, and found to be 76° 35′ 36″, 50° 10′ 30″, and 40° 0′ 10″: what would be the dihedral angles formed by the planes drawn through the eye and each two of the stars?
Ans. 121° 36′ 20″, 42° 15′ 14″, and 34° 15′ 3″.

Polar sub-case.

Given the three angles A, B, C, of a triangle to find the sides a, b, c *.

The solution by (10) is so similar to the preceding case in all its characters, as to be unnecessary to illustrate by even a single operation at length.

The formulæ (10, 11) are adapted to the solution of this case: but, as in the preceding one, the preference will be given to the expressions in (10) for the *tangents of the semi-angles* over those for the sines and cosines: whilst the expressions marked (11) will offer still further facility to expedition and accuracy of operation.

In all these expressions, the radical is essentially +, because $-\cos \mathrm{S}$ is $-\cos (180 + \Sigma) = +\cos \Sigma$, and all the other terms are + in themselves.

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* The sides being determinable in terms of the angles, might, at first sight, seem to constitute an essential distinction between the corresponding cases of plane and spherical trigonometry. It is, however, only in appearance: for though the arcs of the great circles are thus found, the absolute lengths of them, which depend on the linear unit, the radius of the sphere is still as undecided as in plane triangles. *We have, in fact, in both cases, only the ratios of the sides.*

EXAMPLE.

The three angles of a spherical triangle are 130° 3′ 10″, 31° 34′ 26″, and 30° 28′ 12″: find the angles by formulæ (11).

$$\begin{array}{rl}
A = & 130^\circ\ 3'\ 10'' \\
B = & 31\ \ 34\ \ 26 \\
C = & 30\ \ 28\ \ 12 \\
\hline
2\,| & 192\ \ 5\ \ 48 \\
\hline
 & 96\ \ 2\ \ 54
\end{array}$$

$$\begin{array}{rll}
ac\,(-\cos S) = & ac\,(-\cos 96\ \ 2\ \ 54) = & 10{\cdot}9772936 \\
\cos(S-A) = & \cos(-34\ \ 0\ \ 16) = & 9{\cdot}9185657 \\
\cos(S-B) = & \cos\ 64\ \ 28\ \ 28\ = & 9{\cdot}6343904 \\
\cos(S-C) = & \cos\ 65\ \ 34\ \ 42\ = & 9{\cdot}6164217 \\
\hline
 & 2\,| & 40{\cdot}1466714 \\
\hline
 & & 20{\cdot}0733357
\end{array}$$

$$\begin{array}{rll}
\cot\frac{1}{2}a = & 10{\cdot}1547700 & \text{or } a=70^\circ\ 0'\ 2'' \\
\cot\frac{1}{2}b = & 10{\cdot}4389453 & \text{or } b=40\ \ 0\ \ 1 \\
\cot\frac{1}{2}c = & 10{\cdot}4569140 & \text{or } c=38\ \ 30\ \ 2
\end{array}$$

EXERCISES.

1. Given A = 103° 59′ 57″, B = 46° 18′ 7″, C = 36° 7′ 52″, to find a.
Ans. a = 42° 8′ 48″.

2. Find the sides of the triangle whose angles A, B, C, are respectively 120° 43′ 37″, 109° 55′ 42″, and 116° 38′ 33″.
Ans. 115° 13′ 26″, 98° 21′ 40″, and 109° 50′ 22″.

3. An equiangular triangle has each of its angles 70° 31′ 44″, what is the side?
Ans. 60°.

4. What is the magnitude of the side of an equilateral triangle, each of whose angles is 90°?
Ans. 90°.

5. The three sides of a triangle are required so that the angles shall be 86° 4′ 13″, 93° 55′ 47″, and 120°. *Ans.* 83° 10′, 96° 50′, and 120° 28′ 10″.

6. The angles are 118° 55′ 4″, 113° 29′ 40″, 129° 29′ 30″: show that the triangle is quadrantal.

7. The three angles of a spherical triangle are 90°, 90°, and 60°: show that the opposite sides are also 90°, 90°, and 60°.

THE AREAS OF SPHERICAL TRIANGLES.

EXAMPLES.

(*See page* 47).

1. The angles of a spherical triangle, measured on the surface of the earth, are 83° 10′ 57″, 66° 15′ 16″, and 30° 33′ 48½″: find the spherical excess, and the area of the triangle, the earth's diameter being 7957·5 miles, and its form being taken as spherical.

Excess.	*Area.*
$A = 83^\circ\ 10'\ 57''$	$\log (3978{\cdot}75)^2 = 7{\cdot}1994934$
$B = 66\ \ 15\ \ 16$	$\log \varpi = 0{\cdot}4971499$
$C = 30\ \ 33\ \ 48\tfrac{1}{2}$	$\log 1{\cdot}5'' = 0{\cdot}1760913$
$2S = 180\ \ 0\ \ 1\tfrac{1}{2}$	$7{\cdot}8727346$
$\Pi = 180\ \ 0\ \ 0$	$\log (180.60^2) = 5{\cdot}8115750$
$E = 0\ \ 0\ \ 1\tfrac{1}{2}$	$\log 115{\cdot}1223 = 2{\cdot}0611596$

Hence the spherical excess is $1\tfrac{1}{2}''$, and the area is 115·1223 square miles.

2. If the sides of a triangle measured on the earth's surface be 60, 80, and 90 miles, what is the spherical excess?

If we denote the sides of a triangle by a, b, c, the given lengths of which are denoted by α, β, γ, we shall have

$$a = \frac{360\alpha}{7957{\cdot}5\varpi} = 0^\circ\, 51'\, 50{\cdot}4954''$$

$$b = \frac{360\beta}{7957{\cdot}5\varpi} = 1^\circ\, 9'\, 7{\cdot}3272''$$

$$c = \frac{360\gamma}{7957{\cdot}5\varpi} = 1^\circ\, 17'\, 45{\cdot}7431''$$

$$\begin{aligned} \tfrac{1}{2}s &= 49'\ 40{\cdot}891425'' \\ \tfrac{1}{2}(s-a) &= 19\ \ 14{\cdot}643725 \\ \tfrac{1}{2}(s-b) &= 15\ \ \ 7{\cdot}227825 \\ \tfrac{1}{2}(s-c) &= \ \ 6\ \ 18{\cdot}019825 \end{aligned}$$

$$\begin{aligned} \tan \tfrac{1}{2}s &= 8{\cdot}1599518 \\ \tan \tfrac{1}{2}(s-a) &= 7{\cdot}7248518 \\ \tan \tfrac{1}{2}(s-b) &= 7{\cdot}6432940 \\ \tan \tfrac{1}{2}(s-c) &= 7{\cdot}2631612 \\ 2 \,\big|\, & \overline{30{\cdot}7912588} \\ \tan \tfrac{1}{4}E &= 5{\cdot}3956294 \end{aligned}$$

Whence $E = 4 \times 5{\cdot}14'' = 20{\cdot}56''$, as nearly as the tables in ordinary use, and without the employment of second differences, enable us to determine it. This subject will, however, together with some collateral topics, be resumed in the *Geodesy*.

3. The area of a spherical equilateral triangle is one fourth of the surface of the sphere: what are its sides and angles?

Ans. Each side is 111° 28′ nearly, and each angle is 120°.

4. If Δ be the area of a spherical triangle, and Δ' that of its polar triangle, and each of the angles of the first triangle be 120°: show that

$$\tan \tfrac{1}{4}\Delta : \tan \tfrac{1}{4}\Delta' :: 6\sqrt{2} + \sqrt{3} : 2\sqrt{2} - \sqrt{3}.$$

5. The angles of a spherical triangle are in arithmetical progression, whose common difference is 45°, and the area is $\frac{7}{12}\varpi r^2$: what are the angles?

6. To divide a given spherical triangle into two others, which shall have a given ratio, by an arc drawn from one of the angles to the opposite side.

7. If E be the excess of a regular spherical polygon, and a one of its n equal sides, then the area may be found from the equation

$$\cos \frac{\Pi - \frac{1}{2}E}{n} = \cos \frac{\pi}{n} \sec \tfrac{1}{2}a.$$

8. Produce the sides AC, BC of a triangle ABC to such points D and E, so that $\tan \tfrac{1}{2}AC \tan \tfrac{1}{2}BC = \tan \tfrac{1}{2}DC \tan \tfrac{1}{2}EC$, and join DE: then the triangles ACB, DCE will have equal areas.

SOLID ANGLES.

DEFINITION.

If about the angular point of a solid angle as centre, a sphere be described to radius unity, the portion of its surface intercepted between the planes which contain the solid angle is *the measure of the solid angle* *.

As the spherical triangle or polygon varies with the spherical excess, the spherical excess will be the measure of the solid angle; and hence a solid angle contained by n planes is measured by the sum of the n dihedral angles which compose the solid angle, diminished by $(n - 2)\,\Pi$. Taking also the maximum solid angle as that measured by a hemisphere, its expression will be 2Π; and the general expression of measure in reference to this unit will be $\frac{E}{2\Pi}$.

Every class of solid angles is thus compared, even those bounded by conical surfaces, with great facility, provided the given conditions be such as will enable us to find the areas of the spherical surfaces cut off by their boundaries. We shall, however, only give a simple example or two, more for illustration than for the development of new principles or new properties.

1. Take a right prism upon a regular m-gonal base: then each solid angle is contained by three dihedral angles, two of which are right angles, and the third is equal to the angle of the polygonal base. That is, calling each of the equal solid angles, A,

$$A = \tfrac{1}{2}\Pi + \tfrac{1}{2}\Pi + \frac{m-2}{m}\Pi - \Pi = \frac{m-2}{m}\Pi.$$

Comparing this with the maximum solid angle measured by 2Π, we have

$$\frac{\text{solid angle of } m\text{-gonal right prism}}{\text{maximum solid angle}} = \frac{\frac{m-2}{m}\Pi}{2\Pi} = \frac{m-2}{2m}.$$

---

* This method of estimating the magnitudes of solid angles appears to have been first given by *Albert Girard*, in his *Invention Nouvelle en Algebre*, 1629; and would very naturally suggest itself as one of the simplest applications of his theorem for the spherical excess (*prop. xxiv. p.* 19, 20). It, however, never seems to have attracted the slightest attention till re-invented and published by Dr. Gregory in a former edition of this work. The objections which he anticipated and replied to are now obsolete, and the doctrine is in our time universally admitted; and it is, therefore, deemed unnecessary to refer to them further.

Dr. Gregory adopted the number 1000 as the numerical measure of the maximum solid angle; but in this edition it is exchanged for the more natural number 2Π, or the surface of the hemisphere in terms of the unit $\frac{2r^2\varpi}{\Pi}$.

Some continental mathematicians, as *Vincent*, for instance, have introduced this view into treatises on pure geometry. It thus ultimately reduces solid angles, through the medium of dihedral angles, to plane angles only. Of either the logic or the utility of this mode of treating the subject in reference to geometry, apart from algebra, it is not necessary here to offer any opinion.

Giving to *m* the values 3, 4, 5, *etc.* we have the ratio for right prisms on

triangular base $= \frac{1}{6} \cdot 2\Pi$
square base $= \frac{2}{8} \cdot 2\Pi$
pentagonal base $= \frac{3}{10} \cdot 2\Pi$
hexagonal base $= \frac{4}{12} \cdot 2\Pi$
heptagonal base $= \frac{5}{14} \cdot 2\Pi$
octagonal base $= \frac{6}{16} \cdot 2\Pi$
nonagonal base $= \frac{7}{18} \cdot 2\Pi$
decagonal base $= \frac{8}{20} \cdot 2\Pi$

and so on.

2. If there be two right prisms having for bases regular polygons of *m* and *n* sides respectively: then since the angles at the bases are

$$\frac{m-2}{m}\Pi \text{ and } \frac{n-2}{n}\Pi \text{ respectively,}$$

we have the ratios of the solid angles of the two prisms

$$\frac{\text{solid angle of } m\text{-gonal prism}}{\text{solid angle of } n\text{-gonal prism}} = \frac{m-2}{n-2} \cdot \frac{n}{m}.$$

3. Compare the solid angles of the five platonic or regular bodies.

Let there be *m* sides to each face, and *n* planes, which meet in each solid angle; also let A be the common dihedral angle. Then (*p.* 28),

$$\sin \tfrac{1}{2}A = \frac{\cos \frac{1}{n}\Pi}{\sin \frac{1}{m}\Pi}.$$

Giving to *m* and *n* the values already assigned to belong to the particular figures, and working out the numbers, we have

Solid angle of Tetrahedron	=	31°	35′	36″,	where $m = 3, n = 3$
.... Hexahedron	=	90	0	0	.... $m = 4, n = 3$
.... Octahedron	=	77	53	12	.... $m = 3, n = 4$
.... Dodecahedron	=	169	41	42	.... $m = 5, n = 5$
.... Icosahedron	=	150	56	55	.... $m = 3, n = 3$.

4. Let there be two regular pyramids of *m* and *n* sides, having α and β respectively for their dihedral angles: then

$$\frac{\text{vert angle of } m\text{-gonal pyramid}}{\text{vert angle of } n\text{-gonal pyramid}} = \frac{m\alpha - (m-2)\Pi}{n\beta - (n-2)\Pi} = \frac{2\Pi - m(\Pi - \alpha)}{2\Pi - n(\Pi - \beta)}.$$

5. The vertical angles of right cones are also easily compared by corresponding means.

For let the cones be generated by the angles 2α and 2β; and let their vertices be made the centres of spheres, the circular intersections of which with the cones will express the solid angles of the cones. But these surfaces are as the altitudes or versed sines of the circular sections (*vol. i. p.* 488); and hence

$$\frac{\text{angle of cone generated by } 2\alpha}{\text{angle of cone generated by } 2\beta} = \frac{\text{vers}\,\alpha}{\text{vers}\,\beta} = \frac{1 - \cos\alpha}{1 - \cos\beta}.$$

For example, let $\alpha = 30°$ and $\beta = 45°$, giving what is often called the equilateral cone and rectangular cone, from their axial sections being equilateral and right-angled triangles respectively: then

$$\frac{\text{vert angle of equilateral cone}}{\text{vert angle of right cone}} = \frac{2 - \sqrt{3}}{2 - \sqrt{2}}.$$

6. To bisect the solid angle at the vertex of a square pyramid.

First. Let a plane be drawn through the vertex and any two opposite angles of the base, that plane will bisect the solid angle at the vertex; forming two trilateral angles, each equal to half the original quadrilateral angle.

Second. Bisect either diagonal of the base, and draw *any* plane to pass through the point of bisection and the vertex of the pyramid; such plane, if it do not coincide with the former, will divide the quadrilateral solid angle into two equal quadrilateral solid angles. For this plane, produced, will bisect the great circle diagonal of the spherical parallelogram cut off by the base of the pyramid; and any great circle bisecting such diagonal also bisects the equilateral spherical parallelogram, or square. Such plane, therefore, bisects the solid angle.

EXAMPLES FOR EXERCISE.

1. Compare the solid angle of a cube with that of a regular dodecahedron.

Ans. $\dfrac{\text{sol} \angle \text{ of dodec}}{\text{sol} \angle \text{ of cube}} = \dfrac{169^\circ\ 41'\ 42''}{90^\circ\ 0'\ 0''} = 1{\cdot}8855.$

2. Determine the solid angles of a regular pyramid with hexagonal base, the altitude of the pyramid being to each side of the base as 2 to 1.

Ans. Sol $\angle$ at vertex $= 32^\circ\ 14'\ 18''$, $\angle$ at base $= 78^\circ\ 32'\ 56''$.

3. Show that if a tetrahedron and an octahedron be formed of *equal* equilateral triangles, the face of the base of the tetrahedron be applied to the face of the octahedron, the planes of their adjacent faces will form continuous planes.

4. The three diagonals of a parallelopiped are 3, 4, 5: the plane in which 3, 4 are, is inclined to the plane of 4, 5 in angle of 72°10′; the plane 4, 5 is inclined to 5, 3 in angle of 83° 4′ 20″; and the plane 5, 3 to the plane 3, 4 in angle of 63° 10′: show how many parallelopipeds can be constructed with these data; and assign the solid angles of all that can be so formed.

5. A twelve-inch, ten-inch, nine-inch, and six-inch ball all touch each the other three (in the manner of the summit of a triangular pile): what are the solid angles of the tetrahedron formed by joining their centres three and three by four planes? And what soiid angles would the twelve inch ball subtend, if seen from the centres of the other three?

6. A crystal of carbonate of lime has its solid angles contained by parallelograms whose angles are 104° 29′ and 75° 31′: and a crystal of quartz by parallelograms whose angles are 94° 4′ and 85° 56′, they both being parallelopipeds: in which is the sum of all the solid angles the greater, and how much? Show, also, whether there be any difference between the sum of all the dihedral angles (between the planes of their faces) of the one and the other, or not.

7. Three cones have their bases on a parallel of latitude of 80° N.: the one has its vertex at the centre of the earth; the others in the surface of the sphere at the poles: compare their solid vertical angles.

8. The centre of one of the covers of a copy of Hutton's tables is situated at 10 feet horizontal distance from the observer's eye, and the table upon which it rests is 2 feet below his eye; also its nearer, which is also its shorter edge, is inclined to the horizontal line from the book to the observer in an angle of 60°; now if the edges be 6, 9·7, and 1·8 inches respectively, what are the several visual angles subtended by the sides, and the solid angles subtended by the faces.

9. Find the position on the surface of a point on the earth from which a given triangle shall subtend the solid angle of 21° 10′: the latitudes of the three points being 10° 15′, 18° 10′, and 16° 15′, and the longitudes 18° 15′, 25° 18′, and 16° 18′, reckoned from Greenwich.

10. A line is drawn through the centre of a circle and at right angles to its plane; and from two points in this line which are at 10 feet distance from each other, the circle is seen to subtend $100^\circ{\cdot}25$ and $62^\circ{\cdot}5$ solid: find its diameter.

11. The "monster chimney" near Glasgow (a frustrum of a cone) is 450 feet high, and its top and bottom diameters are respectively 13½ and 42½ feet; at what distance above the middle of the shaft in the axis would the top and bottom sections subtend equal conical angles, and what would those angles be?

12. If S be the solid angle of a pyramid, each of whose dihedral angles is 120°, and S′ be the solid angle of the pyramid, each of whose faces is perpendicular to the corresponding edge of the former: show that

$$\frac{S}{S'}=\frac{6\sqrt{2}+\sqrt{3}}{6\sqrt{2}-\sqrt{3}}.$$

THE APPLICATION OF TRIGONOMETRY TO SPHERICAL GEOMETRY.

I. ANALOGIES BETWEEN PLANE AND SPHERICAL GEOMETRY.

THE attentive reader cannot fail to have remarked, that a considerable number of formulæ included amongst those which express properties of spherical triangles, have considerable similarity of general form to corresponding expressions of the properties of plane triangles; and, likewise, that in the enunciations of spherical geometrical theorems, there is a striking analogy to the enunciations of similar theorems in plane geometry [*]. Still, in respect of pure spherical geometry, in the great majority of properties which are similar to plane ones, the similarity is to be found amongst trigonometrical functions of the arcs which form the figure on the sphere; and hence a comparatively small number of such analogies admit of expression, and a still smaller number of investigation, with-

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[*] In general, whatever property of a spherical figure is expressed by an equation containing only the *sines or tangents of the sides and semi-sides* (the functions of the *angles* will not be altered); then, if we omit the functions *sin.* or *tan.*, as the case may be, we shall have a property of the corresponding plane figure expressed by the result. We may instance the spherical and plane formulæ for sin $\frac{1}{2}$A, cos $\frac{1}{2}$A, tan $\frac{1}{2}$ A, *etc.* at p. 35 of this vol. and p. 448 of vol. i. We may, also, instance the properties of the right-angled triangle at p. 28-9; and likewise most of the theorems which follow.

When *complementary functions*, as *cot.*, *cos.*, *etc.*, enter into the expression, they will always admit of transformation so as to involve only direct ones, and thus fulfil the previous condition: but though often necessary, this transformation is not always so, as may be instanced in the expression for the spherical excess at p. 47. In this case, we have the conversion of

$$\sin\tfrac{1}{2}E=\frac{\sin\frac{1}{2}a\,\sin\frac{1}{2}b}{\cos\frac{1}{2}c}\,.\,\sin C,\text{ into }\tfrac{1}{2}E=\tfrac{1}{2}a\,.\,\tfrac{1}{2}b\,.\,\sin C,$$

since from $\frac{1}{2}c$ being extremely small, its cosine is *nearly unity;* and we get as the ultimate result, the usual expression for the area of a plane triangle, of which a, b, C, are given.

The principle of this transformation is that of the evanescent arc, its sine, and its tangent all becoming equal, whilst the cosine and secant respectively become unity.

The legitimate, because safe, mode of transformation, however, adapted to these cases, is, undoubtedly, to substitute in the expressions the values of the functions *sin* θ, *cos* θ, *tan* θ, *etc.*, in terms of the *arc* θ itself. These are (*vol. i. p.* 436),

$$\sin\theta=\theta-\frac{\theta^3}{1.2.3}+\frac{\theta^5}{1.2.3.4.5}-\ldots\ldots$$

$$\cos\theta=1-\frac{\theta^2}{1.2}+\frac{\theta^4}{1.2.3.4}-\ldots\ldots$$

$$\tan\theta=\theta+\frac{2\theta^3}{1.2.3}+\frac{2^4\,\theta^5}{1.2.3.4.5}+\ldots\ldots$$

According to the relative smallness of arc θ in comparison with the entire circle, or of its actual length in comparison with $2r\varpi$, a greater or smaller number of terms of these series will be required to furnish a sufficient approximation: and when taken in this view, θ is so small as to be considered *relatively* evanescent, then all the terms except the first, of each value, may be neglected. This is, in fact, taking the radius infinite, and the spherical surface as a plane; and which agrees with the mode of transformation suggested in the commencement of this note.

out the aid of trigonometrical equations. It is proposed to give in this chapter a few of the most obvious and elementary spherical theorems, as a pattern for the student's imitation in this class of discussions. A few dependent on the right-angled triangle have already been given at p. 28—30.

PROP. I.

In any spherical triangle, $\cos A = \dfrac{-\sin^2\frac{1}{2}a \sec^2\frac{1}{2}b \sec^2\frac{1}{2}c + \tan^2\frac{1}{2}b + \tan^2\frac{1}{2}c}{2 \tan \frac{1}{2}b \tan \frac{1}{2} c}$.

For,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{(1-2\sin^2\frac{1}{2}a)-(1-2\sin^2\frac{1}{2}b)(1-2\sin^2\frac{1}{2}c)}{4 \sin\frac{1}{2}b \sin\frac{1}{2}c \cos\frac{1}{2}b \cos\frac{1}{2}c}$$
$$= \frac{-\sin^2\frac{1}{2}a + \sin^2\frac{1}{2}b \cos^2\frac{1}{2}c + \sin^2\frac{1}{2}c \cos^2\frac{1}{2}b}{2 \sin\frac{1}{2}b \sin\frac{1}{2}c \cos\frac{1}{2}b \cos\frac{1}{2}c}$$
$$= \frac{-\sin^2\frac{1}{2}a \sec^2\frac{1}{2}b \sec^2\frac{1}{2}c + \tan^2\frac{1}{2}b + \tan^2\frac{1}{2}c}{2 \tan\frac{1}{2}b \tan\frac{1}{2}c}$$

In the evanescent state $\sec\frac{1}{2}b$ and $\sec\frac{1}{2}c$ become unity, and $\sin\frac{1}{2}a = \frac{1}{2}a$, *etc.*; and hence we have ultimately

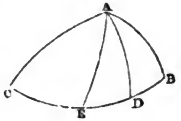
$$\cos A = \frac{-(\frac{1}{2}a)^2 + (\frac{1}{2}b)^2 + (\frac{1}{2}c)^2}{2(\frac{1}{2}b)(\frac{1}{2}c)} = \frac{-a^2 + b^2 + c^2}{2bc}.$$

The same conclusion might have been deduced from the preceding form of the value of A in sines and cosines. These forms correspond to *Euc. ii.* 12, 13[*], or *vol. i. theorems* 36, 37.

PROP. II.

Let the perpendicular AD *be drawn to the base: then*

$$\sin^2\tfrac{1}{2}AC - \sin^2\tfrac{1}{2}AB = \sin^2\tfrac{1}{2}CD - \sin^2\tfrac{1}{2}DB.$$



Denote CD, DB, by a_1 and a_2, and AD by p_1: then by the right-angled triangles, *prop. i. p.* 28, we have

$$\sin^2\tfrac{1}{2}b = \sin^2\tfrac{1}{2}a_1 \cos^2\tfrac{1}{2}p_1 + \cos^2\tfrac{1}{2}a_1 \sin^2\tfrac{1}{2}p_1$$
$$\sin^2\tfrac{1}{2}c = \sin^2\tfrac{1}{2}a_2 \cos^2\tfrac{1}{2}p_1 + \cos^2\tfrac{1}{2}a_2 \sin^2\tfrac{1}{2}p_1$$

Wherefore, by subtraction,

$$\sin^2\tfrac{1}{2}b - \sin^2\tfrac{1}{2}c = \cos^2\tfrac{1}{2}p_1 (\sin^2\tfrac{1}{2}a_1 - \sin^2\tfrac{1}{2}a_2) + \sin^2\tfrac{1}{2}p_1 (\cos^2\tfrac{1}{2}a_2 - \cos^2\tfrac{1}{2}a_1)$$
$$= (\cos^2\tfrac{1}{2}p_1 + \sin^2\tfrac{1}{2}p_1)(\sin^2\tfrac{1}{2}a_1 - \sin^2\tfrac{1}{2}a_2)$$
$$= \sin^2\tfrac{1}{2}a_1 - \sin^2\tfrac{1}{2}a_2.$$

This is the correspondent property to *vol. i. th.* 38, *p.* 311.

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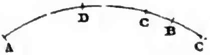
[*] The following theorems relating the various divisions of an arc analogous to those of a straight line in *Euc. ii.* are too simple to require an insertion of their investigations here.

Let the arc AB be divided in any point C, or extended to any point C′, and let it also be bisected in D. Then,



(1) $(\sin AC + \sin CB)^2 = 4 \sin^2 AD \cos^2 CD$ (*Euc. ii.* 4)
(2) $\sin^2 BD - \sin^2 DC = \sin AC \sin CB$ (*ii.* 5)
(3) $\sin^2 DC' - \sin^2 DB = \sin AC' \sin C'B$ (*ii.* 6)
(4) $\sin^2 AB + \sin^2 BC = \sin^2 AC + 2 \sin AB \sin BC$ (*ii.* 7)
(5) $\sin^2 AC + \sin^2 CB = 2 \sin^2 AD + 2 \sin^2 DC$ (*ii.* 9)
(6) $\sin^2 AC' + \sin^2 C'B = 2 \sin^2 AD + 2 \sin^2 DC'$ (*ii.* 10)

Scholium.

This might, however, have been put in a form corresponding to the plane theorem, though not so simple, without employing the semi-arcs.

For $\cos b = \cos p_1 \cos a_1$, and $\cos c = \cos p_1 \cos a_2$; whence, taking the difference of the squares,

$$\sin^2 b - \sin^2 c = \cos^2 p_1 (\sin^2 a_1 - \sin^2 a_2) = \frac{\cos b \cos c}{\cos a_1 \cos a_2} (\sin^2 a_1 - \sin^2 a_2);$$

which, like that above, gives the plane property referred to, in the evanescent state of the arcs.

The same property also applies, duly modified, as will be seen upon performing the process, where the perpendicular AD falls without the triangle.

PROP. III.

(Same figure as prop. ii.)

Draw AE *to bisect the base* BC *of the triangle: then, denoting* AE *by* l_1, *we shall have* $\sin^2 \tfrac{1}{2} b + \sin^2 \tfrac{1}{2} c = 2 \sin^2 \tfrac{1}{4} a + 2 \cos \tfrac{1}{2} a \sin^2 \tfrac{1}{2} l_1$.

$$\text{For } \cos c = \cos l_1 \cos \tfrac{1}{2} a + \sin l_1 \sin \tfrac{1}{2} a \cos \mathrm{AEB},$$
$$\cos b = \cos l_1 \cos \tfrac{1}{2} a + \sin l_1 \sin \tfrac{1}{2} a \cos \mathrm{AEC},$$
$$\text{and } \cos \mathrm{AEB} + \cos \mathrm{AEC} = 0.$$

Whence,

$$2 \cos \tfrac{1}{2} a \cos l_1 = \cos b + \cos c = 2 \cos \tfrac{1}{2} (b + c) \cos \tfrac{1}{2} (b - c)\,^*.$$

Convert these cosines into the sines of the semi-arcs: then

$$(1 - 2 \sin^2 \tfrac{1}{2} l_1)(1 - 2 \sin^2 \tfrac{1}{4} a) = 1 - \sin^2 \tfrac{1}{2} b - \sin^2 \tfrac{1}{2} c, \text{ or}$$
$$\sin^2 \tfrac{1}{2} b + \sin^2 \tfrac{1}{2} c = 2 \sin^2 \tfrac{1}{4} a + 2 (1 - 2 \sin^2 \tfrac{1}{4} a) \sin^2 \tfrac{1}{2} l_1$$
$$= 2 \sin^2 \tfrac{1}{4} a + 2 \cos \tfrac{1}{2} a \sin^2 \tfrac{1}{2} l_1.$$

This proposition is analogous to *Geom. th.* 38. *vol. i. p.* 312.

Corollary 1.

When the triangle is inscribed in a circle, the side a of which is the diameter, we shall have

$$\cos b + \cos c = 2 \cos^2 \tfrac{1}{2} a = 1 + \cos a.$$

Corollary 2.

In an equilateral or equiangular quadrilateral inscribed in a circle, we shall have

$$\sin^2 \tfrac{1}{2} a = 2 \sin^2 \tfrac{1}{2} c,$$

where a is the diagonal, and c one of the equal sides.

Corollary 3.

If there be a spherical parallelogram, that is, a quadrilateral whose opposite sides are equal, and we denote them in succession by a, b, c, d, and if h, h', be the diagonals, then

$$\cos a + \cos b + \cos c + \cos d = 4 \cos \tfrac{1}{2} h \cos \tfrac{1}{2} h'.$$

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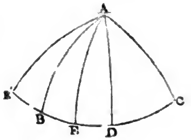
[*] This expression was communicated to me several years ago by Professor Lowry, R. M. College. It has also been given by M. Queret, (*Con. des Temps*, 1822,) and by M. Sorlin, (*Ann. des Math. tom. xv.*)

PROP. IV.



If from the vertex A *of an isosceles triangle* ABC, *the middle of whose base* BC *is* D, *a great circle* AE *be drawn to cut the base in* E, *or the base produced in* E′; *then*

$$\sin^2\tfrac{1}{2}AB - \sin^2\tfrac{1}{2}AE = \sin\tfrac{1}{2}BE\,\sin\tfrac{1}{2}EC,$$
$$\sin^2\tfrac{1}{2}AE' - \sin^2\tfrac{1}{2}AB = \sin\tfrac{1}{2}BE'\,\sin\tfrac{1}{2}E'C.$$

For *(prop. ii.)* we have in the triangles EAC, EAB, where AD is perpendicular to the bases,

$$\begin{aligned}\sin^2\tfrac{1}{2}AC - \sin^2\tfrac{1}{2}AE &= \sin^2\tfrac{1}{2}CD - \sin^2\tfrac{1}{2}DE, \text{ or}\\ &= \sin\tfrac{1}{2}(CD - DE)\sin\tfrac{1}{2}(CD + DE)\\ &= \sin\tfrac{1}{2}BE\,\sin\tfrac{1}{2}EC.\end{aligned}$$

By taking E′AC instead of EAC, the second formula is obtained.

Scholium.

This theorem is analogous to *th.* 39, *vol. i.*: but it may also be put under a different form. For,

$$\cos AB = \cos AD \cos DB, \text{ and } \cos AE = \cos AD \cos DE.$$

The difference of the squares of these being taken, we get

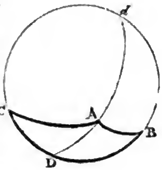
$$\begin{aligned}\sin^2 AB - \sin^2 AE &= \cos^2 AD\,(\sin^2 BD - \sin^2 DE)\\ &= \frac{\cos^2 AB}{\cos^2 BD}\sin BE\,\sin EC.\end{aligned}$$

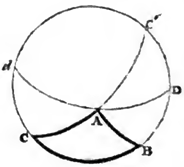
It should also be remarked, that in both formulæ the arcs AB, AC, are supposed to be less than quadrants; as when they are greater, the theorem undergoes some modification, and when they are equal to quadrants, the result is illusory or indeterminate.

PROP. V.

If either the interior or exterior angle at A *be bisected by a great circle arc* DA*d*, *which cuts the base, or base produced, in* D *and* *d*; *then*,

$$\sin BA : \sin AC :: \sin BD : \sin DC,$$
$$:: \sin Bd : \sin dC.$$



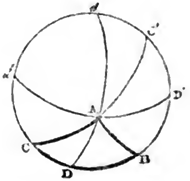


For by the properties of spherical triangles (*eq.* 1. *p.* 31) we have

sin AB : sin BD :: sin ADB : sin BAD, and sin AC : sin CD :: sin ADC : sin DAC; and since sin ADB = sin ADC, sin BAD = sin DAC, sin BD = sin B*d*, and sin DC = sin C*d*, we have at once the conclusion established.

This is analogous to *props.* 2 *and* A *of Euc. vi.*

PROP. VI.



*If d*AD *and d*′AD′ *bisect the interior and exterior angles at* A, *and be produced to meet the base* BC *in* D, *d*, *and* D′, *d*′, *respectively: then*

$$\sin CD : \sin DB :: \sin CD' : \sin D'B, \text{ and}$$
$$\sin Cd : \sin dB :: \sin Cd' : \sin d'B.$$

For, by the preceding proposition, we have

$\sin CD : \sin DB :: \sin CA : \sin AB$, and $\sin CA : \sin AB :: \sin CD' : \sin D'B$, whence $\sin CD : \sin DB :: \sin CD' : \sin D'B$.

Also, since $Cd = \pi - CD$, *etc.* the remaining terms of the stated proposition are justified.

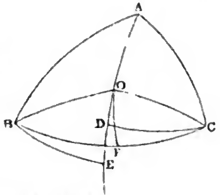
Scholium.

This is analogous to a proposition deducible from *Euc. vi.* 2 *and* A, and which is given under the *Transversals* in this volume.

PROP. VII.

If from the angles B, C *of a spherical triangle* ABC, *perpendiculars* BE, CD *be drawn to the arc* AE, *which bisects the interior angle* A, *and* F *be the point of contact of the inscribed circle: then,*

$$\sin BF \sin FC = \sin BE \sin CD.$$



Let O be the centre of the inscribed circle, and join BO, OC, OF. Then it is clear from the results at *p.* 17, that BOE = COF; and by right-angled triangles we have

$$\sin BE = \sin BO \sin BOE = \sin BO \sin COF,$$
$$\text{and } \sin CD = \sin CO \sin COD = \sin CO \sin BOF;$$

hence, $\sin BE \sin CD = \sin BO \sin CO \sin COF \sin BOF$.

Again, $\sin BF = \sin BO \sin BOF$, and $\sin CF = \sin CO \sin COF$; hence, $\sin BF \sin FC = \sin BO \sin CO \sin COF \sin BOF$; and therefore

$$\sin BF \sin FC = \sin BE \sin CD.$$

Corollary.

$$\tan^2 OF = \tan OD \tan OE.$$

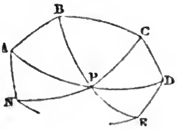
For, $\tan BO : \tan OF :: \tan OC : \tan OD$, and
$\tan OE : \tan BO :: \tan OF : \tan OC$; whence the conclusion follows.

Scholium.

Neither of the theorems corresponding to this proposition or its corollary is found in the ordinary books of elementary geometry. They are, however, familiar to most geometers, and admit of easy investigation by similar triangles.

PROP. VIII.

If from any point P *on the sphere, arcs be drawn to all the angular points* A, B, C, . . . N *of a spherical polygon: then the continued product of the sines of one set of the alternate angles made by these arcs with the sides of the polygon will be equal to the continued product of the sines of the other set of alternate angles: that is*



$$\sin ABP \sin BCP \,..\, \sin NAP = \sin BAP \sin CBP \,..\, \sin ANP.$$

For $\sin ABP \sin BP = \sin BAP \sin AP$,
$\sin BCP \sin CP = \sin CBP \sin BP$,
$\sin CDP \sin DP = \sin DCP \sin CP$,
. . . .
$\sin NAP \sin AP = \sin ANP \sin NP$.

Whence, multiplying these equations, and effacing the common terms, we get the stated result.

Corollary 1.

In the case of the triangle, the converse of this proposition is true: that, if arcs be drawn from the angles A, B, C, making angles with the sides, such that

$$\sin PAB \sin PBC \sin PCA = \sin PBA \sin PCB \sin PAC,$$

then they will meet in the same point P. This may be proved readily, *ex absurdo.*

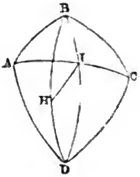
Corollary 2.

If arcs be drawn to bisect the angles of a triangle they will meet in one point; which is a particular case of the preceding corollary.

Scholium.

This is analogous to that at *p.* 450, *vol. i.* The plane theorem had been long known: but the spherical one was first given by Mr. Lowry, in 1793, in the *Math. Repos. O.S. vol. i.* Both have been very generally attributed to Carnot, who published them in 1803 in his *Geometrie de Position.*

PROP. IX.



Let AB, BC, CD, DA *be the sides of a spherical quadrilateral, and denoted by* $2a_1$, $2a_2$, $2a_3$, $2a_4$, *and its diagonals* AC, BD *by* $2d_1$, $2d_2$; *and let the distance* HI *of the middle points of the diagonals be* $2e$: *then*

$$\sin^2 a_1 + \sin^2 a_2 + \sin^2 a_3 + \sin^2 a_4 = 2(1 - \cos d_1 \cos d_2 \cos 2e)$$
$$\cos^2 a_1 + \cos^2 a_2 + \cos^2 a_3 + \cos^2 a_4 = 2(1 + \cos d_1 \cos d_2 \cos 2e)$$

For since AC, BD are bisected in I and H, we have (*prop. iii.*), if $BI = b_1$, and $ID = b_2$,

$$\cos 2a_1 + \cos 2a_2 = 2 \cos b_1 \cos d_1,$$
$$\cos 2a_3 + \cos 2a_4 = 2 \cos b_2 \cos d_1,$$
$$\text{and } \cos b_1 + \cos b_2 = 2 \cos d_2 \cos 2e.$$

Add the first and second of these, and substitute the third in the sum: then

$$\cos 2a_1 + \cos 2a_2 + \cos 2a_3 + \cos 2a_4 = 4 \cos d_1 \cos d_2 \cos 2e.$$

But $\cos 2a_1 = 1 - 2\sin^2 a_1 = 2\cos^2 a_1 - 1$, *etc.* which successively substituted in this equation gives, after slight reduction, the stated equalities.

Scholium.

Though the theorem is more elegant in its present form, there is no familiar plane theorem analogous to it. It may, however, be readily converted into another which has an analogous one in plano, in the following manner.

$$\begin{aligned} \cos d_1 \cos d_2 &= \tfrac{1}{2}\{\cos(d_1 + d_2) + \cos(d_1 - d_2)\} \\ &= 1 - \sin^2 \tfrac{1}{2}(d_1 + d_2) - \sin^2 \tfrac{1}{2}(d_1 - d_2) \\ &= 1 - 2\sin^2 \tfrac{1}{2}d_1 \cos^2 \tfrac{1}{2}d_2 - 2\sin^2 \tfrac{1}{2}d_2 \cos^2 \tfrac{1}{2}d_1. \end{aligned}$$

Hence the second side becomes

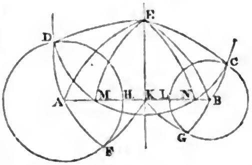
$$2\{1 - \cos 2e + 2\cos 2e(\sin^2 \tfrac{1}{2}d_1 \cos^2 \tfrac{1}{2}d_2 + \cos^2 \tfrac{1}{2}d_1 \sin^2 \tfrac{1}{2}d_2)\}$$
$$= 4\{\sin^2 e + \cos 2e(\sin^2 \tfrac{1}{2}d_1 \cos^2 \tfrac{1}{2}d_2 + \cos^2 \tfrac{1}{2}d_1 \sin^2 \tfrac{1}{2}d_2)\}.$$

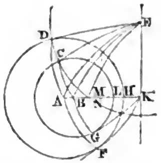
When the ultimate states of the functions are taken, this becomes

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = d_1^2 + d_2^2 + 4e^2;$$

a theorem which, slightly modified, has long been well known; but which (for what reason I know not) has been, by some writers, attributed to Euler.

PROP. X.





Let DHF, GLC *be any two less circles of the sphere, the centres of which are* A *and* B; *then if* E *be such a point that great-circle tangents* EC, ED, *drawn to the two circles be equal to one another, the point* E *will always be situated in a great circle which it is required to determine: and conversely, there is a great circle determinable, such that tangents drawn to the less circles from any point in it will be equal, each to each.*

1. From E draw the great circle EK perpendicular to that passing through A and B, and from K draw the tangents KF, KG to the circles; then join EA, AD, AF, BC, EB, BG.

Then KFA, KGB, EDA, ECB are right angles, and the angles at K are also right angles by construction. Hence

$$\cos ED \cos DA = \cos EA = \cos EK \cos KA, \text{ by trians } EDA \text{ and } EKA,$$
$$\cos EC \cos CB = \cos EB = \cos EK \cos KB, \text{ by trians } ECB \text{ and } EKB.$$

Hence by division, and that cos EC = cos ED, we have

$$\frac{\cos KA}{\cos KB} = \frac{\cos DA}{\cos CB} = \text{a constant ratio.}$$

Hence wherever E be taken to fulfil the condition of the equality of the tangents, the point K is a given point; and hence KE is in a given great circle.

2. Let the arc AB be so divided by the perpendicular great circle in K that cos AK : cos KB : : cos AD : cos BC; then tangents ED, EC drawn from any point E in it, to the two great circles, will be equal.

For, as in the preceding case, and by hypothesis, we have

$$\frac{\cos AD \cos DE}{\cos BC \ \cos CE} = \frac{\cos EK \cos KA}{\cos EK \cos KB} = \frac{\cos KA}{\cos KB} = \frac{\cos AD}{\cos BC}.$$

Hence, also, cos DE = cos CE, and DE = CE, or the tangents are equal.

Corollary 1.

If from any point E in KE a circle be described with a spherical radius equal to EC or ED, it will always cut BC in the same two points M, N equidistant from K, coalescing, however, when the circles are situated as in the second figure.

For by the right-angled triangles EDA, EKA, EKM, we have

$$\cos ED \cos DA = \cos EA = \cos EK \cos KA; \text{ and}$$
$$\cos EK \cos KM = \cos EM = \cos ED; \text{ from which two,}$$
$$\cos KM = \frac{\cos AK}{\cos AD} = \frac{\cos ED}{\cos EK}.$$

Now since AK, AD are constant and given, cos KM, and therefore KM itself is given, and M is therefore a fixed point, independent of the position of E. In like manner N is a given point.

Also, by the triangles EAK, EBK, we have, since EM = EN,

$$\cos MK \cos KE = \cos EM = \cos EN = \cos NK \cos KE,$$

and hence MK = KN, as alleged.

Corollary 2.

When the circles touch each other, H, K, L coincide: whence tangents, drawn from any point in the great-circle tangent at the point of contact of two less circles, are equal.

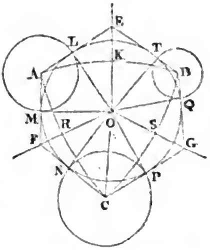
Corollary 3.

This general conclusion holds also when the two given circles cut one another, as by pursuing the preceding reasoning in its application the reader will see to be at once apparent.

Scholium.

These properties throughout have analogous ones in plano, of considerable celebrity. The circle KE corresponds to the *radical axis* of M. Steiner, in plano, and we shall designate it here by the same name. See also the geometry of the present volume, and *Ann. des Math. tom. xvii.*

PROP. XI.



If three circles (A), (B), (C)* *be described on the sphere: their radical axes will pass through the same point* O*; the tangents from* O *to the three circles will be equal; and the circle described from centre* O *with radius equal to one of the equal tangents will cut the other three at right angles.*

1. Let KE be the radical axis of (A) and (B), and RF that of (A) and (C), and let them intersect in O: then the tangents from O to (A) and (B) are equal, and the tangents from O to (A) and (C) are equal, by the preceding proposition. Hence also the tangents from O to (B) and (C), being each equal to those from O to (A), are also equal, and O is a point in the radical axis of (B) and (C). Whence the radical axes to the three circles pass through the same point O.

2. It has been also just proved that the tangents from O to the three circles are equal.

3. Since all the distances LO, MO, NO, PO, QO, TO are equal, a circle passing through one of the points of contact from centre O will pass through all. Join AL; then since OL is a tangent and AL a radius at L, they are perpendicular to one another; and the inclination of two circles on the sphere is measured by that of their great-circle tangents, and this again by that of their linear tangents. Hence (O) is at right angles to (A), and the same with the others.

Scholium.

This, also, has its analogy in plano. The point O is the *radical centre* of the three circles, (A), (B), (C). The name was given to this point by Steiner, in vol. xvii. of the *Annales des Mathematiques.* Some other plane properties connected with this point will be found in a future part of this work.

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* That is the circles whose centres are (A), (B), (C). This notation is very convenient in this class of researches, and in that relating to several spheres.

PROP. XII.

Let OS, OV, OT *be the radical axes of* (A), (B), (C) *taken two and two; and let* (D) *be any other circle on the sphere, the radical axes of* (D) *with each of the other three taken separately will meet two and two in the first three; viz. those* (A), (D) *and* (B), (D) *in that of* (A), (B), *and so on.*

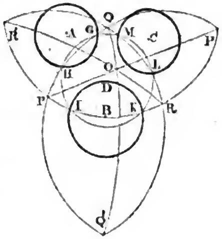
For by the preceding proposition, the radical axes OT, PQ, and QR of (A), (C), (D), taken two and two, meet in one point P; that is, the radical axes of (A), (D) and (B), (D) meet in that of (A), (B); and similarly for the others.

Corollary 1.

A very neat case of the theorem is as follows:—

If there be three circles (A), (B), (C) on the sphere, and these be cut by any number of other circles; then the chords GH, IK, ML, joining the intersections of (A), (B), (C) by (D) will two and two meet in two points Q, Q′, R, R′, P, P′; such that Q, Q′ are in the radical axis of (A), (C), R, R′ in that of (C), (B), and P, P′ in that of (A), (B).

For the chords so drawn are the radical axes of (D) and the other circles separately taken, and hence by the proposition the conclusion is justified.

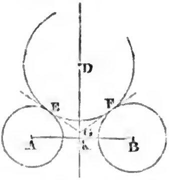


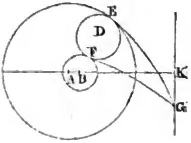
Corollary 2.

If three circles touch one another two and two, and tangents be drawn at the points of contact, these will pass through the same point, and that point will be equidistant in spherical measure from the three points of contact.

Corollary 3.

If any number of circles, as (D), be described to touch two fixed circles (A), (B), and tangents be drawn to them at the points of contact, E and F to meet in G and G′: then G, G′ will always be in the same great circle, the radical axis of (A) and (B).





For EG, FG are the radical axes of (A), (D) and (B), (D); and hence they meet in that of (A), (B).

II. THE HARMONICAL AND ANHARMONIC SECTIONS OF ARCS AND ANGLES.

DEFINITIONS. *(Fig. next page.)*

1. If an arc AB be divided any how in C and D, the three following fractions are called the *anharmonic ratios* of the arc AB; viz.

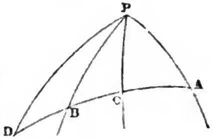
$$\frac{\sin AC \sin BD}{\sin BC \sin AD}; \quad \frac{\sin AC \sin BD}{\sin AB \sin DC}; \quad \frac{\sin AB \sin CD}{\sin AD \sin BC}.$$

2. When angle APB is divided by PC, PD, the fractions following are called the *anharmonic ratios* of the angle APB; viz.

$$\frac{\sin APC \sin BPD}{\sin BPC \sin APD}; \quad \frac{\sin APC \sin BPD}{\sin APB \sin DPC}; \quad \frac{\sin APB \sin CPD}{\sin APD \sin BPC}.$$

3. When any one of these fractions is equal to unity, the section of either arc or angle is called the ***harmonical division*** of that arc or angle.

PROP. XIII [*].



Let any four great circles, PA, PB, PC, PD, *passing through the point* P *be cut by any great circle*, AB, *as in the figure: then*

$$\frac{\sin AC \sin BD}{\sin BC \sin AD} = \frac{\sin APC \sin BPD}{\sin BPC \sin APD}.$$

For, $$\frac{\sin AP \sin APC}{\sin AC} = \sin ACP = \sin BCP = \frac{\sin BP \sin CPB}{\sin CB},$$

and $$\frac{\sin BP \sin BPD}{\sin BD} = \sin BDP = \sin ADP = \frac{\sin AP \sin APD}{\sin AD}.$$

Multiplying the extreme members of these trains of equations, and slightly changing the order of the results, we have the enunciated theorem.

Corollary 1.

In precisely the same manner we obtain the two other equations of anharmonic ratio:—

$$\frac{\sin AC \sin BD}{\sin AB \sin DC} = \frac{\sin APC \sin BPD}{\sin APB \sin DPC}, \text{ and } \frac{\sin AB \sin CD}{\sin AD \sin BC} = \frac{\sin APB \sin CPD}{\sin APD \sin BPC}.$$

The three equations thus obtained are perfectly consonant with those hereafter to be found in plano.

Corollary 2.

If the four arcs PA, PB, PC, PD be given, the section of the arc ABCD will have the same anharmonic ratios, however it may be drawn; for the ratio of the angles is in this case constant.

Corollary 3.

If four arcs, PA, PB, PC, PD, be drawn from four points A, B, C, D in a great circle to any point P whatever, the anharmonic ratios of the angles will be constantly the same wherever P be taken; for these are in all cases the same as those of the angles corresponding thereto.

Corollary 4.

If one of the anharmonic ratios be that of equality, all will be those of equality; and the section will be the ***harmonical***. From this we learn:—

(1.) That if from any harmonically divided arc, ABCD, great circles are so drawn to one point P, the ***angles*** which they form have the harmonical ratio: that is, $\sin APC : \sin CPB :: \sin APD : \sin DPB$.

(2.) That if from the harmonical points A, B, C, D of an arc, great circles be drawn through any point P, then every arc which crosses them is harmonically divided by them.

For other properties see the analogous ones in plano: and a few historical notices will also be found in the same place.

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[*] It is recommended to the student to read with this class of investigations, the corresponding ones in reference to lines situated in a plane, and in reference to lines and planes situated in space, which are given in a future part of this volume.

PROP. XIV.

Denote the parts AB, BC, BD, *of an harmonically divided arc by* κ, θ_1, *and* θ_2, *and the angles* APB, BPC, BPD, *by* K, Θ_1, Θ_2: *then,*

$$(1)\ cot\, \theta_1 - cot\, \theta_2 = 2\, cot\, 2\kappa,$$
$$(2)\ cot\, \Theta_1 - cot\, \Theta_2 = 2\, cot\, 2\mathrm{K};$$
$$(3)\ tan^2\kappa = tan\, (\kappa - \theta_1)\, tan\, (\kappa + \theta_2),$$
$$(4)\ tan^2\mathrm{K} = tan\, (\mathrm{K} - \Theta_1)\, tan\, (\mathrm{K} + \Theta_2).$$

(1.) For by the definition of harmonicals (*p.* 76, *def.* 3) we have

$$\sin (2\kappa - \theta_1) : \sin \theta_1 :: \sin (2\kappa + \theta_2) : \sin \theta_2,$$

or expanding and reducing we get

$$\sin 2\kappa \cot \theta_1 - \cos 2\kappa = \sin 2\kappa \cot \theta_2 + \cos 2\kappa, \text{ or}$$
$$\cot \theta_1 - \cot \theta_2 = 2 \cot 2\kappa.$$

(2.) Since AB is harmonically divided in C, D, the angle APB is harmonically divided (*prop. xiii. Cor.* 4.).

$$\text{or } \sin (2\,\mathrm{K} - \Theta_1) : \sin \Theta_1 :: \sin (2\mathrm{K} + \Theta_2) : \sin \Theta_2, \text{ or}$$
$$\cot \Theta_1 - \cot \Theta_2 = 2 \cot 2\mathrm{K}.$$

(3.) By composition and division of the first proportion, we get

$$\sin (2\kappa - \theta_1) + \sin \theta_1 : \sin (2\kappa - \theta_1) - \sin \theta_1 :: \sin (2\kappa + \theta_2) + \sin \theta_2 : \sin (2\kappa + \theta_2) - \sin \theta_2;$$

that is,

$$\sin \kappa \cos (\kappa - \theta_1) : \cos \kappa \sin (\kappa - \theta_1) :: \cos \kappa \sin (\kappa + \theta_2) : \sin \kappa \cos (\kappa + \theta_2)$$
$$\text{or } \tan^2\kappa = \tan (\kappa - \theta_1) \tan (\kappa + \theta_2)$$

(4.) This is obtained from (2), in the same manner, and on the same principle, that (3) was found from (1).

Scholium.

The properties corresponding to these in plano, as far as the *lines* are concerned, are well known: but those which relate to angles have not yet become familiar to English students.

The following properties are annexed without demonstration, as the proofs are sufficiently simple to be made out by the student himself. They require only the ordinary transformations of the expressions for the functions of two or of three arcs, combined with the theorem here given and the definition from which it is derived.

If the arc AB be harmonically divided in C and D, and likewise bisected in E, the segments will have the following properties :—

$$(1)\ \frac{\sin \mathrm{AD} \sin \mathrm{DB}}{\cos^2\mathrm{ED}} - \frac{\sin \mathrm{AC} \sin \mathrm{CB}}{\cos^2\mathrm{EC}} = \frac{\cos^2\mathrm{EB} \sin^2\mathrm{CD}}{\cos^2\mathrm{EC} \cos^2\mathrm{ED}}$$

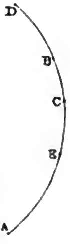
$$(2)\ \sin^2\mathrm{ED} - \sin^2\mathrm{EC} = \sin \mathrm{AD} \sin \mathrm{BC} + \sin \mathrm{AC} \sin \mathrm{BD}$$

$$(3)\ \sin^2\mathrm{ED} - \sin^2\mathrm{EC} = \frac{\sin \mathrm{AC}}{\sin \mathrm{CB}} \left\{ \sin^2\mathrm{DB} + \sin^2\mathrm{CB} \right\}$$

$$(4)\ \sin^2\mathrm{ED} - \sin^2\mathrm{EC} = \frac{\sin \mathrm{CB}}{\sin \mathrm{AC}} \left\{ \sin^2\mathrm{AD} + \sin^2\mathrm{AC} \right\}$$

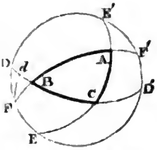
$$(5)\ \left\{ \sin^2\mathrm{ED} - \sin^2\mathrm{EC} \right\}^2 = \left\{ \sin^2\mathrm{DB} + \sin^2\mathrm{CB} \right\} \left\{ \sin^2\mathrm{DA} + \sin^2\mathrm{AC} \right\}$$

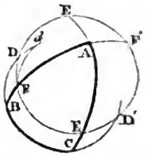
$$(6)\ \sin^2\mathrm{DB} + \sin^2\mathrm{CB} : \sin^2\mathrm{DA} + \sin^2\mathrm{AC} :: \sin^2\mathrm{BC} : \sin^2\mathrm{CA}.$$



III. SPHERICAL TRANSVERSALS.

Definition. A *spherical transversal* is any figure traced on the sphere to cut the component parts of any great circle, triangle, or polygon.





PROP. XV.

Let the sides of the triangle ABC be cut by the transversal DEF in D, E, F: then will there be this relation amongst the segments :—

$$\sin AF \sin BD \sin CE = \sin FB \sin DC \sin EA.$$

For $\frac{\sin AF}{\sin AE} = \frac{\sin E}{\sin F}$, $\frac{\sin BD}{\sin BF} = \frac{\sin F}{\sin D}$, $\frac{\sin CE}{\sin CD} = \frac{\sin D}{\sin E}$;

and multiplying these, we have

$$\frac{\sin AF \sin BD \sin CE}{\sin AE \sin BF \sin CD} = \frac{\sin E \sin F \sin D}{\sin F \sin D \sin E} = 1 \text{; and hence}$$

$$\sin AF \sin BD \sin CE = \sin FB \sin DC \sin EA.$$

Corollary.

As the transversal DEF cuts the sides a second time in D′, E′, F′, and the resulting expression is in *sines*, it will follow that for any of these segments their supplements may be substituted.

PROP. XVI.

If points D, E, F, be taken in the sides of a spherical triangle, so that either only one or else all three be taken in prolongation, and that there be the following relation amongst the segments into which the sides are divided,

$$\sin AF \sin BD \sin CE = \sin FB \sin DC \sin EA\,;$$

then will the points D, E, F, be in one great circle.

For if not, let the great circle through E and F cut the side BC in some point d different from D. Then, by the foregoing proposition and the hypothesis,

$$\frac{\sin dB}{\sin dC} = \frac{\sin FB \sin EA}{\sin AF \sin CE} = \frac{\sin DB}{\sin DC}.$$

Whence (*vol. i. p.* 426, *eq.* 17), and that $Cd - dB = CB = CD - DB$,

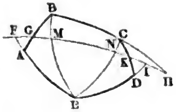
$$\frac{\tan \frac{1}{2}(Cd + dB)}{\tan \frac{1}{2}(Cd - dB)} = \frac{\tan \frac{1}{2}(CD + DB)}{\tan \frac{1}{2}(CD - DB)}, \text{ and therefore}$$

$Cd + dB = CD + DB$, which is impossible, except d coalesce with D. The three points D, E, F, are therefore in one great circle.

Corollary.

The three points D′, E′, F′, diametral to D, E, F, are also in the same great circle. For if three points, D, E, F, on the surface of the sphere be in a diametral plane, the diameters themselves, and therefore their opposite extremities, D′, E′, F′, will be in that plane.

PROP. XVII.



If a spherical polygon, as for instance the pentagon ABCDE, *be cut by a great-circle transversal in* F, G, H, K, I; *then the sides will be divided at the points of section, so that the product of the sines of all the one set of alternate segments will be equal to the product of the sines of the other set of alternate segments; that is*

sin AG *sin* BH *sin* CK *sin* DI *sin* EF = *sin* GB *sin* HC *sin* KD *sin* IE *sin* FA.

For join BE, CE, meeting the transversal in M and N. Then, by the triangles ABE, BEC, we have

$$\frac{\sin AF \sin BG}{\sin FE \sin GA} = \frac{\sin BM}{\sin ME} = \frac{\sin NC \sin HB}{\sin EN \sin CH}.$$

From this, again, and from the triangle CDE, we have

$$\frac{\sin AF \sin BG \sin CH}{\sin FE \sin GA \sin HB} = \frac{\sin NC}{\sin NE} = \frac{\sin KD \sin IE}{\sin KC \sin ID}.$$

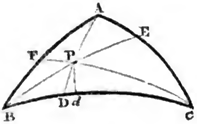
The first and last terms of which, upon cancelling the denominators, form the equation alleged in the proposition.

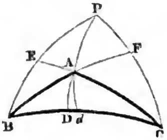
In the same manner may the demonstration be extended to a polygon of any number of sides.

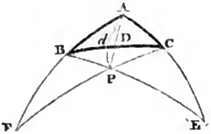
PROP. XVIII.

If through a point P on the surface of the sphere great circles be drawn from the angles A, B, C, of a spherical triangle ABC, to cut the opposite sides in D, E, F: then will

sin AF sin BD sin CE = sin FB sin DC sin EA.







For the triangle BAD cut by the transversal FPC and the triangle DAC cut by the transversal BPE, give

$$\sin AP \sin DC \sin BF = \sin PD \sin CB \sin FA,$$
$$\text{and } \sin PD \sin BC \sin EA = \sin AP \sin DB \sin CE.$$

Multiply these, and efface the common terms: then there results the stated equation.

PROP. XIX.

If points D, E, F, be taken in the sides BC, CA, AB, of a spherical triangle, so that either two or none of them be in the extension of the sides, and so as to have the relation

sin AF sin BD sin CE = sin FB sin DC sin EA;

then the arcs AD, BE, CF, will pass through the same point P on the surface of the sphere.

For if not, let two of them, BE, CF, intersect in P, and draw the great circle APd meeting BC in d. Then, by *prop. xv.* and *hyp.*

$$\frac{\sin \mathrm{B}d}{\sin d\mathrm{C}} = \frac{\sin \mathrm{BF} \sin \mathrm{AE}}{\sin \mathrm{FA} \sin \mathrm{CE}} = \frac{\sin \mathrm{BD}}{\sin \mathrm{DC}}.$$

Wherefore, reasoning as in *prop. xvi.*, we have

$$\mathrm{CD} - \mathrm{DB} = \mathrm{C}d - d\mathrm{B},$$

which is impossible, except d coalesce with D; and in this case AD, BE, CF, pass through one point, P, as affirmed.

This proposition, like its corresponding one in plano, is exceedingly efficient in the proof of theorems, which allege that three specified arcs pass through the same point on the surface of the sphere; as, for instance, those given at p. 22 for proof, by means of pure spherical geometry, and many others of a similar kind. A few are annexed in the way of corollaries.

Corollary 1.

The arcs AD, BE, CF, drawn from the angles of a triangle ABC to bisect the opposite sides in D, E, F, all pass through one point P.

For AF = FB, BD = DC, and CE = EA; whence the sines of these fulfil the criterion.

Corollary 2.

The arcs AD, BE, CF, bisecting the interior angles A, B, C, pass through one point.

For, sin AF : sin FB : : sin AC : sin CB (*prop. xvii. p.* 79),
sin BD : sin DC : : sin BA : sin AC,
sin CE : sin EA : : sin CB : sin BA;

whence, multiplying *etc.*, we have the criterion, viz.

$$\sin \mathrm{AF} \sin \mathrm{BD} \sin \mathrm{CE} = \sin \mathrm{FB} \sin \mathrm{DC} \sin \mathrm{EA}.$$

In like manner, the same may be shown, if one of the arcs AD bisect an interior angle, and the other two, BE, CF, bisect the other two exterior ones.

This, however, has been otherwise proved by another spherical general theorem.

Corollary 3.

The arcs drawn from the angles A, B, C, to the points of contact of the inscribed or escribed circle, all pass through one point.

For, in both cases, AF = AE, BF = BD, and CD = CE; whence the product of the sines of one alternate set of segments is equal to that of the other, and the criterion is fulfilled.

In like manner, referring to *fig. p.* 17, we should find that arcs AD_1, BE_2, CF_3, would pass through the same point. For

$$\begin{array}{l|l|l} \mathrm{CD}_1 = \mathrm{BD} & \mathrm{BF}_3 = \mathrm{AF} & \mathrm{AE}_2 = \mathrm{CE} \\ \mathrm{D}_1\mathrm{B} = \mathrm{DC} & \mathrm{F}_3\mathrm{A} = \mathrm{FB} & \mathrm{E}_2\mathrm{B} = \mathrm{EA}. \end{array}$$

$$\text{Whence } \sin \mathrm{CD}_1 \sin \mathrm{BF}_3 \sin \mathrm{AE}_2 = \sin \mathrm{BD} \sin \mathrm{AF} \sin \mathrm{CE},$$

$$\text{and } \sin \mathrm{D}_1\mathrm{B} \sin \mathrm{F}_3\mathrm{A} \sin \mathrm{E}_2\mathrm{B} = \sin \mathrm{DC} \sin \mathrm{FB} \sin \mathrm{EA}.$$

Now as the right sides of these equations are equal, the left are so too; and the resulting equation is the criterion which has been deduced in the proposition itself, of the great circles passing through the same point.

Corollary 4.

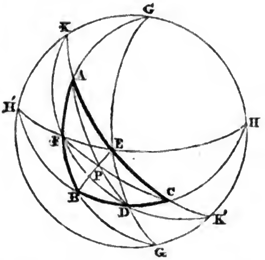
Perpendiculars from the angles A, B, C, upon the opposite sides, BC, CA, AB, meet in one point.

For denote the perpendiculars by p_1, p_2, p_3; then by right-angled triangles we have

$$\begin{array}{l|l} \sin BD = \cot B \tan p_1 & \sin DC = \cot C \tan p_1 \\ \sin CE = \cot C \tan p_2 & \sin EA = \cot A \tan p_2 \\ \sin AF = \cot A \tan p_3 & \sin FB = \cot B \tan p_3. \end{array}$$

The products of the second sides of these two sets of equations are identical, and hence those of the first sides are equal, and constitute the criterion.

PROP. XX.



If through any point P *in a great circle* AD, *drawn from the vertex* A *to cut the base of the triangle* ABC *in* D, *arcs* BPE, CPF *be drawn to cut* AC, AB, *the sides of the triangle in* E, F*: then the arc* FE *will always cut* BC *in the same two points,* H *and* H′*; and* BC *will be so divided that*

$$\sin BD : \sin DC :: \sin BH : \sin HC :: \sin BH' : \sin HC'.$$

For, by *props. xvi.* and *xv.* we have,

$$\sin BD : \sin DC :: \sin BF \sin AE : \sin FA \sin EC :: \sin BH : \sin HC.$$

Also, $\sin BH = \sin BH'$ and $\sin HC = \sin CH'$: whence the proportion is the same, wherever P be taken in the arc AD, which divides BC in a constant ratio.

PROP. XXI.

If three great circles, AD, BE, CF, *passing through the same point* P *on the sphere and the angles of the triangle* ABC, *cut the opposite sides in* D, E, F*; and if the great circles,* DE, EF, FD, *be drawn to cut the sides* AB, BC, CA *in* G, G′, H, H′, K, K′*: these last six points are in the circumference of one great circle. (See preceding figure.)*

For by the preceding proposition we have

$$\begin{aligned} \sin BD : \sin DC &:: \sin BH : \sin HC, \\ \sin CE : \sin EA &:: \sin CK : \sin KA, \\ \sin AF : \sin FB &:: \sin AG : \sin GB; \end{aligned}$$

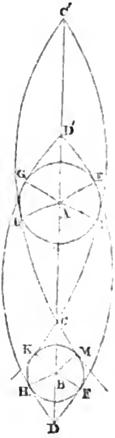
and multiplying these, in columns, the first and second columns become equal, by *prop. xvii.*; and hence the third and fourth will be equal; that is

$$\sin BH \sin CK \sin AG = \sin HC \sin KA \sin GB.$$

Also, all the sides of the triangle ABC being cut in extension, and having the relation just deduced, the three points of section G, H, K will be in one great circle, by *prop. xvi.*

Also, the points G′, H′, K′ being at the distance of a semicircle from G, H, K respectively, are at opposite extremities of the diameters through G, H, K, and as these are in one plane, the opposite points G′, H′, K′ are in that plane; and the six points being in one plane and on the surface of the sphere, they are in one great circle.

PROP. XXII.



Let two circles of the sphere (A), (B) *have common great circle tangents drawn to them, viz.* EF, GH *meeting in* D, D′, *and* KI, LM *meeting in* C, C′: *then the four points* C, C′, D, D′ *will be in the great circle* AB, *and there will exist the relations*

$$\textit{sin}\,\mathrm{AC} : \textit{sin}\,\mathrm{CB} :: \textit{sin}\,\mathrm{AD} : \textit{sin}\,\mathrm{DB},$$
$$\textit{sin}\,\mathrm{AC'} : \textit{sin}\,\mathrm{C'B} :: \textit{sin}\,\mathrm{AD'} : \textit{sin}\,\mathrm{D'B}.$$

1. For join AD, BD by great circles: then in the right-angled triangles GAD, EAD we have

$$\sin \mathrm{ADG} = \sin \mathrm{AG} \operatorname{cosec} \mathrm{AD},$$
$$\text{and } \sin \mathrm{ADE} = \sin \mathrm{AE} \operatorname{cosec} \mathrm{AD};$$
$$\text{and } \mathrm{AG} = \mathrm{AE}: \text{ whence } \mathrm{ADG} = \mathrm{ADE},$$

or AD bisects the angle EDG. In the same manner BD bisects it. Hence the point D is in AB; and it may be similarly shown that C, D′, and C′ are in AB.

2. Again, $\sin \mathrm{BF} = \sin \mathrm{BD} \sin \mathrm{BDF}$,

$$\text{and } \sin \mathrm{AE} = \sin \mathrm{AD} \sin \mathrm{ADE};$$

from which $\sin \mathrm{AE} : \sin \mathrm{BF} :: \sin \mathrm{AD} : \sin \mathrm{DB}$.

Similarly $\sin \mathrm{AI} : \sin \mathrm{BM} :: \sin \mathrm{AC} : \sin \mathrm{CB}$.

Whence, since AE = AI and BF = BM, these two give

$$\sin \mathrm{AC} : \sin \mathrm{CB} :: \sin \mathrm{AD} : \sin \mathrm{DB}.$$

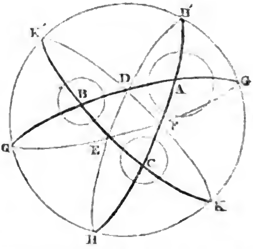
The other proportion follows in a similar manner.

Scholium.

The points C, D, C′, D′ are called the *centres of similitude,* or *poles of similitude* of the circles (A), (B).

PROP. XXIII.

If great circle tangents be drawn to three circles, two and two, the six pairs of points of intersection, or centres of similitude, will be ranged upon four great circles, three pairs upon each.



Let (A), (B), (C) be three given circles, G, G′, the points in which the exterior tangents to (A), (B) intersect, and D, D′ (D′ being on the other side of the sphere, and invisible in reference to the part of the figure exhibited) those which the interior tangents intersect. Also, let H, H′ and F, F′ be the corresponding parts in respect of (A) and (C), and K, K′ and E, E′ those in respect of (B), (C). Then the theorem asserts that there are three pairs of these intersections situated on each of the great circles GH, FE, DE, FD.

Let a, b, c denote the radii of the circles (A), (B), (C): then the triangle ABC is so cut at the points A, D, E, F, G, H, K, and their diametrals, that

$\sin \mathrm{AD} : \sin \mathrm{DG} :: \sin a : \sin b$ .... (1)
$\sin \mathrm{CF} : \sin \mathrm{FA} :: \sin c : \sin a$ .... (2)
$\sin \mathrm{BE} : \sin \mathrm{EC} :: \sin b : \sin c$ .... (3)
$\sin \mathrm{AG} : \sin \mathrm{GB} :: \sin a : \sin b$ .... (4)
$\sin \mathrm{CH} : \sin \mathrm{HA} :: \sin c : \sin a$ .... (5)
$\sin \mathrm{BK} : \sin \mathrm{KC} :: \sin b : \sin c$ .... (6).

Combining these in triads so as to have the third and fourth terms equal, viz. (1, 2, 6), (1, 3, 5), (2, 3, 4), and (4, 5, 6), we shall have, by equating the first and second terms of the results,

$$\begin{aligned}\sin AD \sin CF \sin BK &= \sin DG \sin FA \sin KC,\\ \sin AD \sin BE \sin CH &= \sin DG \sin EC \sin HA,\\ \sin CF \sin BE \sin AG &= \sin FA \sin EC \sin GB,\\ \sin AG \sin CH \sin BK &= \sin GB \sin HA \sin KC.\end{aligned}$$

The first of these shows that K, F, D, and hence their diametrals K′, F′ D′, are in one great circle, the criterion of *prop. xvi.* being fulfilled. In like manner H, E, D and their diametrals H′, E′, D′; G, E, F, and their diametrals G′, E′, F′; and K, G, H, with their diametrals K′, G′, H′, are respectively in great circles of the sphere.

EXERCISES IN SPHERICAL GEOMETRY.

(To be effected by the aid of trigonometry.)

1. If from a fixed point on the surface of the sphere, arcs be drawn to cut a given less circle, each in two points; and if these four points be joined two and two by arcs, and produced (if necessary) to meet: then the points of intersection will always be in the same great circle, which is perpendicular to the arc drawn from the given point to the centre of the given circle.

2. If two spherical quadrilaterals be one inscribed in a circle, and the other circumscribed about it, the latter having as points of contact the angular points of the former: then the two pairs of diagonals of these quadrilaterals will intersect in the same point.

3. If the pairs of opposite sides of the two quadrilaterals defined in the last exercise be produced to meet, each pair in two points; then the eight points of intersection will be situated in one great circle of the sphere.

4. If the opposite sides of a spherical hexagon inscribed in a circle be produced to meet, each opposite pair in two points: then the six points of section will be in one great circle.

5. If great circle arcs be drawn to join the opposite angular points of a spherical hexagon circumscribed about a circle, they will all three pass through the same two points, diametrally situated with respect to each other on the sphere.

6. Let ABCD be a quadrilateral situated on the sphere; produce AB, CD, to meet in E, E′, and AD, BC, to meet in F, F′, and draw the circle EFE′F′; let P, P′, be the intersections of AC, and BD, Q, Q′, those of BD and EF, and R, R′, those of EF and AC; draw also EP and FP to cut the opposite sides AD, BC, and AB, CD: then will every side AB, BC, CD, DA, and each diagonal AC, BD, EF, be harmonically divided at the points of mutual section of these several great circles.

7. The triangle ABC is cut by a transversal circle, AB in D and E, BC in F and G, and CA in H and K: it is required to show that

$$\frac{\tan \tfrac{1}{2}AD \tan \tfrac{1}{2}BF \tan \tfrac{1}{2}CH \tan \tfrac{1}{2}AE \tan \tfrac{1}{2}BG \tan \tfrac{1}{2}CK}{\tan \tfrac{1}{2}DB \tan \tfrac{1}{2}FC \tan \tfrac{1}{2}HA \tan \tfrac{1}{2}EB \tan \tfrac{1}{2}GC \tan \tfrac{1}{2}KA} = 1;$$

and to generalise the theorem by the substitution of any spherical polygon for the triangle ABC.

8. In *fig. prop. viii. p.* 71, denote the angles at A, B, C, *etc.*, by α, β, γ, *etc.*; the angles NAP, ABP, BCP, *etc.*, by α_1, β_1, γ_1, *etc.*, and PAB, PBC, PCD, *etc.*, by α_2, β_2, γ_2, *etc.*: then,

$$\frac{\cot \alpha_1 - \cot \alpha}{\cot \alpha_2 - \cot \alpha} \cdot \frac{\cot \beta_1 - \cot \beta}{\cot \beta_2 - \cot \beta} \cdot \frac{\cot \gamma_1 - \cot \gamma}{\cot \gamma_2 - \cot \gamma} \cdots \frac{\cot \nu_1 - \cot \nu}{\cot \nu_2 - \cot \nu} = 1.$$

SPHERICAL ASTRONOMY.

Observations and experiments of various kinds show that the earth differs very little in its figure from a sphere; and hence the principal inquiries and calculations of practical astronomy are conducted on the hypothesis of its being accurately spherical. Moreover, since, to an observer on the earth, the heavens appear as a large concave sphere (all the fixed stars being so remote that the diameter of the earth would subtend at them only an insensible angle), every part of which is equidistant from the observer, it has been found expedient, both for the purposes of reasoning and calculation, to imagine various circles to be described upon the earth, and their planes extended to intersect the celestial sphere in corresponding and similarly situated circles. The observer is supposed to be situated at the centre of this celestial sphere, and therefore that the planes of all of these circles pass through his eye. There are also certain points and circles thus conceived to exist on the celestial sphere which have no correspondent ones on the terrestrial, some of which are permanent, and others variable with the observer's position on the earth's surface. The principal points, lines, and circles, which enter into this department of astronomy, are described in the following

DEFINITIONS.

1. The *axis of the earth* is a line passing through the centre, and terminating at the points which are quiescent during its daily revolution.

2. These quiescent points are the *poles of the earth*, and are called the *north pole* and the *south pole*.

3. That great circle of the earth, the plane of which is perpendicular to the axis (or, the great circle of which those points are the poles), is called the *equator*. It divides the earth into two *hemispheres*, the *northern* and the *southern*, so called according as they contain the north or south poles.

4. Secondaries to the equator are called the *meridians* of the places through which they pass.

5. *Terrestrial longitude* of a place is the arc of the equator intercepted between the intersection of the meridian of that place with the equator, and some fixed point in the equator. On the earth it is usually reckoned east and west from the initial point 0° to 180° east and 180° west. The longitude estimated one way or the other is called *east longitude* and *west longitude* *.

6. The *latitude* of a place on the surface of the earth is its distance from the equator reckoned upon the meridian of the place. It is *north latitude* or *south latitude*, according as the place is situated in the northern or the southern hemisphere.

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* Different countries estimate longitude from different initial points, viz. from the points in which the meridians of their national observatories respectively intersect the equator. Thus, the English and Americans estimate from the meridian of the Royal Observatory of Greenwich, the French from the Observatory of Paris, the Germans from that of Berlin, the Russians from that of Petersburgh, and the Danes from that of Copenhagen. A very simple computation serves to reduce from one origin to any other, as the relative longitudes of all the principal observatories have been carefully determined.

7. A *parallel of latitude* is a less circle of the sphere, passing through any point of the earth which has a specified latitude, parallel to the equator, and cutting off equal arcs from all the meridians.

8. The *horizon* of a place, or the *sensible horizon*, is the indefinitely extended tangent plane to the earth at that point.

9. The axis of the earth being conceived to be prolonged to meet the celestial sphere, intersects it in the *celestial poles*, or poles *of the heavens*. This prolonged axis is sometimes called the *axis of the heavens;* and the elevated pole is denoted by P in all illustrative figures.

10. The terrestrial equator being produced to meet the celestial sphere, cuts it in a circle called the *celestial equator*, or the *equinoctial*.

11. The meridian plane of a place being similarly extended to meet the celestial sphere, gives the *celestial meridian* of that place.

12. A plane through the centre of the earth parallel to the sensible horizon cuts the celestial sphere in the *rational horizon*.

13. The smallness of the earth's radius, in comparison with the distances of the fixed stars, causes the introduction of no sensible error into the calculations or reasonings to arise from our considering the rational and sensible horizon to be coincident. In the case of the sun, moon, and planets, all the error introduced can be rendered a subject of separate computation, and the result of the hypothesis of coincidence be thence corrected. This correction is called the *parallax*.

14. The poles of the rational horizon are called the *zenith* and the *nadir*, the former the pole above the observer's head, and the latter that beneath him.

15. Circles drawn secondary to the horizon are called *vertical circles*. The *altitudes* of celestial bodies from the horizon are estimated upon them; they are also called *azimuth circles*, or *amplitude circles*, for a reason explained in definitions 17 and 18.

16. That vertical circle which is at right angles to the meridian of the place of observation is called the *prime vertical*. It cuts the horizon in its eastern and western points.

17. The *azimuth* of a celestial body is the angle formed by a vertical circle and the meridian of the observer. It is measured by the arc of the horizon intercepted between the meridian and the vertical circles.

18. The *amplitude* of a celestial object is the angle formed by the vertical circle passing through it and the prime vertical. It is measured by the arc of the horizon intercepted between the east or west point and the vertical circle.

19. The circle of the horizon being divided into four quadrants by the meridian and its prime vertical, each of these quadrants is further sub-divided into eight smaller portions, the points of division of which constitute what are usually termed the *points of the compass*. These, however, are never used in astronomy, properly so called; but they are extremely useful in describing approximately the azimuth or amplitude of an object. For this purpose, they are extensively used in navigation and nautical astronomy.

20. *Almacantars*, or *parallels of altitude*, are less circles of the sphere, the planes of which are parallel to the horizon. All the points of any one of them are at equal altitudes above the horizon.

21. The real motion of the earth round the sun in a year, gives to the sun an apparent motion round the earth in the same time. The apparent path of the sun is in one plane, and the real motion of the earth, therefore, is in the same plane. This plane cuts the celestial sphere in a circle called the *ecliptic*. This plane differs from the equinoctial, and the angle which they make with each

other is called the *inclination of the ecliptic*, or the *obliquity of the ecliptic*. Its elevated pole is marked P′ in the figures.

22. The points in which the ecliptic and equator intersect, are the *equinoxes;* and one of them is technically called the *first point of Aries*, or *vernal equinox*, the other being the *autumnal equinox*.

23. A secondary to the equinoctial through any celestial object cuts the equator in a point, the arc between which and the first of Aries, is called the *right ascension* of that object. It is reckoned entirely round the circle of the equinoctial from 0° to 360°. There is also another mode of estimation, as in the next definition.

24. These secondaries themselves are called *hour lines*, since they become the meridians of any given place at equal successive times, arising from the uniform rate of the earth's rotation on its axis. The angle which any two of them form at the pole, or the dihedral angle of their planes, is called the *hour angle* of those secondaries, or of the objects through which they pass.

25. This uniform rotation has caused it to become much more general to express the *right ascension in time* than in degrees. The right ascension in time is, however, convertible into degrees at the rate of 15° to 1 hour.

26. The complement of the declination is the *polar distance*, and is north or south, according to the pole reckoned from. It is also called the *co-declination*.

27. The distance of the object from the equinoctial, estimated on the secondary, is the *declination;* and is north or south, according as the point is situated in the celestial hemisphere, which contains the north or south celestial pole, the division being made by the equinoctial.

28. A secondary to the ecliptic through any celestial body cuts the ecliptic in a point, the arc between which and the first of Aries is called the *longitude* of the object. It is usually by astronomers reckoned entirely round the ecliptic from 0° to 360°.

29. The distance of the object from the equinoctial, estimated on the secondary, is called the *latitude* of the object; and is *north* or *south*, according as the point is situated in the celestial hemisphere, which contains the north or south celestial pole made by the ecliptic. The complement of this is called the *colatitude*.

30. The parallels to the ecliptic and equinoctial are respectively called *parallels of celestial latitude* and *parallels of declination*.

31. The parallels of declination which touch the ecliptic are called the *tropics;* of which that in the northern equatorial hemisphere is the *tropic of Cancer*, and the other the *tropic of Capricorn*. The points of contact are called the *summer, and the winter, solstice* respectively.

32. The two secondaries to the equinoctial which pass through the solstices, and the equinoxes, are called respectively the *solstitial colure* and the *equinoctial colure*.

33. That portion of the circle of declination passing through any celestial object which is above the horizon, is the *diurnal arc* of that object, and the part below the horizon its *nocturnal arc*.

34. Those bodies whose parallels of declination do not cut the horizon at all, are called *circumpolar stars*, and at their passage of the meridian of a place, have their *greatest and least altitudes*.

35. The parallels of declination which pass through the poles of the ecliptic are called the *polar circles*, the *arctic* in the northern and the *antarctic* in the southern equatorial hemispheres respectively.

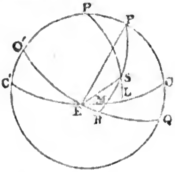
36. The secondaries to the ecliptic and equator which pass through any celestial object, form an angle which is called the *angle of position*: and the celestial body being marked S, the triangle PSP′ is called the *first astronomical triangle.*

37. The triangle ZPS formed of the arcs joining the zenith, the pole of the equinoctial, and a star is the *second astronomical triangle.*

PROBLEM I.

Of the six following magnitudes, taken in reference to a celestial object, any three being given to find the other three: viz.

i = *obliquity of the ecliptic*
a = *right ascension*
d = *declination*
p = *angle of position*
l = *latitude*
λ = *longitude.*



Let S be the position on the sphere in reference to the equator EQ and ecliptic EC of the celestial body; draw the perpendicular arcs SR, SL to EQ and EC, and produce them to P and P′ till RP and LP′ become quadrants; also describe the great circle through P, P′ to meet EC, EQ in C′, Q′, C, Q, and draw EP.

Then by the definitions we have ER the right ascension, RS the declination, EL the longitude, and LS the latitude of S; and RSL or PSP′ is the angle of position, and CEQ the obliquity of the ecliptic. From the structure of the figure, too, it will be obvious that each of the six parts of the *first astronomical triangle* P′SP (*def.* 36) is a function of a single and different one of the six parts enunciated in the proposition: viz.

$$PP' = CEQ = i \quad | \quad PS = \tfrac{1}{2}\pi - d \quad | \quad SP'P = LC = \tfrac{1}{2}\pi - \lambda$$
$$P'SP = RSL = p \quad | \quad P'S = \tfrac{1}{2}\pi - l \quad | \quad SPP' = RQ' = \tfrac{1}{2}\pi + a.$$

Now of these six parts of the triangle P′SP, any three being given, the other three can be found by the rules already given for the solution of spherical triangles. The following general formulæ, expressive of the connection of the parts, with recapitulations of particular and the most usual processes of solution, may, however, be advantageously annexed.

From (*p.* 32, *eq.* 4.) we have, merely putting changed letters for A, B, C, *a*, *b*, *c*.

$$\cos P'S = \cos P'P \cos PS + \sin P'P \sin PS \cos P'PS$$
$$\cos PS = \cos P'P \cos P'S + \sin P'P \sin P'S \cos PP'S,$$

in which inserting the values above shown to belong to these, we have,

$$\sin l = \cos i \sin d - \sin i \cos d \sin a \, \ldots\ldots\ldots\ldots \, (1)$$
$$\sin d = \cos i \sin l + \sin i \cos l \sin \lambda \, \ldots\ldots\ldots\ldots \, (2).$$

Again, from (*p.* 33, *eq.* 6.) we get, employing the same set of values,

$$\cot PP'S \sin P'PS = \cot SP \sin PP' - \cos P'PS \cos P'P,$$
$$\cot P'PS \sin PP'S = \cot SP' \sin PP' - \cos PP'S \cos PP'; \text{ or}$$
$$\tan \lambda = \tan a \cos i + \tan d \sin i \sec a \, \ldots\ldots\ldots\ldots\ldots\ldots \, (3)$$
$$\tan a = \tan \lambda \cos i - \tan l \sin i \sec \lambda \, \ldots\ldots\ldots\ldots\ldots\ldots \, (4)$$

The most frequent data are those which include the obliquity with either the right ascension and declination, or the latitude and longitude; whichever pair be given the other pair to be found. The ordinary mode of effecting it by a subsidiary angle is, to assume θ and ω, so that

$$\tan \theta = \frac{\sin a}{\tan d}, \text{ and } \tan \omega = \frac{\sin \lambda}{\tan l}: \text{ then}$$

$$\sin l = \frac{\sin d \cos(\theta + i)}{\cos \theta} \ldots (a) \qquad \sin d = \frac{\sin l \cos(\omega - i)}{\cos \omega} \ldots (c)$$

$$\tan \lambda = \frac{\tan a \sin(\theta + i)}{\sin \theta} \ldots (b) \qquad \tan a = \frac{\tan \lambda \sin(\omega - i)}{\sin \omega} \ldots (d)$$

Again, in nearly the same manner that (3, 4) were obtained, we may find the expressions for p in i, a, d, or in i, λ, l; viz.

$$\cot p = \sin d \tan a + \cos d \sec a \cot i \ldots\ldots (5)$$

$$\cos p = \sin \lambda \tan l + \cos \lambda \sec l \cot i \ldots\ldots (6)$$

and their solutions effected by a subsidiary angle as before.

These, however, may be superseded by the following ones:—

$$\sin p = \frac{\sin i \cos a}{\cos l} \ldots (e) \qquad \sin p = \frac{\sin i \cos \lambda}{\cos d} \ldots (f)$$

which are the immediate results of (*p.* 31, *eq.* 1.) applied to PSP′.

In the same manner we also get (or by right-angled triangles),

$$\cos a \cos d = \cos \lambda \cos l \ldots\ldots (7)$$

which becomes, when as practically the case for the sine, $\cos l = 1$,

$$\cos a = \cos \lambda \sec d \ldots\ldots (8)$$

Also, by the right-angled triangles EMR, SML, we have

$$\tan a = \tan \lambda . \cos i \ldots (9) \qquad \sin d = \sin \lambda \sin i \ldots (11)$$

$$\cos \lambda = \cos a \cos d \ldots (10) \qquad \sin \lambda = \sin d \operatorname{cosec} i \ldots (12)$$

The cases of the problem arising from every combination of the six conditions in four at a time, are contained in the preceding equations.

Scholium.

It is to be remarked, that the figure and the corresponding signs of the functions involved in the expressions above given, are so framed as to be adapted to the first quadrant of right ascension and longitude: when the point S has these in quadrants different from the first, there will be corresponding changes of sign in the values of the expressions; and the principles laid down at *p.* 424, *vol. i.* will always apply to the determination of arcs or angles from these, or of these from the arcs or angles.

EXAMPLES.

1. The right ascension of the star Aldebaran being 67° 40′ 30″, and its declination 16° 8′ 20″ N.: and it is required to find the latitude, longitude, and the angle of position, the obliquity of the ecliptic being 23° 27′ 49″.

Ans. lat = 5° 40′ 18″ S., long. = 68° 29′ 28″, ang. pos. = 8° 44′ 26″.

2. It is required to find the latitude and longitude of the moon when her right ascension is 304° 21′, and her declination 22° 57′ S.

Ans. lat. = 8° 8′ 46″ S., long. = 301° 21′ 54″.

3. When the longitude of the moon is 1ˢ 7° 41′ 23″, and her latitude 3° 49′ 57″ S., what are her right ascension and declination?

Ans. R. Asc. = 36° 36′, decl. = 10° 28′ N. nearly.

4. When the sun's declination is 19° 18′ 20″ N., what are his right ascension and longitude? *Ans.* R. Asc. = 53° 5′ 6″, long. = 55° 25′ 43″.

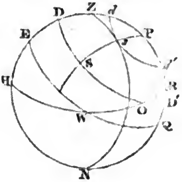
5. Given the sun's longitude 6ˢ 8° 9′ 36″, to find the right ascension and declination. *Ans.* R. Asc. = $12^h\ 29^m\ 58^s$, decl. = 3° 14′ 24″ S.

6. The latitude of the moon's centre being 4° 0′ 34″ N., her longitude 7ˢ 14° 26′ 21″, and the obliquity of the ecliptic 23° 27′ 48″: to find her right ascension and declination.

Ans. R. Asc. = 223° 11′ 11″, decl. = 12° 21′ 14″ N.

PROBLEM II.

To find the expressions which connect the risings and settings, azimuths, and altitudes of the heavenly bodies with the latitude of the place of observation.



Let HR be the horizon, and Z the zenith; P the elevated pole of the heavens, EQ the equator cutting the horizon in W. Let S be the heavenly body: then PS is its polar distance, or complement of the declination SX; ZS is the zenith distance or complement of the altitude SV; VR (or the angle SZP) is the azimuth, or complement of the amplitude VW; PR is the altitude of the pole above the horizon, DO the semi-diurnal arc, and OD′ the semi-nocturnal arc; and finally, ZPS is the hour angle from the meridian.

Besides the notation of the last proposition, put

$VS = H =$ altitude
$ZS = \zeta =$ zen. dist.
$VR = A =$ azimuth
$SP = \delta =$ polar dist.

$SPZ = P =$ hour ∠ from mer.
$EZ = L =$ lat. of place
$OD = D =$ semi-diurnal arc
$OD' = D' =$ semi-nocturnal arc.

1. *The latitude.* Since E is in the equator, and P is its pole, $EP = \frac{1}{2}\pi$; and since Z is the pole of the horizon $RZ = \frac{1}{2}\pi$: wherefore,

$$EZ + ZP = EP = \tfrac{1}{2}\pi = RZ = RP + PZ, \text{ or } EZ = RP: \text{ that is,}$$

the latitude of the place is equal to the elevation of the pole above the horizon.

Again, let the parallel of declination through S meet the meridian in D and D′. Then, in the case of D′ being *below* the horizon, we have $PD' = PD$, and

$$EZ = RP = RD - DP = RD - PS = \textit{meridian alt.} - \textit{co-declination}$$
$$= \textit{meridian altitude} + \textit{declination} - \textit{a quadrant}.$$

Or in the case *dsd′*, where both *d* and *d′* are above the horizon, or the star is circum-polar, we have $Pd = Pd'$, and

$$EZ = RP = \tfrac{1}{2}(RP + Pd) + \tfrac{1}{2}(RP - Pd') = \tfrac{1}{2}(Rd + Rd')$$
$$= \textit{half the sum of the meridian altitudes of the object.}$$

Or again,

$$EZ = ED + DZ = \textit{declination} + \textit{zenith distance on the meridian.}$$

2. *The azimuth and altitude on the six-hour circle.* In the triangle ZPS we have

$$\cos ZS = \cos PS \cos PZ + \sin PS \sin PZ \cos ZPS, \text{ or}$$
$$\cos \zeta = \sin d \sin L + \cos d \cos L \cos P \quad \ldots\ldots\ldots\ldots\ldots\ldots (13),$$

which when the six-hour circle is taken, gives $P = \frac{1}{2}\Pi$, and hence

$$\cos \textit{zen. dist.} = \cos \zeta\ (= \sin H) = \sin d \sin L \quad \ldots\ldots\ldots\ldots (14).$$

In the right-angled triangle SZP, which, when S is on the six-hour circle, the triangle becomes, we have

$$\cot SZP = \cot PS \sin PZ, \text{ or } \cot \textit{azim.} = \cot A = \tan d \cos L \ \ldots\ldots (15).$$

3. *The hour-angle and the azimuth of a body on the horizon.* In this case, $\zeta = \frac{1}{2}\pi$, and we have equation (13) converted into

$$\cos \textit{hour angle} = \cos P = -\tan d \tan L \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (16).$$

This theorem serves for finding the time of rising and setting (abating the effects of refraction and parallax) of the heavenly bodies, whether the sun, stars, or planets.

It will be remarked, that if d and L be both of the same name (N. or S.), the angle P will be greater than 90°, and if of different names, less than 90°. That is, in the former case the semi-diurnal arc corresponds to more, and in the latter to less, than six hours.

Again, in the quadrantal triangle, which ZSP becomes when S is in the horizon, we have

$$\cos PZS \sin PZ = \cos PS, \text{ or } \cos \textit{azim.} = \cos A = \sin d \sec L \;\ldots\ldots \;(17).$$

4. *The hour-angle and altitude of a body on the prime vertical.* In this case $PZS = \frac{1}{2}\pi$, or the triangle PZS is right-angled: whence, by Napier's circular parts, we have

$$\cos \textit{hour ang} = \cos P = \tan ZP \cot SP = \cot L \tan d \ldots\ldots\ldots\ldots (18)$$
$$\cos SZ \cos PZ = \cos PS, \text{ or } \sin H = \sin d \operatorname{cosec} L \;\ldots\ldots\ldots\ldots (19).$$

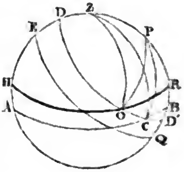
5. *The hour-angle, zenith distance, and azimuth of a body, at the instant of its entire motion being vertical to the horizon.* In this case, the motion in dsd' must coincide with that of the vertical circle ZS′, which touches dsd' in S. Draw Ps': then Ps' being the spherical radius of the circle dsd', and Zs' a tangent at s', the angle ZS′P is a right one, and the triangle Zs'P is right-angled. Hence by the rules for the circular parts, we have

$$\cos P = \tan L \cot d \ldots\ldots\ldots\ldots (20)$$
$$\sin H = \sin L \operatorname{cosec} d \ldots\ldots\ldots\ldots (21)$$
$$\sin A = \sec L \cos d \ldots\ldots\ldots\ldots (22).$$

Scholium 1.

It will be at once obvious, that as the whole of the motion of the star is vertical, its vertical rate will be greater at the point s' than in any other on the hemisphere containing the semi arc $ds'd'$.

Scholium 2.



The observations made regarding the refractive power of the atmosphere upon the rays of light passing through it, have led to the conclusion that the crepusculum or twilight continues during the time that the sun takes in traversing the arc OC from the horizon to the parallel AB, situated 18° of vertical distance below it. We can compute the angles OPR, CPR, viz. P and P′, for we have ZS = 108°, and the rest as in the preceding problem: then the time of passing over this angular space, at 15° to the hour, will give, as a computing formula,

$$t = \tfrac{1}{15}(P - P').$$

EXAMPLES.

1. To a place having north latitude 52° 12′ 35″, when does the sun rise and set, and what are the rising and setting azimuths, the sun's declination being 16° 0′ 46″, no account of its daily variation or of refraction being taken?

Ans. time from noon = $7^h\ 26^m\ 53\frac{1}{3}^s$, azim = 63° 14′ 44″ from north.

2. Abating refraction, what time does the sun rise at the equator on the 20th of July and the 5th of September, 1843? *Ans.* at 6 o'clock on *all* days.

3. When does the sun rise due east at Woolwich, its latitude being 51° 28′ 29″?
Ans. at the equinoxes.

4. At Cambridge, in lat. 52° 12′ 35″, the sun's declination is 15° 54′ 25″, and his altitude 40°: what is his azimuth, and the time of observation from noon?
Ans. azim. = 119° 53′ 8″ from north, time from noon = $2^h\ 55^m$.

5. Find the azimuth, at 9 in the morning, of the last question.

6. The declination of the star *a*, *Ursa Majoris*, was known to be on a certain day 62° 50′, at what time was his apparent motion vertical to a person situated at N. lat. 52° 32′; and what were his azimuth and altitude at the time?
Ans. time = $3^h\ 11^m\ 51^s$, the azim. 48° 38′ 28″ from N., alt. 63° 8′ 29″.

7. At what time on the 15th of September, 1842, did twilight begin and end at Woolwich?

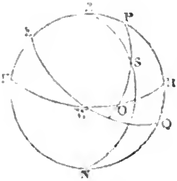
8. In lat. 51° 32′ N. the sun's declination on the shortest day being taken as 23° 28′ S., it is required to find the time of daybreak and the end of evening twilight.

PROBLEM III.

Of these six things, any three being given, to find the other three:—

κ = *colat. of place*
δ = *codecl. of body*
ζ = *zenith dist.*

V = *angle of variation*
A = *azimuth*
P = *horary angle.*



The sides and angles of the *second astronomical triangle* constitute the several things given and sought in this problem: viz.

$$ZP = \kappa,\ PS = \delta,\ SZ = \zeta;$$

$$\text{and } PSZ = V,\ SZP = A,\ ZPS = P:$$

whence, any three of these being given to find the other three, is an applied statement of the general problem of spherical trigonometry; and the solution has been given of all the cases that can arise, with all desirable minuteness, in the former part of this volume, that of the second general case, which required a subsidiary angle, being given in a variety of ways. There can be no doubt, however, that it would contribute to convenience, were the entire solutions of all the cases tabulated in reference to the *second astronomical triangle (def.* 37*)*, and the notation selected to express its several parts. Dr. Pearson has given nearly one-half of such table, at *p.* 459, *vol. ii. of his Practical Astronomy;* but as he only inserted as much as could be put in one single page, it may be useful to the student to work out the entire system for convenient reference.

EXAMPLES.

1. The zenith distance of the sun's centre on the 20th of September, 1828, was deduced from the requisite correction of observations made upon it, to be 58° 53′ 3·57″, the colatitude of the place of observation 38° 58′ 22″, and the correct polar distance of the sun, corrected from the Nautical Almanac, was 58° 56′ 45″: it is required to find the time?

In this we have, *Sph. Trig. p.* 35,

$$\tan \tfrac{1}{2} \mathrm{P} = \sqrt{\frac{\sin \frac{1}{2}(\zeta + \delta - \kappa) \sin \frac{1}{2}(\zeta - \delta + \kappa)}{\sin \frac{1}{2}(-\zeta + \delta + \kappa) \sin \frac{1}{2}(\zeta + \delta + \kappa)}}, \text{ or}$$

$$\sin \tfrac{1}{2} \mathrm{P} = \sqrt{\frac{\sin \frac{1}{2}(\zeta + \delta - \kappa) \sin \frac{1}{2}(\zeta - \delta + \kappa)}{\sin \delta \sin \kappa}}.$$

When the time only, without either of the other angles being required, the second is considered to be the simpler form. From either formula,

$$\mathrm{P} = 36^\circ\ 9'\ 4'' = 2^{\mathrm{h}}\ 24^{\mathrm{m}}\ 36{\cdot}3^{\mathrm{s}}.$$

2. The sun's declination being 19° 39′ N., and his altitude when on the prime vertical 25° 26′: it is required to find the latitude of the place of observation, and the time. *Ans.* lat. = 51° 32′ N. time from noon = $4^{\mathrm{h}}\ 54^{\mathrm{m}}\ 5^{\mathrm{s}}$.

3. In lat. 51° 32′ the sun's altitude, when on the prime vertical, was 25° 26′: what was his declination, and the time?

Ans. dec. = 19° 39′ N. time from noon = $4^{\mathrm{h}}\ 54^{\mathrm{m}}\ 5^{\mathrm{s}}$.

4. At what time was the star Regulus due east of Greenwich (lat. 51°28′40″N.) on the 20th of November, 1796, its right ascension and declination then being $4^{\mathrm{h}}\ 24^{\mathrm{m}}\ 14^{\mathrm{s}}$, and 16° 5′ 19″ N.? and what was its altitude at the time?

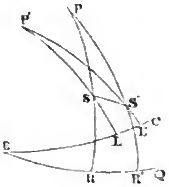
Ans. East at $7^{\mathrm{h}}\ 28^{\mathrm{m}}\ 13^{\mathrm{s}}$, altitude = 16° 39′ 24″.

5. In latitude 51° 32′ N. the sun's true altitude was found to be 46° 20′ when his declination was 23° 28′ N. Find his azimuth, and the time of observation.

Ans. azim. = 113° 3′ from N., time from noon = $2^{\mathrm{h}}\ 55^{\mathrm{m}}\ 20^{\mathrm{s}}$.

PROBLEM IV.

Given the right ascensions and declinations, or the latitudes and longitudes of two celestial bodies, to find their distance.



Let S, S_1 be the bodies EQ, EC, the equator and ecliptic: then there are given

$$\left.\begin{aligned} \mathrm{EL} &= \lambda,\ \mathrm{EL}' = \lambda_1, \\ \mathrm{LS} &= l,\ \mathrm{L'S'} = l_1 \end{aligned}\right\}$$

$$\text{or } \left.\begin{aligned} \mathrm{ER} &= a,\ \mathrm{ER}' = a_1 \\ \mathrm{RS} &= d,\ \mathrm{R'S'} = d_1 \end{aligned}\right\}$$

to find the distance SS′.

Here we obviously have in the former case in the triangle SP′S′,

$$\mathrm{SP'} = \tfrac{1}{2}\pi - l,\ \mathrm{P'S'} = \tfrac{1}{2}\pi - l_1, \text{ and } \mathrm{SP'S'} = \lambda_1 - \lambda;$$

that is, two sides and the included angle, to find the base, which is the *primary sub-case of case ii. Sph. Trig. p.* 57.

In the latter case we have

$$\mathrm{SP} = \tfrac{1}{2}\pi - d,\ \mathrm{PS'} = \tfrac{1}{2}\pi - d_1, \text{ and } \mathrm{SPS}_1 = a_1 - a;$$

viz. two sides, and the included angle, to find the base, as before.

Corollary.

If λ and l be considered as terrestrial longitude and latitude, the same process applies to finding the distance of the two places, S, S′, on the earth's surface, in degrees. This may be converted into English miles, at the rate of 69½ miles to one degree.

EXAMPLES.

1. It is required to find the distance between the stars Capella and Procyon,

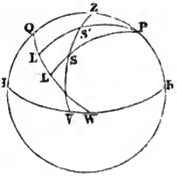
the right ascensions being respectively 75° 21′ 19″, and 112° 6′ 47″, and their declinations 45° 46′ 15″, and 5° 45′ 3″, both north. *Ans* 51° 6′ 56″.

2. The latitudes of two stars are 22° 51′ 57″ N., and 15° 58′ 14″ S; and their longitudes 2ˢ 18° 57′ 57″, and 15° 58′ 14″ S.: what is their distance? *Ans.* 51° 36′ 39″.

3. Supposing the Paris observatory to be in lat. 48° 50′ 14″ N., and that of Pekin to be in lat. 39° 54′ 13″ N., and their difference of longitudes to be 114° 7′ 30″: how distant are they? *Ans.* 73° 56′ 40″, or 5139 miles.

PROBLEM V.

Two stars, S, S′, whose right ascensions and declinations are known, are observed to be on the same known vertical angle at the same time: it is required to determine the latitude of the place of observation.



LET S, S′ be the stars, and ZV the vertical on which they are situated. Then, producing PS, PS′ to meet the equator QW in L and L′, we have given $LS = d$, $L'S' = d_1$, and the angle SPS′ is measured by $LL' = a_1 - a$. Whence the two sides $\frac{1}{2}\pi - d$, $\frac{1}{2}\pi - d_1$, and included angle $a_1 - a$ of the triangle SPS′, are given. Hence SS′ is also known, together with the angles S and S′, being *primary sub-case ii. Sph. Trig. p.* 57.

Again, the vertical ZV being known in position (its azimuth), we have given the angles Z, S (or S′), and side SP (or S′P): from which ZP, the colatitude of the place can be found by *case i. p.* 56.

Scholium.

The problem admits of many variations amongst the data; but there must always be one (and one only) which has reference to the observer's position on the earth. They are so obvious, however, in their trigonometrical character, as to require no specific enumeration here *.

EXAMPLE.

Two stars, the right ascensions of which are 78° 24′, and 104° 52′, and declinations 27° 25′, and 28° 30′, either both N. or both S., are distant from each other 28° 30′ (either by calculation as in *prob. iv.* or observation), are observed to be on a vertical whose azimuth QZV is 73° 36′: what is the latitude of the place? *Ans.* 32° 23′ 18″ of the same name as that of the stars.

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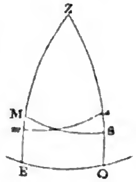
* It must be apparent, that the ordinary problems that occur for solution are only direct and simple applications of the rules of spherical trigonometry; and hence, that the few preceding examples are quite sufficient for the student's exercise.

Whatever be the conditions of the question proposed for solution, a rude sketch of the figure which represents these conditions in connexion with the quæsitum, will always indicate the trigonometrical character of the problem. It will, however, very much conduce to the student's comprehension of the subject, to read, in connexion with what is here given, some treatise on descriptive astronomy. Sir John Herschel's, in the Cabinet Cyclopædia; Dr. Brinkley's Treatise on Astronomy; or Dr. Gregory's Lessons, may be mentioned as well adapted to the purpose.

PROBLEM VI.

Given the apparent altitudes of the sun and moon, or of the moon and a star, together with their distance, and likewise granted, that knowing the apparent altitudes we can find the true; it is required thence to find the true distance.*



LET s and m be the apparent places of the sun and moon, Z the zenith of the observer, ms the apparent distance, SM the true places of the sun and moon, situated in the same vertical circles Zs, Zm. Then it is required to find the true distance MS.

In this there are given, mZ, Zs, the complements of the altitudes mE, sQ, and the base ms, from which the angle Z can be obtained. Then, having the corrected sides ZS, MZ, and the included angle Z, we can find the base MS, or true distance sought by *primary sub-case ii. Sph. Trig. p.* 57.

This problem, however, being that involved in one of the best and most generally practicable methods of determining the longitude of a place on the earth's surface, numerous attempts have been made by astronomers to shorten and simplify the processes, by combining both the operations spoken of above in one only. Three methods, concluding with that of Borda, are given here; but in a foot note on next page, references are made to numerous others, and in the *Journal of the Royal Institution, no. ix.*, will be found a comparison of the relative advantages of a considerable number of them.

Put h, h_1, the apparent, and H, H_1, the true altitudes; d the apparent, and D the true distance.

Then because Z is common to the two triangles MZS, mZs, we have

$$\frac{\cos D - \sin H \sin H_1}{\cos H \cos H_1} = \cos Z = \frac{\cos d - \sin h \sin h_1}{\cos h \cos h_1}.$$

Resolving which for cos D, we get

$$\cos D = \frac{(\cos d - \sin h \sin h_1) \cos H \cos H_1}{\cos h \cos h_1} + \sin H \sin H_1$$

$$= \frac{\{\cos d + \cos (h + h_1) - \cos h \cos h_1\} \cos H \cos H_1}{\cos h \cos h_1} + \sin H \sin H_1$$

$$= \frac{2 \cos \tfrac{1}{2} (h + h_1 + d) \cos \tfrac{1}{2} (h + h_1 - d) \cos H \cos H_1}{\cos h \cos h_1} - \cos (H + H_1)$$

$$= \left\{ \frac{2 \cos \tfrac{1}{2} (h + h_1 + d) \cos \tfrac{1}{2} (h + h_1 - d) \cos H \cos H_1}{\cos h \cos h_1 \cos (H + H_1)} - 1 \right\} \cos (H + H_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Put $\dfrac{\cos \frac{1}{2} (h + h_1 + d) \cos \frac{1}{2} (h + h_1 - d) \cos H \cos H_1}{\cos h \cos h_1 \cos (H + H_1)} = \cos^2 F$: then

$$\cos D = (2 \cos^2 F - 1) \cos (H + H_1) = \cos 2F \cos (H + H_1) \quad \ldots\ldots (2).$$

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* The apparent altitude is not the true one for two reasons, which have been called parallax and refraction: the former arising from our observations being made at the surface of the earth instead of the centre, causing the body to appear *lower* than it is, and the latter from the atmosphere refracting the rays of light out of straight lines, causing the body to appear higher than it is. The moon is *depressed* by their conjoint action—all other bodies *elevated*, but always in vertical circles. These will be explained under *Practical Astronomy*.

Again, subtract each side of (1) from unity: then,

$$\sin^2\tfrac{1}{2}D = \cos^2\tfrac{1}{2}(H + H_1) - \frac{\cos\frac{1}{2}(h+h_1+d)\cos\frac{1}{2}(h+h_1-d)\cos H\cos H_1}{\cos h\cos h_1}$$

$$= \cos^2\tfrac{1}{2}(H + H_1)\left\{1 - \frac{\cos\frac{1}{2}(h+h_1+d)\cos\frac{1}{2}(h+h_1-d)\cos H\cos H_1}{\cos h\cos h_1\cos^2\frac{1}{2}(H+H_1)}\right\}$$

Put $\sin^2\theta = \dfrac{\cos\frac{1}{2}(h + h_1 + d)\cos\frac{1}{2}(h + h_1 - d)\cos H\cos H_1}{\cos h\cos h_1\cos^2\frac{1}{2}(H + H_1)}$;

then we get *Borda's theorem* *, viz.:—

$$\sin^2\tfrac{1}{2}D = \cos^2\tfrac{1}{2}(H + H_1)\cos^2\theta, \text{ or } \sin\tfrac{1}{2}D = \cos\tfrac{1}{2}(H + H_1)\cos\theta \dots\dots \quad (3)$$

Scholium.

This method of investigating Borda's theorem is taken from Young's Trigonometry, and is one of the most elegant that has been given. Professor Young, however, considers that, for the purposes of actual computation, the formula (1) is preferable to Borda's; and I believe he is right in considering it to be original.

EXAMPLES.

1. Suppose the apparent distance between the centres of the sun and moon to be 83° 57′ 33″, the apparent altitude of the moon's centre 27° 34′ 5″, that of the

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* The following are the authorities referred to in the introduction to this solution:—

App. Requisite Tables	Maskelyne
Selections, Naut. Alm. (p. 151) 1813	Lyons
App. Req. Tab	Witchell
Selections from N. Alm. (p. 35)	Dunthorne
Intr. to Spherics, 1805	Kelly
Naut. Alm. 1819, 1825	Brinkley
Comp. Coll. of Tables, 1805	Mendoza Rios
Con. des Temps, 1775	Borda
Additions to Biot's Astr. 1811	Rossel
Journ. Science, ix. 362	Adams
Tables for correcting App. Dist. 1772	Shepherd
Nova Acta Petrop. 1791	Krafft.
Linear Long. Tables and Horary Tables, 1790	Margetts
On the Long. 1810	Mackay
Navigation, 1821	Inman
Longitude in Rees's Cycl.	Turner
Ditto ditto	Garrard
Brande's Journ. iii. N. S.	Wiseman
Lunar and Horary Tables	Thompson
Naut. Tables, 1825; Horary Tables, 1828	Lynn
Tables to be used with Naut. Alm., and Naut. Alm. 1829	Lax
Brande's Journ. xix. 117	Blackburne
Astronomy, vol. ii. 612	Pearson
Lunar Tables	Mrs. Taylor.
Lat. and Long. Computer	Beverley.
A privately printed paper	Gordon.
Naut. Astr., Encyc. Metrop	Kater.

Also, Con. des Temps, 1775, 1785, 1796, 1807, 1808.

The first twelve methods have been compared by Dr. Young, in vol. ix. Quart. Journ. p. 350—371. In point of simplicity the method of Beverley is unquestionably superior to all others.

sun's centre to be 48° 27′ 32″, and the true altitudes of their centres to be respectively 28° 20′ 48″, and 48° 26′ 49″: what is the true distance of their centres? *Ans.* 83° 20′ 54″.

2. Given the apparent and true altitudes of the moon's centre, 8° 26′ 13″ and 9° 20′ 45″, those of a star 35° 40′ and 35° 38′ 49″, and their apparent distance 31° 13′ 26″, to find the true distance. *Ans.* 30° 23′ 56″.

3. The apparent distance of the moon's centre from the star Regulus being 63° 35′ 13″ when the apparent altitude of the moon's centre was 28° 29′ 44″, and of the star 45° 9′ 12″; the moon's correction for parallax and refraction 48′ 1″ (additive), and the star's for refraction 57″ (subtractive): required the true distance. *Ans.* 63° 5′ 8″.

4. The apparent distance of the moon's centre and a star was 2° 20′ when the apparent altitudes of the moon's centre and of a star were 9° 39′ and 11° 14′, and their corrections respectively 51′ 30″ and 4′ 40″: how far were they really apart? *Ans.* 1° 49′ nearly.

5. On the 14th of March, 1818, the apparent altitudes of the centres of the sun and moon were observed to be 48° 55′ 49″ and 24° 43′ 37″, and the measured distance of their centres to be 86° 54′ 25″; also the corrected zenith distances of the centres of the sun and moon were 41° 4′ 55″ and 64° 29′ 7″: what was the true distance of the centres? *Ans.* 86° 16′ 44·2″.

MISCELLANEOUS EXERCISES IN SPHERICAL ASTRONOMY.

1. GIVEN the obliquity of the ecliptic 23° 27′ 42″, and the right ascension of the sun 10° 39′ 40″, to find the sun's longitude, declination, and angle of position. *Ans.* 11° 35′ 49″, 4° 35′ 26″, and 66° 58′ 1″.

2. Given the colatitude of the place 62° 12′, and the zenith distance at 6 o'clock of the sun 71° 39′ 37″: find the sun's declination and azimuth.
Ans. 20° 50′ 12″, and 79° 56′ 4″ from the north.

3. Given the colatitude of the place of observation 37° 47′ 18″, the codeclination of the sun 74° 51′ 50″, and the sun's zenith distance 50° 44′ 32″, to find the time from noon, the azimuth, and the angle of variation.
Ans. time $2^h\,56^m\,45^s$, azimuth 119° 55′ 6″, variation 32° 22′ 45″.

4. In what north latitude will the shortest day be three-fifths of the longest? *Ans.* 41° 23′ 7″.

5. At what time in the month of May, when the sun's declination is 20° 16′ 32″, will the shadow of a perpendicular object be equal to its length?
Ans. $9^h\,13^m\,16^s$, A.M.

6. In what latitude, on the 1st of June[*], will the sun's altitude, when due east, be double his altitude at 6 o'clock? *Ans.* 49° 6′ S.

7. On the 24th of May, in lat. 50° 12′ N., required the time it will take the body of the sun to rise out of the horizon. *Ans.* $3^m\,58^s$.

8. In N. lat. at $11^h\,10^m$, and at $12^h\,40^m$ by chronometer, the corrected altitude

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[*] It is to be understood that the information to be obtained from the Nautical Almanac is within reach of the pupil; as in this example the sun's declination on the 1st of June; in the next, the declination and semi-diameter; and so on. That given for the nearest time in the almanack to that in the question is also in these to be taken as correct, disregarding the slight changes those quantities might have undergone in the elapsed time. These corrections will be given in a future part of this volume.

of the sun's centre was 26° 55′, and his declination 5° 17′ S; required the latitude of the place of observation. *Ans.* 57° 9′ N.

9. The north latitudes of three places are, of Moscow (M) 55° 30′, of Vienna (V.) 48° 12′, and of Gibraltar (G) 35° 30′, and they all lie in the same great circle; the difference of longitude of the first and second is equal to that of the second and third: find the true bearing of each place from the other two, their distances and differences of longitude.

Ans. common diff. long. = 14° 13′ 27″;
V and G bear from M, S. 56 4 W.
V from M and G .... N. 35 16 E.
M from V .... N. 44 50 W.
G from V .... S. 44 50 E.

Distances.
VG = 16° 29′ = 1146 Eng. miles.
VM = 11 23 = 791 ....
MG = 27 52 = 1937 ....

10. Required the latitude of the place and declination of the sun when the length of the day is to that of the night as 3 to 2, and the sun's mid-day altitude to his midnight depression as 2 to 1.

Ans. lat. = 61° 57′ N. dec. = 9° 21′ N.

11. In N. lat. 51° 30′ I observed the sun to rise on one point of the compass, and eleven hours after my shadow was cast towards the same point: what was the sun's declination? *Ans.* 9° 54′ 41″ N.

12. To what height must a person be raised above St. Paul's on the 21st of June, to just see the sun's upper limb, the earth's radius being 3980 miles, neglecting the effect of refraction? *Ans.* 155·4283 miles.

13. In the latitudes of 30° and 50° N. on the same meridian, on the 21st of June (forenoon) the sun's altitude will be equal at a particular instant: when is that, and what is this altitude?

Ans. time from noon = $3^h\ 55^m\ 16^s$, alt. = 37° 37′ 17″

14. On what day of the year, and in what latitude, does the sun rise at half-past 5, and appear due east at 10?

Ans. N. or S. lat. = 21° 13′, and N. or = S. dec. 18° 35′.

15. Where is the sun's altitude at 6 o'clock on the longest day equal to the colatitude? *Ans.* lat. = 65° 17′ 12″.

16. Let ζ_1 ζ_2 be the observed azimuths of a star, ρ_1 ρ_2, the observed zenith distances, at two different times of observation: then l, the latitude of the place of observation, is computible from

$$\tan l = -\frac{\sin \rho_1 \cos \zeta_1 - \sin \rho_2 \cos \zeta_2}{\cos \rho_1 - \cos \rho_2}.$$

17. The hour angles from the meridian are h_1 and h_2, l the latitude of the place, d the declination of a star: then the change in star's altitude will be the greatest possible in passing from one to the other of these circles, when

$$\tan d = \frac{\sin \frac{1}{2}(h_1 - h_2)}{\sin \frac{1}{2}(h_1 + h_2)} \tan l.$$

18. Explain in what regions of the ecliptic the sun has greater longitude than right ascension, and why.

19. The sun's azimuth was observed to be 12° 55′ at 6 o'clock, and his altitude when due east to be 26° 48′: find the declination and latitude of the place of observation.

Ans. decl. = 16° 23′ or 21° 31′, and the latitudes corresponding are 38° 42′ and 54° 25′.

20. If M, S, be the apparent and M', $S,'$ the true altitudes of the moon and sun respectively; also, if D D' be the apparent and true distances, then:

$$\text{versin}\, D' = \text{vers}\,(M' - S') + \frac{\cos M' \cos S'}{\cos M \cos S}\{\text{vers}\, D - \text{vers}\,(M - S)\}.$$

21. On a certain night the azimuth of a certain star was found to be $50^\circ 11' 47''$ from the north, and when it had increased its altitude by $32^\circ 31' 48''$ its azimuth was $84^\circ 49' 47''$ from the south: required the latitude of the place, and the declination of the star that was observed, it being equi-distant in the two observations from the branches of the meridian below and above the pole.

Ans. lat. $= 53^\circ 56' 10''$, decl. $= 28^\circ 42' 52''$.

MISCELLANEOUS EXAMPLES FOR EXERCISE IN SPHERICAL TRIGONOMETRY.

1. In the right-angled triangle of which A is the right angle, there are given $a = 78^\circ 20'$, and $c = 76^\circ 52'$, to find B, C, and b.

Ans. $B = 27^\circ 45' 5''$, $C = 83^\circ 55' 36''$, $b = 27^\circ 7' 49''$.

2. In a spherical triangle right-angled at C are given, $B = 117^\circ 34'$, and $A = 31^\circ 51'$ to find the other parts.

Ans. $a = 76^\circ 52' 10''$, $B = 27^\circ 7' 44''$, $A = 83^\circ 55' 37''$.

3. The spherical triangle BAC is right-angled at A; and in it there are given $b = 117^\circ 34'$, $C = 31^\circ 51'$, to find the other parts.

Ans. $a = 113^\circ 54' 53''$, $c = 28^\circ 50' 30''$, $B = 104^\circ 7' 54''$.

4. Given $b = 27^\circ 6'$, and $c = 76^\circ 52'$, to find the other parts.

Ans. $a = 78^\circ 20' 11''$, $B = 27^\circ 43' 11''$, $C = 83^\circ 56' 11''$.

5. Given $b = 42^\circ 12'$, $B = 48^\circ$, to find the other parts.

Ans. $a = 64^\circ 40' 26''$, $c = 54^\circ 43' 46''$, $C = 64^\circ 35' 18''$,
or $a = 115\ 19\ 34$, $c = 125\ 16\ 14$, $C = 115\ 24\ 42$.

6. Given $B = 48^\circ$, $C = 64^\circ 45'$, to find the rest.

Ans. $a = 64^\circ 40' 4''$, $b = 42^\circ 11' 50''$, $c = 54^\circ 43' 19''$.

7. In the quadrantal triangle given the side $c = 90^\circ$, $C = 64^\circ 40'$, and $A = 42^\circ 12'$, to find the rest.

Ans. $b = 115^\circ 25' 15''$, $a = 48^\circ 0' 14''$, $B = 125^\circ 16' 53''$.

8. In an oblique-angled spherical triangle are given the three sides $a = 56^\circ 40'$, $b = 83^\circ 13'$, $c = 114^\circ 30'$, to find the angles.

Ans. $A = 48^\circ 31' 15{\cdot}34''$, $B = 62^\circ 55' 42''$, $C = 125^\circ 18' 57''$.

9. Given two sides equal to $114^\circ 30'$ and $56^\circ 40'$ respectively, and the angle opposite the former equal to $125^\circ 20'$, to find the other parts.

Ans. $a = 83^\circ 11' 55''$, $A = 48^\circ 30' 24''$, $B = 62^\circ 53' 59''$.

10. Given the two angles $48^\circ 30'$ and $125^\circ 20'$, and the side opposite the former $65^\circ 10'$, to find the angle opposite the former. *Ans.* $98^\circ 46' 17''$.

11. Given two sides and their included angle $114^\circ 30'$, $56^\circ 40'$, and $62^\circ 54'$ respectively, to find the rest.

Ans. $b = 83^\circ 11' 48'$ $A = 48^\circ 30' 25''$, $C = 125^\circ 19' 59''$.

12. Given two angles A, B, and side c, equal to $125^\circ 20'$, $48^\circ 30'$, and $83^\circ 12'$, to find the rest. *Ans.* $a = 56^\circ 39' 32''$, $b = 114^\circ 29' 54''$, $C = 62^\circ 53' 55''$.

13. In a spherical triangle, the angles A, B, C, are $48^\circ 31'$, $62^\circ 56'$, and $125^\circ 20'$, it is required to find the sides respectively opposite to them.

Ans. $a = 56^\circ 39' 43''$, $b = 83^\circ 14' 23''$, $c = 114^\circ 31' 34''$.

14. Given two angles, A, B = 50° 12′, and 58° 8′, and the side a opposite the former 62° 42′, to find the remaining parts.

Ans. A = 130° 54′ 33″, c = 119° 3′ 32″, b = 79° 12′ 13″,
or, A = 156 16 32, c = 152 14 14, b = 100 47 47.

15. The excess of the three angles of a triangle measured on the earth's surface above two right angles is 1″: what is its area, supposing the earth's diameter to be 7957·75 miles? *Ans.* 76·75299 miles, or nearly 76¾ miles.

16. Determine the solid angles of a regular pyramid with a hexagonal base, the altitude of the pyramid being to each side of the base as 2 to 1.

Ans. dihedral angle of any two faces = 125° 22′ 35″
.... of base and surface = 66 35 12.

17. Find the distance between the observatories of Paris and Greenwich, supposing their latitudes to be 48° 50′ 14″ and 51° 28′ 40″, and their longitudes 2° 20′ 24″ E. and 0° 0′ 0″. *Ans.* 3° 2′ 21″.

18. Find the inclinations of the faces of the regular tetrahedron and octahedron; and show that they are the supplements of each other.

Ans. $2 \sin^{-1}\sqrt{\tfrac{1}{2}}$, and $2 \sin^{-1}\sqrt{2}$; or 70° 31′ 43½, and 109° 28′ 16½″.

19. Two great circles, AB, AC, on the globe make an angle at $A = \alpha$, and a point D is also taken on the sphere, so that drawing the perpendicular arcs DE, DF, to AB and AC, we have $AE = \beta$, and $ED = \gamma$: what are the lengths of the arcs AF, FD?

Ans. $\sin DF = \sin \omega \sin \phi$, $\cos AF = \cos \phi \sec DF$; where $DA = \phi$, $DAF = \omega$, $\cos \phi = \cos \beta \cos \gamma$, and $\tan (\alpha + \omega) = \tan \gamma \operatorname{cosec} \beta$.

20. The right ascensions of two stars (A) (B) are 26° 10′ 15″ E. and 12° 15′ 18″ W.; and their declinations are 33° 10′ 15″ N., and 12° 16′ 45″ S.: it is required to find their differences of latitude and longitude; and their respective distances from each other, and from a third star whose position is *lat.* 48° 10′ 15″ N., and *long.* 108° 18′ 30″ W.

Ans. Diff. lat. = 27° 14′ 39″; diff. long. = 52° 35′ 40″; dist. = 58° 22′ 21″; and the other two distances, 104° 11′ 43″ and 96° 18′ 10″.

21. In the last example, if α = 23° 28′, β = 42° 18′, and γ = 29° 6′, what are the arcs AF, FD, AD, FE?

Ans. AF = 48° 36′ 4″, FD = 12° 14′ 8″, AD = 49° 44′ 20″, FE = 17° 47′ 9″.

22. Given the altitudes of two stars 33° 10′ 15″ and 28° 18′ 45″, and their azimuths (or angles made by vertical planes passing through them with the meridian) equal to 42° 10′ 18″ E. and 3° 15′ 18″ W.: to find their difference of longitude and difference of right ascension, the inclination of the equator to the ecliptic being supposed 23° 28′: and the latitude of the place of observation (the Royal Military Academy, Woolwich) 51° 28′ 29″ N.

23. The crown of an arch of Aurora Borealis was observed from Woolwich Common to have the altitude 15° 10′ E. and the azimuth 18° 12′ W. The same arch was seen at the same time from Bath under an azimuth of 3° 2′ E. What was its altitude in the Bath observation, and to what place would it appear vertical; and how high was it above the earth?

Note. The latitude of Bath is 51° 22′ 30″ N. and its longitude 2° 21′ 31″ W.: while those of Woolwich are 51° 28′ 29″ N. and 0° 3′ 34″ E.

24. It is required to find a place equally distant from three given places on the earth's surface: say Bath in lat. 51° 22′ 30″ N. long. 2° 21′ 30″; Constan-

tinople in lat. 41° 1′ N. long. 29° E; and Cape Horn lat. 55° 58′ S. and long. 67° 20′ W.

Ans. lat. 11° 13′ 2″ S., long. 6° 52′ 4″ W., dist. from each place 63° 5′ 50″.

25. Demonstrate the following properties :—

(1.) $\sin^2\frac{1}{2}(A-B)\sin^2\frac{1}{2}c + \sin^2\frac{1}{2}(A+B)\cos^2\frac{1}{2}c = \cos^2\frac{1}{2}C$;

(2.) $\cos^2\frac{1}{2}(A-B)\sin^2\frac{1}{2}c + \cos^2\frac{1}{2}(A+B)\cos^2\frac{1}{2}c = \sin^2\frac{1}{2}C$;

(3.) $\sin^2\frac{1}{2}(a-b)\cos^2\frac{1}{2}C + \sin^2\frac{1}{2}(a+b)\sin^2\frac{1}{2}C = \sin^2\frac{1}{2}c$; and

(4.) $\cos^2\frac{1}{2}(a-b)\cos^2\frac{1}{2}C + \cos^2\frac{1}{2}(a+b)\sin^2\frac{1}{2}C = \cos^2\frac{1}{2}c$;

(5.) $\sin^2 C \sin(a+b)\sin(a-b) = \sin^2 c \sin(A+B)\sin(A-B)$;

(6.) $\dfrac{\sin(A+B)}{\sin C} = \dfrac{\cos\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)}{\cos^2\frac{1}{2}c}$;

(7.) $\dfrac{\sin(A-B)}{\sin C} = \dfrac{\sin\frac{1}{2}(a+b)\sin\frac{1}{2}(a-b)}{\sin^2\frac{1}{2}c}$;

(8.) $\dfrac{\sin(a+b)}{\sin c} = \dfrac{\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{\sin^2\frac{1}{2}C}$;

(9.) $\dfrac{\sin(a-b)}{\sin c} = \dfrac{\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{\sin^2\frac{1}{2}C}$.

26. Prove by means of the equations that connect the sides and angles of a spherical triangle, that

(1.) Any two sides of a triangle are together greater than the third.

(2.) The difference of two sides is less than the third.

(3.) If three sides of a triangle be equal to three sides of another, the angle and area of the one are equal to those of the other.

(4.) If two sides and included angle, or two angles and included side, of one triangle, be equal to those of another, the remaining parts are equal each to each.

(5.) The greatest side is opposite to the greatest angle, and conversely.

(6.) Two triangles, which are either equiangular or equilateral, or have part for part equal in any corresponding manner on the same spheres are equal in all respects.

27. In a spherical triangle ABC, if $A = 179°\,59'$, $B = 179°\,58'$, and $C = 179°\,57'$, prove that $a = 180°$, $b = 180°$, $c = 0$; or universally, that if $A = 180° - \theta_1$, $B = 180° - \theta_2$, and $C = 180° - (\theta_1 + \theta_2)$, then $a = 180°$, $b = 180°$, and $c = 0$.

28. To solve the spherical triangle, there are given the following triads of data: where l_1, l_2, l_3, are the arcs from A, B, C, to the middles of a, b, c, and the other notations as before:

(1.) $A, B, a-b$
(2.) $A-B, a, b$
(3.) $a-b, b-c, B$
(4.) C, c, r
(5.) R, r, c
(6.) R, r, C
(7.) A, a, p_1
(8.) p_1, p_2, p_3
(9.) l_1, l_2, l_3
(10.) p_1, l_1, a
(11.) p_1, l_1, A
(12.) $a-b, b+c, B$
(13.) $\sin a \cos b, \sin a, C$
(14.) $\tan a - \tan b, \tan A - \tan B, \cot C$
(15.) $\dfrac{\sin a \sin b}{\operatorname{cosec} c}, \dfrac{\tan A \tan B}{\cot C}, \dfrac{\sin a}{\sin A}$
(16.) $\sin a \sin A, \sin b \sin B, \sin c \sin C$
(17.) S, s, p_1
(18.) $s-a, s-b, s-c$.

29. Find the distances D, D_1, D_2, D_3, of the centres of the four circles inscribed in the associated system of triangles, from the centre of the circle circumscribing the fundamental triangle of the system.

30. Find the angles, the diagonals, the angle under which the diagonals intersect, and the area of a spherical quadrilateral inscribed in a circle, the four sides of which are given.

31. Investigate the following systems of equations as expressing properties of spherical triangles:—

(1.) $$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}(a+b)}.$$

(2.) $$\cot c\cos A+\frac{\cot B\sin A}{\sin c}=\cos b=\cot a\cos C+\frac{\cot B\sin C}{\sin a}.$$

(3.) $$\sin(a-c)=\sin b\,\frac{\cos C-\cos A}{2\cos^2\frac{1}{2}B}.$$

(4.) $$\sin(a+c)=\sin b\,\frac{\cos C+\cos A}{2\sin^2\frac{1}{2}B}.$$

(5.) $$\frac{\cot A}{\cot a}=\frac{\cos a\sin b-\sin a\cos b\cos C}{\cos A\sin B+\sin A\cos B\cos c}.$$

(6.) $$\sin B\sin C-\cos B\cos C\cos a=\sin b\sin c+\cos b\cos c\cos A.$$

32. In the following expressions n signifies as in the preceding investigations (*p.* 38); and it is required to investigate them.

(1.) $$\tan\tfrac{1}{2}(\ \ A+B+C)=-\frac{1+\cos a+\cos b+\cos c}{n}.$$

(2.) $$\tan\tfrac{1}{2}(-A+B+C)=\frac{1+\cos a-\cos b-\cos c}{n}.$$

(3) $$\tan\tfrac{1}{2}(\ \ A-B+C)=\frac{1-\cos a+\cos b-\cos c}{n}.$$

(4.) $$\tan\tfrac{1}{2}(\ \ A+B-C)=\frac{1-\cos a-\cos b+\cos c}{n}.$$

Also determine the values of $\cot\frac{1}{2}(a+b+c)$, *etc.* in terms of the angles A, B, C; both independently and by means of the polar triangle.

33. Let BDE be the base of a tetrahedron, having $BD=7$, $DE=8$, $EB=9$, and let the edges drawn to the vertex A, be $BA=10$, $DE=11$, $EA=12$: to find the dihedral and solid angles of the figure.

34. In every tetrahedron the following relation subsists:

Let the angle formed by AE, and a line through A parallel to BD, be called the angle (AE, BD), and so with the others; then $AE\,.\,BD\sin(AE, BD)=AB\,.\,DE\sin(AB, DE)=AD\,.\,BE\sin(AB, DE)$.

35. Given two of the adjacent triangular faces of a tetrahedron and their mutual inclination, to find all the other angles, plane, dihedral, and solid.

36. Find the radius of the sphere which circumscribes a tetrahedron, whose edges, taken in a given order, are given.

37. Find the radii of the spheres inscribed and escribed to a tetrahedron, whose edges, taken in a given order, are given.

38. Given the three diagonals of a parallelopiped equal to $2a$, $2b$, $2c$, and the inclinations of their planes respectively equal to C, B, and A, to find the edges, the volume, and the plane, dihedral, and solid angles of the figure.

39. Given the three adjacent edges equal to a, b, c, and the three plane angles which they mutually form equal to C, B, A respectively, to find the diagonals and all the other angles of the parallelopipedon.

40. In a triquadrantal triangle, if α, β, γ be the distances of a point from the three angles A, B, C, show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

41. If a spherical quadrilateral have each of two opposite sides equal to a, and each of the two others equal to b, whilst each of its diagonals is equal to d, show that

$$\cos^2\tfrac{1}{2}d = \cos\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b).$$

42. Given the three arcs bisecting the angles *externally*, and terminating in the opposite sides, to find the sides, angles, and area.

43. Of the four circles inscribed within the associated system of spherical triangles, any three are given to find the fourth.

44. Of the circumscribing circles, any three are given to find the fourth.

45. If through the vertex and through the extremities of the base of a spherical triangle, two equal and parallel less circles be described: then all the spherical triangles which have the same base, and their vertices in the circle through the vertex of the given one will have the same area.

46. If the vertical angle of a spherical triangle, and the sum of the three sides be given, the base will always be a tangent to a circle which may be found.

47. A person is going from London to Constantinople, and intends to touch at the equator. Which is the shortest route and distance he must travel to effect it?

48. A cone has its base on a given circle of the sphere, and its vertex at a given point within or without the sphere: show that it will cut the sphere a second time in a circle, and find the position of its centre and the magnitude of its radius, by a general formula.

49. Find each circumscribed radius in terms of the inscribed radii, and each inscribed radius in terms of the circumscribed radii.

50. If d be the length of the arc which bisects C and terminates in c, then

$$\tan d \sin(a+b) = 2\sin a \sin b \cos\tfrac{1}{2}C,$$
$$\cos d \cos\tfrac{1}{2}c = \cos\tfrac{1}{2}(b+a)\cos\tfrac{1}{2}(b-a).$$

51. If a plane triangle whose sides are a, b, c is placed in a sphere of radius r, show that the angle between the arcs of great circles of which a, b, are chords, will be a right angle when $ab = 2r\sqrt{a^2+b^2-c^2}$.

52. If two great circles ABC, abc, be cut by three others which pass through the same point P in A, B, C, and a, b, c: then

$$\frac{\sin Bb}{\sin Pb}\cdot\sin AC = \frac{\sin Aa}{\sin Pa}\cdot\sin BC + \frac{\sin Cc}{\sin Pc}\cdot\sin AB.$$

53. If arcs be drawn from the angles A, B, C, of a triangle through any point O to meet the opposite sides in a, b, c, and P be pole and R the radius of the circumscribed circle: then

$$\frac{\sin Oa}{\sin Aa} + \frac{\sin Ob}{\sin Bb} + \frac{\sin Oc}{\sin Cc} = \frac{\cos OP}{\cos R}.$$

54. If P be the pole of the circle which circumscribes the equilateral triangle ABC, and Q any point on the sphere:

$$\cos QA + \cos QB + \cos QC = 3\cos PQ \cos PA.$$

55. If O be the pole of the circle about the triangle ABC;

$$\cos\tfrac{1}{2}AOB = \sin\tfrac{1}{2}a \sin\tfrac{1}{2}b + \cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos C.$$

56. If CD be an arc drawn to bisect the angle C and meet AB in D; then

$$\cos \mathrm{CD} = \frac{\sin \mathrm{AD} \cos \mathrm{BC} + \sin \mathrm{BD} \cos \mathrm{AC}}{\sin \mathrm{AB}}:$$

57. When CD bisects the base AB, or the central angle C, the following relations subsist:—

$$\cos \mathrm{CD} = \frac{\sin (\mathrm{A} + \mathrm{B})}{\sin \mathrm{C}} \cos \tfrac{1}{2}c$$

$$\tan \mathrm{CD} = \frac{\sin \mathrm{A} \sin \mathrm{B} \sin c}{2 \sin \frac{1}{2}(\mathrm{A} + \mathrm{B}) \cos \frac{1}{2}(\mathrm{A} - \mathrm{B}) \cos \frac{1}{2}\mathrm{C}}.$$

58. If h, k, l, be the distances of a point O from the angles A, B, C, of a spherical triangle: then

$$\sin (a - b) \cos l + \sin (b - c) \cos h + \sin (c - a) \cos k = 0.$$

59. Given a spherical triangle to find the radius of the circle which passes through the three angular points of the three triangles associated with it which are not common to the given triangle.

60. A hemisphere BC B′C′ is divided into four triangles by two great circles which intersect at A on its surface; show that if E, E_1, E_2, E_3, be the spherical excesses of these four triangles, the angles A, B, C, of that which the excess is E, and a, b, c, the sides; then

$$\mathrm{A} = \tfrac{1}{2}(\mathrm{E} + \mathrm{E}_1); \qquad \mathrm{B} = \tfrac{1}{2}(\mathrm{E} + \mathrm{E}_2); \qquad \mathrm{C} = \tfrac{1}{2}(\mathrm{E} + \mathrm{E}_3).$$

$$\tan^2 \tfrac{1}{2}a = \frac{\sin \frac{1}{2}\mathrm{E} \sin \frac{1}{2}\mathrm{E}_1}{\sin \frac{1}{2}\mathrm{E}_2 \sin \frac{1}{2}\mathrm{E}_3}; \quad \tan^2 \tfrac{1}{2}b = \frac{\sin \frac{1}{2}\mathrm{E} \sin \frac{1}{2}\mathrm{E}_2}{\sin \frac{1}{2}\mathrm{E}_3 \sin \frac{1}{2}\mathrm{E}_1}; \quad \tan^2 \tfrac{1}{2}c = \frac{\sin \mathrm{E} \sin \mathrm{E}_3}{\sin \mathrm{E}_1 \sin \mathrm{E}_2}.$$

61. If chords be drawn to the two sides of a spherical triangle, they will contain a less or a greater angle than that of the spherical triangle, according as that spherical angle is greater or not greater than a right angle.

62. Let ABCD be a spherical quadrilateral, AB, CD being the opposite sides, upon which, as bases, describe any two triangles DLC, AMB, the vertices L and M being in AB and CD respectively: also draw the diagonals AC, BD, meeting in H, and let AM, DL, intersect in E, and BM, CL, in F: it is required to prove that E, F, H, are in the same great circle.

63. Let there be three arcs, α, β, γ, such that $\alpha + \beta + \gamma = 180°$: then it is required to prove that

$$\begin{aligned}
\sin \alpha + \sin \beta + \sin \gamma &= 4 \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \\
-\sin \alpha + \sin \beta + \sin \gamma &= 4 \cos \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \\
\sin \alpha - \sin \beta + \sin \gamma &= 4 \sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \\
\sin \alpha + \sin \beta - \sin \gamma &= 4 \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \cos \tfrac{1}{2}\gamma
\end{aligned}$$

$$\begin{aligned}
\sin 2\alpha + \sin 2\beta + \sin 2\gamma &= 4 \sin \alpha \sin \beta \sin \gamma \\
-\sin 2\alpha + \sin 2\beta + \sin 2\gamma &= 4 \sin \alpha \cos \beta \cos \gamma \\
\sin 2\alpha - \sin 2\beta + \sin 2\gamma &= 4 \cos \alpha \sin \beta \cos \gamma \\
\sin 2\alpha + \sin 2\beta - \sin 2\gamma &= 4 \cos \alpha \cos \beta \sin \gamma
\end{aligned}$$ [*].

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[*] The following values, adapted to any values of α, β, γ, are often useful:—

$$\begin{aligned}
\sin(\alpha+\beta+\gamma) &= \sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\beta\cos\gamma+\cos\alpha\cos\beta\sin\gamma-\sin\alpha\sin\beta\sin\gamma \\
\sin(-\alpha+\beta+\gamma) &= -\sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\beta\cos\gamma+\cos\alpha\cos\beta\sin\gamma+\sin\alpha\sin\beta\sin\gamma \\
\sin(\alpha-\beta+\gamma) &= \sin\alpha\cos\beta\cos\gamma-\cos\alpha\sin\beta\cos\gamma+\cos\alpha\cos\beta\sin\gamma+\sin\alpha\sin\beta\sin\gamma \\
\sin(\alpha+\beta-\gamma) &= \sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\beta\cos\gamma-\cos\alpha\cos\beta\sin\gamma+\sin\alpha\sin\beta\sin\gamma
\end{aligned}$$

$$\begin{aligned}
\cos(\alpha+\beta+\gamma) &= \cos\alpha\cos\beta\cos\gamma-\cos\alpha\sin\beta\sin\gamma-\sin\alpha\cos\beta\sin\gamma-\sin\alpha\sin\beta\cos\gamma \\
\cos(-\alpha+\beta+\gamma) &= \cos\alpha\cos\beta\cos\gamma-\cos\alpha\sin\beta\sin\gamma+\sin\alpha\cos\beta\sin\gamma+\sin\alpha\sin\beta\cos\gamma \\
\cos(\alpha-\beta+\gamma) &= \cos\alpha\cos\beta\cos\gamma+\cos\alpha\sin\beta\sin\gamma-\sin\alpha\cos\beta\sin\gamma+\sin\alpha\sin\beta\cos\gamma \\
\cos(\alpha+\beta-\gamma) &= \cos\alpha\cos\beta\cos\gamma+\cos\alpha\sin\beta\sin\gamma+\sin\alpha\cos\beta\sin\gamma-\sin\alpha\sin\beta\cos\gamma.
\end{aligned}$$

THE CONIC SECTIONS.

DEFINITIONS.

1. A *conic section* is the figure made by the intersection of a right cone * with a plane. It receives different names according to the position of a plane drawn through the vertex of the cone parallel to the plane which makes the section. A fundamental and general property of the conic section is, in each case, deduced from the plane and cone conjointly; after which are deduced from this one property of each kind of section, considered in plano, all the other properties we may require of it, by means of lines and points in the *plane* of the curve itself.

2. The plane drawn through the vertex parallel to the section-plane is called the *directing plane*.

3. The cone itself is to be considered as extended, or capable of extension, indefinitely both ways; that is, below the circular base and beyond the vertex, and constituting two *opposite cones*, or, more appropriately, *the opposite sheets* of the same cone.

4. When the directing plane touches the cone along one of its edges, the section made in it is called the *parabola*. When the directing plane cuts the sheets of the cone, the section is called the *hyperbola*—or, as both sheets of the cone are then also cut by the sectional plane, the figures formed in the opposite sheets are often called *opposite hyperbolas*. They are, however, treated only as two branches of the same curve, in the investigation of the properties. When

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* There are other surfaces besides the right cone, all plane sections of which are curves identical in all their characters with the plane sections of the right cone. The oblique cone is one of them. The others are more general still, as the oblique cone is a generalization of the right one. They are the ellipsoid, prolate, and oblate, to which the sphere is intermediate; the only sections of which are ellipses or circles: the hyperboloïd of one sheet and the hyperboloïd of two sheets, to which the cone is intermediate; but from which every class of conic sections can be cut: and the paraboloïd, the elliptic, and hyperbolic respectively, intermediate to the ellipsoïd and hyperboloïd of two sheets, and to the ellipsoid and hyperboloïd of one sheet. The want of any *geometrical* definition which shall include all these surfaces, and at the same time be independent of the properties of any of the particular sections themselves, has hitherto defeated every attempt to give a general definition of the conic sections materially different from that in the text. The mere extension to the oblique cone, is too trivial, when compared with the number of other surfaces having the same properties, to merit particular attention. It is, therefore, unquestionably the most logical mode of proceeding, to consider the properties under the simplest manner of their genesis, consistent with their names of conic sections; and to treat the several surfaces above spoken of by means of the properties of the curves whose sections they are proved to be. A fine instance of this mode of proceeding may be seen in the Memoir of M. Chasles, on the *general properties of the Cones of the second degree*, published in the Memoirs of the Brussels Academy, which has been translated, with interesting and valuable notes, by Mr. C. Graves, of Trinity College, Dublin. (*Dublin*, 1841.) Any attempt to treat the surfaces above mentioned in a corresponding manner, has not yet been made: but M. Chasles has, in the case of the hyperboloïd of one sheet, (Quetelet's *Cor. Math. tom. ii.* N.S.) shown that the subject under another aspect—that of anharmonic ratio—is rich in interesting and valuable consequences. His researches, however, tend only to the substitution of this surface in space for the straight line (instead of the plane, as we may be at first sight led to suppose the most appropriate), in the enunciation of a considerable number of theorems relative to space of three dimensions, corresponding to those which are known to be true in plano. A course so brief as this must necessarily be, is not, however, the place to introduce or discuss these interesting topics.

the directing plane neither cuts nor touches the sheets of the cone, the sectional plane makes the *ellipse* *.

5. The *transverse plane* is that plane which passes through the axis of the cone, and is perpendicular to the sectional or to the directing plane.

6. The intersection of the transverse plane and the sectional plane is called the *transverse axis* of the conic section. Where this line meets the section in two points (as in the ellipse and hyperbola), the portion intercepted by those intersections is called the *transverse diameter:* and where the transverse axis meets the curve but in one point (as in the parabola), the portion lying within the concavity of the curve is called the *principal diameter of the parabola.*

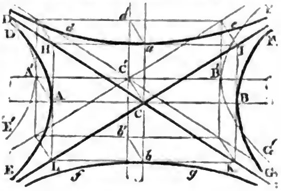
7. The points of section of the diameters with the curve are called the *vertices of those diameters.*

8. The middle point of the diameter of the ellipse or hyperbola is called the *centre of the curve;* and a line drawn through the centre at right angles to the transverse axis is the *conjugate axis.* The *conjugate diameter* is a part of the conjugate axis (half on each side of the centre) which is a mean proportional between the diameters of the circular sections of the cone, which pass by the vertices of the transverse diameter.

9. The *angle of a cone* is the angle made by the edges when cut by a plane through the axis; and two cones are said to be *supplementary* when their angles are so.

It is obvious that the common tangents to the two supplementary cones, when they are placed in contact, will be perpendicular to the plane in which their axes are situated.

10. Let LCI and HCK be the lines of contact of the conjugate cones, and AB, *ab*, their axes; take any point I in one of the lines of contact, and draw the perpendiculars IH, IK cutting the axes in *a* and B, and being themselves consequently bisected in those points. Then if the cones whose axial sections are HCI and ICK, with their opposite sheets, be cut by planes parallel to the plane of the axes, and at distances from it equal to I*a* and IB respectively, two pairs of opposite hyperbolic branches will be produced in the cones by these planes. If now these planes, with their sections, be moved down till they coincide with the plane of the axes, and their intersections with the common tangent planes coincide with HK and LI respectively, they will form the annexed figure, in which the pairs of branches are called the *conjugate hyperbolas.* That is, the branches FBG, DAE are *conjugate* to the branches *fbg, dae;* and mutually *fbg, dae* are called *conjugate* to FBG, DAE.



These four branches of the curve, though not cut from the cone by the *same* plane, are yet to be viewed as forming part of the same system. Indeed, though for convenience of verbal description they have been cut by means of two planes, it is easy to see that by a different method of placing the axes of the conjugate cones, the conjugate sections might have been made by a single plane—namely, by depressing the axis of the obtuse angled cones below the present plane of the axis (the cone still having the same tangent planes perpendicular to the plane of the paper) by a distance equal to the excess of IA above IB.

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* As varieties of these curves, the circle is a case of the ellipse, and is formed when the sectional plane is parallel to the base. The point is also another case, viz. when the sectional and directing planes coincide. When the directing plane of the hyperbolic section coincides with the sectional plane, the section itself is reduced to two straight lines. When the cone by its increasing acuteness becomes a cylinder, the parabolic section becomes two parallel lines; or when the sectional plane is coincident with the directing plane of the parabola, the parabola itself is reduced to a single straight line. These are all the special varieties that can occur.

When the cones are right-angled, the two lines aI, IB will be equal, and the construction will require the axes aC, CA to be in the same plane and the sectional plane parallel to it. The conjugate hyperbolas are then equal in all respects; and in this case, either of them is called a ***rectangular hyperbola***, or an ***equilateral hyperbola***, these terms being used indiscriminately.

11. The lines HK, LI are called the ***asymptotes of the hyperbola:*** for the reason to be shown in *prop.* A, *cor.* 7.

12. ***A tangent to a conic section*** is a straight line which meets the curve, but being produced, does not cut it.

13. Any line drawn through the centre of a conic section and terminated by the curve, is called a *diameter* of the conic section. In the parabola, since the centre is infinitely distant, all the diameters are, in fact, parallel to the principal diameter.

14. If a tangent and a diameter of a conic section be drawn through the same point in the curve, they are said to be ***conjugate to each other:*** and ***any lines parallel to these*** are also said to be conjugate to each other. If they be lines terminated by the curve, and meet a point ***within*** it, they are called ***conjugate chords;*** if they meet the curve and intersect at a point ***without*** it, they are ***conjugate secants;*** if they both touch it, they are ***conjugate tangents;*** if they both pass through the centre, they are ***conjugate diameters;*** and if one be a diameter, and the other a chord, the latter is called an ***ordinal chord to that diameter;*** and the half-chord is called an ***ordinate*** to that diameter *.

15. The distance from the point in which an ordinal chord or ordinate cuts the diameter to the vertex of that diameter, is called the ***abscissa*** or ***absciss.*** The ellipse and hyperbola have two abscisses for each ordinate, the distances from each vertex of the diameter; but the parabola has only one absciss for each ordinate.

16. The absciss and ordinate, when spoken of together, are called the ***co-ordinates*** of the point or points in which the ordinal chord cuts the curve.

17. The ***parameter of any diameter*** of a conic section is the third proportional to that diameter and its conjugate, in the ellipse and hyperbola; and to any absciss and its ordinate in the parabola.

18. The ***focus*** of a conic section is that point in the transverse diameter, at which the ordinal chord, or double ordinate, is equal to the parameter of the transverse.

19. The ***subtangent*** is that portion of the diameter of the curve intercepted between the tangent and the ordinate, whether the ordination be rectangular or oblique.

20. The ***normal*** is the portion of the perpendicular drawn to the tangent from the point of contact, which is intercepted between that point and the transverse axis: and the ***subnormal*** is the portion of the transverse intercepted between the normal and right ordinate of the point.

21. If tangents be drawn at the extremities of any two diameters conjugate to each other, they are shown in the subsequent part of the work to form a parallelogram; and this is called the ***conjugate parallelogram.*** When it is right-angled

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* It will be shown, in the course of the work, that the point of bisection of the ***ordinal chord*** is its point of intersection with the diameter.

it is called the *conjugate rectangle.* Also, when a parallelogram is formed by joining the points of contact of the conjugate parallelogram, this is also conjugate, and the two are distinguished by the terms *circumscribed* and *inscribed* conjugate parallelograms.

22. In the ellipse and hyperbola, the *focal distance of the centre* signifies the distance of either focus from the centre; and the *excentricity* is the ratio of the focal distance to the semi-transverse diameter. In the parabola, the focal distance is infinite, and the excentricity is a ratio of equality.

23. If in the ellipse and hyperbola there be taken a fourth proportional, (all estimated from the centre on the transverse axis,) to the semi-transverse diameter, the focal distance, and the distance of the right ordinate of a point; the extremity of this fourth proportional is called *the dividing point.*

24. If a third proportional, estimated from the centre of an ellipse or hyperbola upon the transverse axis, be taken to the focal distance and semi-transverse diameter; the perpendicular to the transverse drawn through the extremity of the third proportional, is called the *directrix.* In the parabola, the directrix is drawn perpendicular to the principal diameter from a point in the diameter produced, which is at the same distance from the vertex that the focus is.

In the ellipse and hyperbola there will be two directrices, one related to each focus: in the parabola only one.

25.*If two straight lines be drawn from any point in the curve, one to either of the foci, and the other perpendicular to the corresponding directrix, these lines will have the same ratio wherever in a given curve the point be taken *(prop. iii. Par. prx. Ell. and Hyp.)*, and this ratio is called the *determining ratio.*

26. The distance of any point in the curve of a conic section from the focus is called the *focal distance:* any chord passing through the focus and terminated by the curve, is called a *focal chord:* if the chord be an ordinate, it is called the *focal ordinate;* and the tangents to the curve at the extremities of such an ordinate, the *focal tangents.*

27. By an *edge of the cone,* is meant any one of the straight lines by which it was in the position of generatrix of the cone itself, conformably to *def.* 19, *p.* 348, *vol. i.*

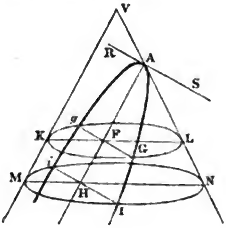
THE PARABOLA.

SECTION I.

PROPERTIES RELATIVE TO THE AXIS, AND LINES CONNECTED WITH IT.

PROP. I.

In the parabola, the abscisses are to one another as the squares of their ordinates.



Let AVM be the transverse plane, iAI the parabolic sectional plane; AH the transverse axis KGLg, MINi, any two sections of the cone perpendicular to its axis, which also cut the sectional plane within the cone; and let KL, MN be the intersections of these with the transverse plane, and GFg, IHi, those with the sectional plane.

Then since KGLg, MINi are sections perpendicular to the axis of the right cone, they are circles; and since the transverse plane passes through the axis of the cone, it passes through the centres of these circles, and the lines KL, MN are diameters of the circles KGL*g*, MIN*i*.

Again, since the plane AVM passes through the axis of the right cone, it is perpendicular to the base and to each of its parallels, as KGL*g*, MIN*i*, *etc.*; and it is perpendicular to IA*i* by definition (*def.* 5): wherefore, the two planes KGL, IA*i* being each perpendicular to MVB, their common section G*g* is perpendicular to MVB, and hence to every line, as AH, KL, situated in it.

Now KL being a diameter of the circle KGL*g* and G*g* perpendicular to it, this line G*g* will be bisected in F. In the same manner, I*i* is perpendicular to AI and MN, and is therefore also bisected in H.

The tangent plane to the cone at AV, and likewise the sectional plane IA*i*, being perpendicular to the transverse plane AVM, their intersection RS is perpendicular to the plane AVM, and hence also to the line AH in it. Wherefore, in the sectional plane IA*i*, the lines RS, G*g*, I*i* being perpendicular to AH in it, are parallel to one another.

But RSV being a tangent plane to the cone, touches it only in the line AV; and hence the straight line RAS meets the cone only in the point A, and therefore, also, only meets the section in that point. It is, therefore, a tangent at A, the vertex of the principal diameter; and hence, again, G*g*, I*i* are ordinal chords to that diameter, and FG, HI are ordinates.

Now, by similar triangles FAL, HAN, and the equals KF, MH (AH being parallel to VM), we have

$$AF : AH :: FL : HN :: KF \,.\, FL : HM \,.\, HN.$$

But by the properties of the circles KGL, MIN, we have

$$KF \,.\, FL = FG^2, \text{ and } MH \,.\, HN = HI^2;$$

$$\text{wherefore } AF : AH :: FG^2 : HI^2.$$

Corollary 1.

If there be taken a third proportional to any absciss and its ordinate, this will be of the same magnitude, wherever in the curve, the points to which the abscissa and ordinate refer, are taken.

For, by the proposition, and *Euc.* vi. 1,

$$FG^2 : HI^2 :: AF : AH,$$

$$:: p.AF : p.AH.$$

But (*hypoth.*),

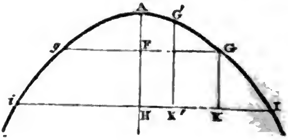
$$AF : FG :: FG : p;$$

$$\text{or } p \,.\, AF = FG^2.$$

Whence $FG^2 : HI^2 :: FG^2 : p.AH$;

$$\text{or } p \,.\, AH = HI^2;$$

$$\text{or again, } AH : HI :: HI : p.$$



It follows, therefore, that the third proportional to AF, FG, is equal to the third proportional to AH, HI; and so on for any other absciss and ordinate. This third proportional is, then, the *parameter* (*def.* 17).

Corollary 2.

The distance of the vertex from the focus is equal to one-fourth of the parameter, or to one-half the ordinate of the focus.

Let F be the focus: then $AF = \frac{1}{2}FG = \frac{1}{4}p$; or $p = 4AF$.

For, (*th. and cor.* 1,) $AF : FG :: FG : p$;

and (*def.* 18) $FG = \frac{1}{2}p$: whence also $AF = \frac{1}{2}FG = \frac{1}{4}p$.

PROP. II.

As the parameter of the axis is to the sum of any two ordinates, so is the difference of those ordinates to the difference of the abscisses.

LET G and I be any two points of the parabola, of which AF, AH are the abscisses, and FG, HI the ordinates: then

$$p : \mathrm{HI} + \mathrm{FG} : \mathrm{HI} - \mathrm{FG} : \mathrm{FH}.$$

For (*cor. i.*) $\mathrm{HI}^2 = p\,.\,\mathrm{AH}$, and $\mathrm{FG}^2 = p\,.\,\mathrm{AF}$: hence

$$\mathrm{HI}^2 - \mathrm{FG}^2 = p.\mathrm{AH} - p.\mathrm{AF}, \text{ or } (\mathrm{HI}+\mathrm{FG})\,(\mathrm{HI}-\mathrm{FG}) = p\,(\mathrm{AH}-\mathrm{AF}) = p\,.\,\mathrm{KG}:$$

$$\text{whence } p : \mathrm{HI} + \mathrm{FG} :: \mathrm{HI} - \mathrm{FG} : \mathrm{FH}.$$

Corollary 1.

Produce HI to meet the curve at i: then $p : \mathrm{K}i :: \mathrm{KI} : \mathrm{FH}$.

For $i\mathrm{K} = \mathrm{HI} + \mathrm{FG}$, and $\mathrm{KI} = \mathrm{HI} - \mathrm{FG}$: wherefore

$$i\mathrm{K}\,.\,\mathrm{KI} = (\mathrm{HI}+\mathrm{FG})\,(\mathrm{HI}-\mathrm{FG}) = \mathrm{HI}^2 - \mathrm{FG}^2 = p\,.\,\mathrm{AH} - p\,.\,\mathrm{AF} = p\,.\,\mathrm{FH}:$$

$$\text{whence } p : \mathrm{K}i :: \mathrm{KI} : \mathrm{FH}.$$

Corollary 2.

If a double ordinate be divided by lines parallel to the axis, it is divided by them into segments whose rectangles are as the dividing lines themselves.

That is: if the double ordinate $\mathrm{I}i$ be divided by lines GK, G′K′, parallel to the diameter AH, then

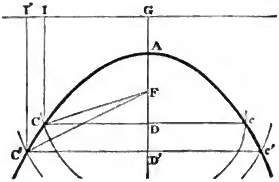
$$i\mathrm{K}\,.\,\mathrm{KI} : i\mathrm{K}'.\,\mathrm{K}'\mathrm{I} :: \mathrm{GK} : \mathrm{G}'\mathrm{K}'.$$

For $p\,.\,\mathrm{GK} = i\mathrm{K}\,.\,\mathrm{KI}$, and $p\,.\,\mathrm{G}'\mathrm{K}' = i\mathrm{K}'\,.\,\mathrm{K}'\mathrm{I}$: hence

$$i\mathrm{K}\,.\,\mathrm{KI} : i\mathrm{K}'\,.\,\mathrm{K}'\mathrm{I} :: p\,.\,\mathrm{GK} : p\,.\,\mathrm{G}'\mathrm{K}' :: \mathrm{GK} : \mathrm{G}'\mathrm{K}'.$$

PROP. III.

In the parabola, the determining ratio is one of equality: or the line drawn from the focus to any point in the curve is equal to the perpendicular drawn from the same point to the directrix.



THAT is, if F be the focus, and in the axis AF produced the opposite way be taken AG = AF, and the perpendicular GI be drawn to GF: then, if from any point C in the parabola a line CI be drawn perpendicular to the directrix GI (*def.* 24), and CF be joined, the lines FC, CI will be equal.

For draw the ordinate at C. Then since CDF is a right angle,

$$\begin{aligned}\mathrm{CF}^2 &= \mathrm{FD}^2 + \mathrm{DC}^2 = \mathrm{FD}^2 + p\,.\,\mathrm{AD} &&= \mathrm{FD}^2 + 4\mathrm{AF}\,.\,\mathrm{AD}\ (\textit{prop. i. cor. } 2.)\\ &= \mathrm{FD}^2 + (\mathrm{FD} + \mathrm{FA})\,.\,2\mathrm{GF} &&= \mathrm{FD}^2 + 2\mathrm{GF}\,.\,\mathrm{FD} + 2\mathrm{GF}\,.\,\mathrm{FA}\\ &= \mathrm{FD}^2 + 2\mathrm{GF}\,.\,\mathrm{FD} + \mathrm{GF}^2 &&= (\mathrm{FD} + \mathrm{FG})^2.\end{aligned}$$

Whence $\mathrm{CF} = \mathrm{GF} + \mathrm{FD} = \mathrm{GD} = \mathrm{CI}$.

The same demonstration applies, *mutatis mutandis*, when D is between A and F: but the steps are left for the student's exercise.

Corollary 1.

Hence any number of points in the parabola whose parameter is given may be easily constructed.

For, taking AF = AG = ¼ parameter, and drawing lines through any points in the axis, AD, such as DC: then with F as centre and distances equal to DG, D′G, *etc.* describing circles to cut DC, D′C, in C, *c*, C′, *c*′, *etc.* the points C, C′, *c*, *c*′, *etc.* will be in the curve. For CI = DG, *etc.* A sufficient number, therefore, of such points being constructed, the curve may be approximately traced through them, sufficiently accurate for all the purposes of constructive practice.

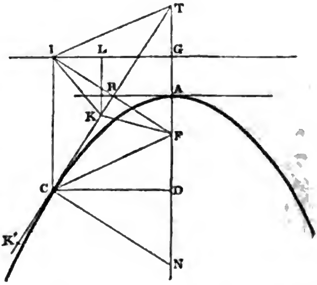
Corollary 2.

A line drawn through the vertex A, perpendicular to the axis AF, will be a tangent to the parabola at A.

For *any other point* in this line, as H, is at a greater distance from F than A is, and hence at a greater distance from the focus than from the directrix GI; and is, therefore, *without the curve.* The line AH, therefore, meets the curve in only one point A, and is hence (*def.* 12) a tangent at A.

PROP. IV.

If a line be drawn to bisect the angle made by two lines, one of which is drawn from a point in the parabola to the focus, and the other from the same point perpendicular to the directrix, that line will be a tangent to the parabola at that point.



Let C be the point in the parabola, F the focus, GI the directrix, perpendicular to which CI is drawn, then the line CT drawn to bisect the angle ICF will be a tangent at C.

For since C is a point in the curve, CT meets the curve: and if it be not a tangent, it will meet the curve again in some point, K. Then also K is a point in the parabola. Join KF, and draw KL perpendicular to the directrix, and join IK.

Then K being a point in the parabola, KF = KL (*prop. iii.*): and since IC = CF, and by hypothesis ICK = KCF, and KC common to the triangles ICK, KCF, the base IK is equal to KF. But KF=KL, and therefore IK=KL. Whence the angles KIL, KLI are also equal.

But KLI is a right angle by construction, and therefore two angles of the triangle IKL are equal to two right angles: which is impossible (*Euc. i.* 17, or *Geom. theor.* 17, *cor.*) Hence the point K is not in the parabola.

In the same way it may be shown that no other point, K′ in the line CT is in the parabola: and therefore that CT does not cut the parabola, or it is a tangent to the parabola at C. Q. E. D.

Corollary 1.

Let the tangent CT and the axis AF be produced to meet at T: then will CF = FT.

For ICT = FCT (by *theor.*), and ICT = CTF (*Euc.* i. 29, or *Geom. th.* 12): wherefore CTF = TCF, and CF = FT.

Corollary 2.

Draw the ordinate CD: then will the sub-tangent DT be bisected in A, or DA = AT.

For (*cor.* 1.) AF + AT = FT = FC = CI = GD = GA + AD = FA + AD; whence, taking FA from both, we have DA = AT.

Corollary 3.

Draw the normal CN to meet the axis in N: then will F be the centre of the circle described through T, C, N.

For TCN is a right angle (*def.* 20), and hence C is in the semicircle on TN. Moreover, (*Euc. vi.* 8, or *Geom. th.* 87,) TCD = CND; and since CF = FT, we have TCF = CTF = CTN = DCN. Wherefore CNF = CND = DCT = DCF + FCT = DCF + DCN = FCN: and hence also FN = FC = FT; or F is the centre of the circle through T, C, N.

Corollary 4.

The sub-normal DN is always of the same magnitude; viz. equal to half the parameter.

For since TCN is a right angle, (*def.* 20,) we have

DT (= 2AD) : DC :: CD : DN (*Euc. vi.* 8, *cor.* 1, or *Geom. th.* 87).
Also AD : DC :: CD : p (*def.* 17).
Whence DN = $\frac{1}{2}p$ = 2AF.

Corollary 5.

Join FI meeting the tangent CT in H: then HA is coincident with the tangent at A, and a mean proportional between AF and AD.

For since CF = CI, and CT bisects the angle FCI, (*th. iv.*) IF, CT will be perpendicular to one another; and hence FH will be parallel to NC, and NF : FT :: CH : HT. But NF = FT, and hence CH = HT. Again, as the lines DT, TC are bisected in H and A, the line HA is parallel to CD; that is, perpendicular to DA, and therefore a tangent at A (*th. iii. cor.* 2).

Also, (*Euc. vi.* 8, *cor.* 1,) FA : AH :: AH : AT (= AD).

Corollary 6.

The perpendicular FH is a mean proportional between FA and FT, or FA and FC; as is obvious from the familiar properties of the triangle (*Euc. vi.* 8, or *Geom. th.* 87).

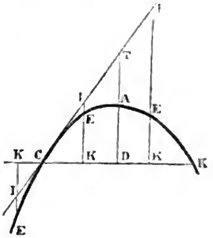
Corollary 7.

Join IT: it will be parallel to FC, and the figure CITF will be a rhombus.

For, since IC = CF = FT, and IC, FT are parallel, the lines IT, CF joining the extremities of the equal lines IC, FT towards the same parts are parallel and equal. Wherefore IC = CF = FT = TI, which is the defining property of a rhombus.

PROP. V.

If there be a tangent drawn to any point of a parabola, and a double ordinate to the axis be drawn from the point of contact: then any line drawn parallel to the axis will be divided into segments by the tangent, curve, and double ordinate, which have the same ratio as the segments of the double ordinate made by that line.



LET CT be a tangent to the parabola at C, and CL the double ordinate to the diameter AD; and let any line IK parallel to AD be divided by the curve in E: then will

$$\mathrm{IE : EK :: CK : KL.}$$

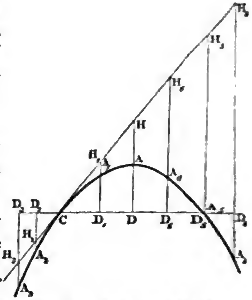
For let DA be produced to meet the tangent at T, making the triangles CKI, CDT similar.

Then, $\mathrm{CK : KI :: CD : DT} = 2\mathrm{DA}$ (*pr. iv. cor.* 2),
and $\mathrm{CD} : 2\mathrm{DA} :: p : \mathrm{CL}$ (*pr. i. cor.* 1);
whence $\mathrm{CL : KI} :: p : \mathrm{CK}$ (*alt.* and *Euc. v.* 11);
but $p : \mathrm{CK :: KL : KE}$ (*pr. ii. cor.* 1);
whence $\mathrm{KI : KE :: CL : LK}$ (*Geom. v.* 11, and *alt.*),
or $\mathrm{IE : EK :: CK : KL}$ (*div.*)

Corollary 1.

If any number of such lines be drawn in any positions whatever relative to the point C, then there will exist between the external parts of those lines HA, H_1 A_1, H_2 A_2, *etc.*, and the corresponding segments of the tangent, CH, CH_1, CH_2, *etc.* the relation

$$\mathrm{AH : A_1H : A_2H_2} : etc. :: \mathrm{CH^2 : CH_1^2 : CH_2^2} : etc.$$



For taking IK of the figure in the proposition as a specimen, we have by alternation,

$$\mathrm{IE:CK::EK:KL}; \text{or} \, \mathrm{P.IE:CK^2::P.EK:CK.KL}.$$

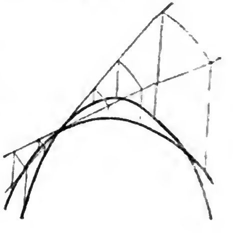
But (*pr. ii. cor.* 1) $p.\mathrm{KE} = \mathrm{CK.KL}$; whence also $p.\mathrm{IE} = \mathrm{CK}^2$.

Now as this conclusion applies to any one of the lines H_1 D_1, H_2 D_2, *etc.* of the figure of the corollary, we have generally

$$p.\mathrm{AH} = \mathrm{CH}^2, p.\mathrm{A_1\,H_1} = \mathrm{CH_1}^2, etc.$$

$$\text{or } \mathrm{AH : A_1\,H_1 : A_2\,H_2} : etc. :: \mathrm{CH^2 : CH_1^2 : CH_2^2}, etc.$$

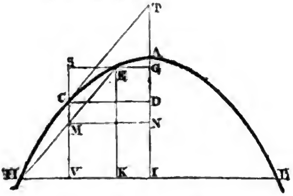
Corollary 2.

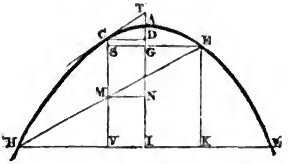


As this property is independent of the position of the tangent in the curve, or its inclination to the axis of the figure, it will hold good for every alteration that can be made in the inclination of the tangent. If, therefore, the lines HA, H_1A_1, H_2A_2, *etc.* be appended to the line CH, and free to move about the points of appendage H, H_1, H_2, *etc.*: then, so long as they are all parallel to one another, whatever the angle CHD may be, their extremities A, A_1, A_2, *etc.*, will always be situated in the curve of *some* parabola, provided they be so in *any one* case. The annexed figure shows two positions of the tangent, the lines not changing their original direction: and it will be equally simple to sketch for the case of the tangent retaining its position, and the appended parallels changing their direction.

PROP. VI.

If any chord be drawn parallel to a tangent, to meet the curve in two points; and if from its extremities and from the point of contact ordinates be drawn to the axis: then double the ordinate of the point of contact will be equal to the sum or difference of the other two, according as they are situated on the same or different sides of the axis.





Let any chord HE parallel to the tangent CT, and the ordinates EG, CD, HI, be drawn: then

$$HI + EG = 2CD, \text{ in } fig.\ 1.$$
$$HI - EG = 2CD, \text{ in } fig.\ 2.$$

For draw EK parallel to the axis AI, and produce HI to meet the curve again in L.

Then by similar triangles EHK, TCD, and *pr. ii. cor.* 1,

$$2AD\ (= DT) : CD :: EK : KH :: KL : p;$$

or, *alt.* and *def.* 17, this gives

$$2AD : KL :: CD : p :: 2AD : 2CD.$$

Wherefore 2CD = KL in both figures. Taking them separately, we get

$$2CD = KL = LI + IK = HI + IK = HI + EG \text{ in } fig.\ 1.$$
$$2CD = KL = LI - IK = HI - IK = HI - EG \text{ in } fig.\ 2.$$

Corollary 1.

Any chord, as EH, parallel to a tangent CT, will be bisected in M by the diameter CM drawn through C.

Let CM intersect HI, EG in V, S, and draw MN parallel to the ordinates. Then we have $HI - CD = CD \mp EG$, as *fig.* 1 *or* 2 is taken: but in both this equation expresses HV = VK = ES, by the properties of parallels. Hence the triangles HVM, EMS, are similar, and have HV = ES; wherefore, HM = ME.

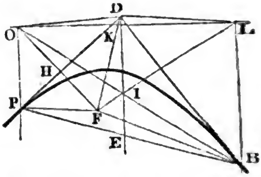
Corollary 2.

If two parallel chords be bisected by a line, that line will be a diameter, and the tangent at its extremity will be parallel to the chords. This follows from the last, *ex abs.*

Corollary 3.

If a diameter be drawn to bisect any chord, and a tangent be drawn at its intersection with the curve, the chord and tangent will be parallel.

PROP. VII.



If two tangents to a parabola be drawn to meet, and from the point of intersection there be drawn a diameter and a line to the focus; and if also there be drawn the chord of contact and lines from the points of contact to the focus: then,

(1.) *The line from the intersection of the tangents to the focus will make equal angles with those from the points of contact to the focus.*

(2.) *The line from the intersection of the tangents to the focus, and the diameter through the intersection of the tangents, make equal angles with the tangents.*

(3.) *The diameter through the intersection of the tangents bisects the chord of contact.*

Let the tangents PD, BD intersect in D; and draw PF, BF, DF to the focus F; also draw the chord of contact PB, and the diameter DE of the parabola through E: then

DFB = DFP; PDF = BDE; and PE = EB.

For draw PO, BL perpendicular to the directrix, and join DO, DL, OF, LF, OB; and let H be the intersection of OF with PD, K that of the directrix OL with the diameter DE, and I that of OB, FL. Then

First. Since PH bisects OF at right angles, the triangle ODF is isosceles, and OD = DF. For a similar reason FD = DL; and hence also OD = DL, and DLO = DOL, or adding the right angles LOP, OLB, we have DOP = DLB. But, manifestly, DFP=DOP, and DFB=DLB; wherefore, also, DFP=DFB.

Second. Since ODL is an isosceles triangle, and DK perpendicular to the base OL, it bisects the vertical angle ODL, making ODK = KDL.

Now ODK = ODP + PDF + FDK = 2PDF + FDK,
and KDL = LDB + BDF − FDK = 2BDF − FDK;
wherefore 2PDF = 2BDF − 2FDK, or PDF = BDF − FDK = BDE.

Third. By similar triangles, and that OL is bisected in K, we have

OK : KL :: OI : IB :: PE : EB.

Wherefore, PE = EB.

Corollary 1.

Perpendiculars from D the intersection of two tangents DB, DP of a parabola to the lines BF, PF drawn from the focus F to the points of contact B, P are equal to one another; make equal angles with the line DF from the intersection of the tangents to the focus; and make equal angles with the tangents BD, PD themselves.

Corollary 2.

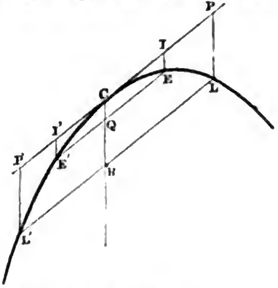
Perpendiculars from D the intersection of the tangents upon those PO, BL drawn from the points of contact to the directrix, are equal to one another.

SECTION II.

PROPERTIES DEPENDING ON OBLIQUE ORDINATES.

PROP. VIII.

The abscisses of any diameter are to one another as the squares of their ordinates.



Let P′CP be the tangent at C, CR the diameter, and E′E, L′L the double ordinates to the abscisses CQ, CR: then shall CQ : CR :: QE^2 : RL^2.

For, draw L′P′, E′I′, EI, LP parallel to CR; then, since E′E, L′L are double ordinates to the diameter CR, they are bisected in Q and R (*pr. vi. cor.* 1); also, CQ = IE, CR = PL, CI = QE, CP = RL, by the parallelograms CE, CL.

Then, (*pr. v. cor.* 1,) we have

$$IE : PL :: CI^2 : CP^2, \text{ or } CQ : CR :: QE^2 : RL^2.$$

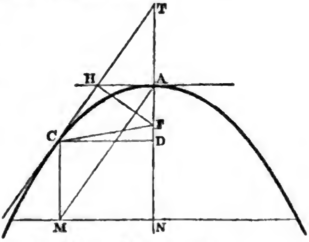
The same reasoning, obviously, applies to the points E′, L′ on the other side of the diameter CR.

Scholium.

Hence, as the abscisses of any diameter and their ordinates have the same relations as those of the axis, namely, that the ordinates are bisected by the diameter, and their squares proportional to the abscisses; so all the other properties of the axis and its ordinates and abscisses, before demonstrated, will likewise hold good for any diameter and its ordinates and abscisses. And also those of the parameters, understanding the parameter of any diameter, as a third proportional to any absciss and its ordinate (*def.* 17). Some of the most material of these are demonstrated in the following propositions.

PROP. IX.

The parameter of any diameter is equal to four times the distance of the vertex of that diameter from the focus of the parabola.



Let CM be any diameter different from the axis AN; then the parameter of that diameter is $p_1 = 4CF$.

For, draw the perpendicular FH upon the tangent CT, and the ordinate AM from the vertex of the axis parallel to CT; also the ordinate CD.

Then, the right-angled triangles CDT, FHT having their angles at T common are similar. Also, by parallels and *pr. i. cor.* 1,

$$p \cdot AD = p \cdot CM = CD^2, \text{ and } p_1 \cdot CM = MA^2 = CT^2.$$

Wherefore, and by similar triangles and *pr. iv. cor.* 6,

$$p : p_1 :: CD^2 : CT^2 :: FH^2 : FT^2 :: FA \cdot FT : FT^2 :: FA : FT.$$

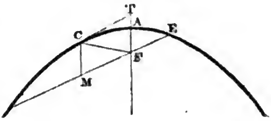
But $p = 4FA$, and hence $p_1 = 4FT = 4FC$ (*pr. iv. cor.* 1).

Corollary.

Hence the parameter p_1 of the diameter CM is equal to 4FA + 4AD, or to p + 4AD, that is, the parameter of the axis added to 4AD.

PROP. X.

That abscis of any ordinate which passes through the focus of the parabola is equal to one-fourth of the parameter of that diameter, and its ordinate is one-half of the same parameter.



Let F be the focus, EM the ordinate drawn through F to the diameter CM: then CM = $\frac{1}{4}p_1$, and EM = $\frac{1}{2}p_1$.

For join FC, and draw the tangent CT. Then by *parallels, pr. iv. cor.* 1, and *pr. ix.* we have

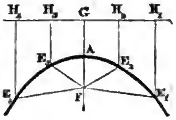
$$CM = FT = FC = \tfrac{1}{4}p_1.$$

Again, (*def.* 17,) CM : ME :: ME : p_1, or $ME^2 = p_1 . CM = \frac{1}{4}p_1^2$; and hence ME = $\frac{1}{2}p_1$.

Corollary 1.

Hence, of any diameter, the double ordinate which passes through the focus is equal to the parameter, or to quadruple its absciss.

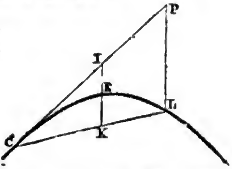
Corollary 2.



Hence, and from *prop. iii. cor.* 1, and *prop. ix.*, it appears, that if the directrix GH be drawn, and any lines HE, HE, parallel to the axis; then every parallel HE will be equal to EF, or $\frac{1}{4}$ of the parameter of the diameter to the point E.

PROP. XI.

If from the point of contact of a tangent to the parabola, any chord be drawn, and another line parallel to the axis to meet the chord, curve, and tangent: this line will be divided by them in the same ratio that it divides the chord.



If CP be a tangent, and CL a chord to the parabola, and any line IK be parallel to the axis meeting the curve in E: then

$$IE : EK :: CK : KL.$$

For draw LP parallel to IK:

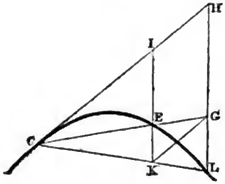
Then IE : PL :: $CI^2 : CP^2 :: CK^2 : CL^2$ (*pr. v. cor.* 1, and *sim. trians.*)
and IK : PL :: CK : CL :: CK^2 : CK . CL (*sim. trians.* and G. 79);
whence IE : IK :: CK . CL : CL^2 :: CK : CL,
and IE : EK :: CK : KL (*div.*)

Corollary.

When CK = KL, we have also IE = EK = $\frac{1}{2}$IK; or the sub-tangent KI is double the abscissa KE.

PROP. XII.

If from the point of contact of a tangent with the parabola, two lines be drawn to the vertices of any two diameters, each to intersect the other diameter: then the line joining these two points of intersection will be parallel to the tangent.



Let CE, CL be drawn from the point of contact C to meet the diameters HL, IE in G and K: then GK will be parallel to the tangent CH at C.

For by *pr. xi.* and *parallels,* we have

$$IE : EK :: CK : KL :: CE : EG.$$

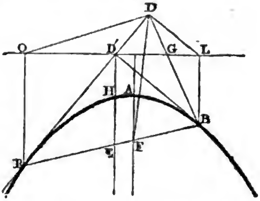
Hence, the lines CG, IK being similarly divided, the lines KG, CI are parallel.

Scholium.

This theorem appears to have been first given by Toricelli, the inventor of the barometer, as subservient to the construction of a problem in gunnery, to which reference will be made under the head of *Projectiles.*

PROP. XIII.

If the chord of contact of two tangents to a parabola pass through the focus, the tangents will intersect in the directrix and at right angles.



Let PB be any chord through the focus F, and PD, BD be tangents at P and B: then the point D will be in the directrix, and PDB will be a right angle.

For if not, let the tangent at P cut the directrix in D′, draw PO, BL perpendicular to the directrix, join DF, and draw the diameter D′HE, and draw D′B, DO, DL.

Then, by the reasoning employed in the demonstration of *pr. vii.*, we have $DFP = DOP$, and $DFB = DLB$. Also, by the conclusion of that theorem, $DFB = DFP$. But (*hyp.*) BFP is a straight line, and hence PFD and DFB are together equal to two right angles; and being equal, each of them is a right angle: wherefore, also, DOP, DLB are right angles. Also, since D′ is in the directrix, and PO, BL perpendicular to it, the angles D′OP, D′LP are right angles: and hence $D'OP = DOP$, and $D'LB = DLB$; the less to the greater in both cases. The point of intersection of the tangents to the focal chord is therefore in the directrix, and in the figure above is represented by D′.

Then (*pr. vii.*) $PE = EB$; and PB will be conjugate to this diameter. Wherefore (*pr. x.*) PE, ED′, EB will each be equal to the semi-parameter of the diameter HE, and therefore equal. They will hence be in the circle PD′B of which PB is a diameter; and, consequently, PD′B in a semicircle is a right angle.

Corollary 1.

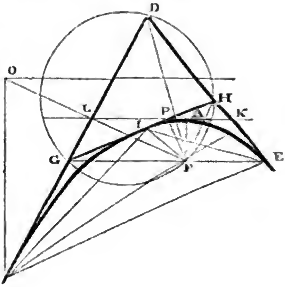
The intersection of tangents which are perpendicular to one another, will always be in the directrix, and the chords of contact will all pass through the focus.

Corollary 2.

Every tangent limited by the directrix and point of contact will subtend a right angle at the focus.

PROP. XIV.

If three tangents to a parabola by their intersections two and two form a triangle, the circle circumscribing this triangle will pass through the focus.



Let DGH be the triangle formed by the intersection of the three tangents CD, DE, GH of a parabola, and F the focus: then the circle described about DGH will pass through F.

For draw the perpendiculars FL, FP, FK to the three tangents: then each of these intersections L, P, K, is in the tangent drawn to the vertex A of the parabola (*pr. iv. cor.* 5).

Now since FLD, FKD are right angles, the four points F, L, D, K are in a circle: and $FLK = FDK$, or $FLP = FDH$. Again, for a similar reason F, L, G, P are in a circle, and therefore $FLP = FGP = FGH$. Consequently $FGH = FDH$, and the points F, G, D, H are therefore in a circle. Whence the circle GDH passes through the focus.

Corollary 1.

The two triangles CFD, DFE are equiangular to LFK and to one another.

For draw CO perpendicular to the directrix: then O, L, F are in one line (*pr. iv. cor.* 5), and $FLA = OCL = FCL$, and CLF, LAF are right angles: wherefore $LCF = FLA = FGP$. Also it has been proved that $LDF = FKA = FHP$: and hence two of the angles FCD, FDC of the triangle CFD are equal to two of the angles FGP, FHP of the triangle GFH. The third angles CFD, GFH, are therefore equal, and the triangle CFD is equiangular to GFH.

In the same way it is proved that the triangle DFE is equiangular to GFH: and it follows, also, that CFD, DFE are equiangular.

Corollary 2.

If a third tangent GI vary its position, but always cuts the two fixed tangents DC, DE in two points, as G, H: then the angle GFH will always be of the same magnitude.

For it has just been shown that GFH is always equiangular to the two CFD, DFE, which under the conditions are constant.

Corollary 3.

If three straight lines which intersect in G, D, H, are tangents to a parabola, in the manner of the figure, then

$$GD \cdot DH = CG \cdot HE,\ DG \cdot GH = DC \cdot HI, \text{ and } DH \cdot HG = DE \cdot GI.$$

For, by the pairs of similar triangles FCG, FDH, and FDG, FEH, we have

$$\mathrm{GC : HD :: GF : FH :: GD : HE, \text{ or } HD . DG = CG . HE};$$

and in the same way the other two cases may be proved.

Corollary 4.

If two tangents CD, DE be drawn to a parabola at C and E to meet in D, and GH be a third tangent meeting them in G, H, and touching the parabola in I, then according as C and E are on the same or different sides of GH,

$$\mathrm{DG . DE \pm DC . DH = DC . DE.}$$

For (*Cor.* 3) GC : HD :: GD : HE, we have (*comp.*) CD : CG :: DE : HD, or CD . DH = CG . DE. Add this to DG . DE: then

$$\begin{aligned}\mathrm{DG . DE + CD . DH} &= \mathrm{CG . DE + DG . DE}\\ &= \mathrm{CD . DE.}\end{aligned}$$

In a similar manner the other case may be proved.

Corollary 5.

If the three tangents CD, DE, GH be drawn as before, and any fourth tangent LK be drawn cutting the first three in L, K, P, and touching the parabola at A: then

$$\mathrm{KP : PL :: HI : IG;\ PL : LK :: CG : CD;\ and\ LK : KP :: ED : DH.}$$

For since the three tangents at C, A, I intersect, and likewise the three at E, I, A also intersect, we have

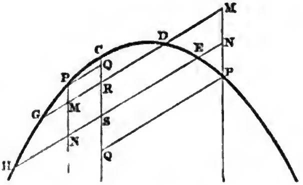
$$\mathrm{AL : LP :: PG : GI, \text{ or } \mathit{div.}\ AP : PL :: PI : IG}$$
$$\mathrm{PK : KA :: IH : HP, \text{ or } \mathit{div.}\ AP : PK :: PI : IH.}$$

Whence, *ex. eq.* KP : PL :: HI : IG.

And in the same way the other two stated conclusions are deduced.

PROP. XV.

If a diameter cut parallel chords in a parabola, the segments of this diameter intercepted by them will be as the rectangles of the segments made in them by the diameter.



If the parallel chords GD, HE be cut by *any* diameter PN in M and N: then

$$\mathrm{PM : PN :: GM . MD : HN . NE.}$$

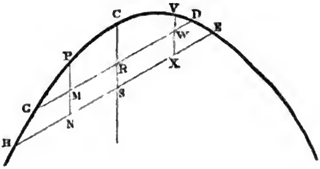
For draw the diameter CS bisecting GD, HE, and likewise the ordinate PQ of the vertex P. Then if p_1 denote the parameter of the diameter CS, we have

$$p_1 . \mathrm{CS} = \mathrm{SH}^2,\ p_1 . \mathrm{CR} = \mathrm{GR}^2, \text{ and } p_1 . \mathrm{CQ} = \mathrm{PQ}^2:$$
$$\text{whence } p_1 (\mathrm{SC} - \mathrm{CQ}) = \mathrm{SH}^2 - \mathrm{PQ}^2 = \mathrm{SH}^2 - \mathrm{SN}^2, \text{ or}$$
$$p_1 . \mathrm{QS} = p_1 . \mathrm{PN} = \mathrm{HN . NE.}$$
$$\text{Similarly, } p_1 . \mathrm{RQ} = p_1 . \mathrm{PM} = \mathrm{GM . MD.}$$
$$\text{Hence } \mathrm{PM : PN :: GM . MD : HN . NE.}$$

The same proof, excepting merely what depends on the change of signs, applies to the case where PM cuts the parallel chords GD, HE in prolongation, as in the case figured to the *right* of the diameter CQ.

Corollary 1.

If any chord be cut by two diameters, the intercepted segments of these diameters will be to one another as the rectangles of the segments into which they respectively divide the chord.



For let HE be cut by the two diameters PN, VX, and let p_1 be the parameter of CS taken as before. Then

$p_1 . PN = HN . NE$, and $p_1 . VX = HX . XE$; wherefore

$$PN : VX :: HN . NE : HX . XE.$$

Corollary 2.

If two parallel chords be cut by any two diameters, the segments of these diameters respectively cut off by the chords, will be as the rectangles of the segments into which the chords are respectively divided.

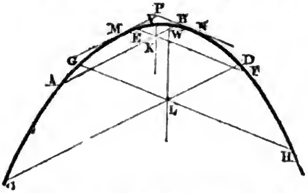
For let VX cut HE in X, and PN cut GD in M; then, by the *theorem* and *corollary*,

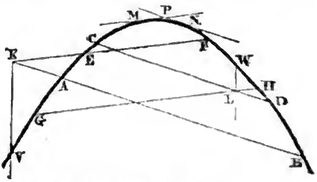
$$PM : PN :: GM . MD : HN . NE$$
$$\text{and } PN : VX :: HN . NE : HX . XE;$$
$$\text{hence } PM : VX :: GM . MD : HX . XE.$$

The same will, evidently, be the case when M, N, W, X, or any of them are *without* the field of the curve line.

PROP. XVI.

If two lines which meet one another (chords or secants) be parallel to two others which meet one another: the rectangles of the segments of one pair of parallels will be as the rectangles of the segments of the other pair.





That is, if EF, AB intersect in K, and are parallel to GH, CD which intersect in L; then

$$EK . KF : AK . KB :: GL . LH : CL . LD.$$

For draw the diameters KV, LW; then we have at once from *pr. xv. cor.* 2,

$$EK . KF : GL . LH :: VK : WL :: AK . KB : CL . LD,$$

from which the enunciated property flows by alternation.

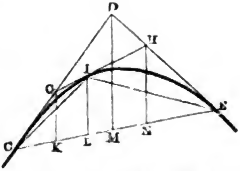
Corollary.

Draw the tangents MP, NP parallel to AB, EF respectively. Then in this case, rectangles of the segments become the squares of the tangents themselves, and the constant ratio then becomes that of these squares; or

$$EK . KF : AK . KB :: NP^2 : MP^2.$$

PROP. XVII.

If three tangents to the parabola intersect two and two, they will thus be divided, estimated from their points of contact to those of intersection, into segments which have the same ratio.



Let CD, DE, GH, touch the parabola in C, E, I, and meet each other in D, H, G: then

$$CG : GD :: DH : HE :: GI : IH.$$

For through G, I, D, H, draw diameters of the parabola, and join CI, IE.

Then GK bisects CI, (*pr. vi. cor.* 1,) and therefore it also bisects CL; and, in like manner, HN bisects LE, and DM bisects CE. Whence

$$KM = CM - CK = \tfrac{1}{2}CE - \tfrac{1}{2}CL = \tfrac{1}{2}(CE - CL) = \tfrac{1}{2}LE = LN = NE.$$
$$MN = ME - NE = \tfrac{1}{2}CE - \tfrac{1}{2}EL = \tfrac{1}{2}(CE - EL) = \tfrac{1}{2}CL = CK = KL.$$

But by parallels $GI : IH :: KL : LN$

$$CG : GD :: CK : KM \ (= KL : LN)$$
$$DH : HE :: MN : NE \ (= KL : LN),$$

wherefore the property is established.

Corollary 1.

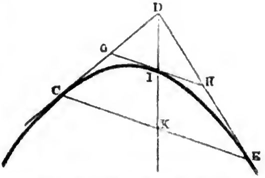
When the tangent GH is parallel to the chord of contact of the other two, it will bisect CD, DE in G and H. For in this case we have

By *theorem*, $CG : GD :: DH : HE$, or *comp.* $CD : DE :: GD : HE$;

but by parallels CE, GH, we have $CD : DE :: GD : DH$;

wherefore $DH = HE$, and therefore also $CG = GD$.

Corollary 2.



A tangent to the parabola parallel to the chord of contact, is bisected in its own point of contact.

For it bisects CD and DE by last *cor.* and by *th.* itself,

$$GI : IH :: CG : GD :: DH : HE.$$

Corollary 3.

The diameter which passes through the intersection of the tangents is conjugate to the chord of contact.

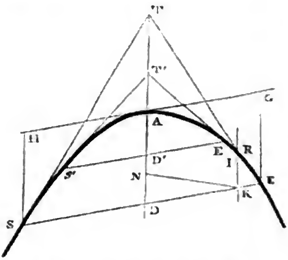
For, CE is parallel to GH the tangent at the extremity I of the diameter IK; and hence the diameter IK and the tangent GH at I being conjugate, the chord CE, which is parallel to GH, is also conjugate to IK.

Scholium.

The property itself has been often attributed to Dr. Halley, but it is found in the *Conics* of Apollonius. The uses made of it in the demonstration of theorems are numerous; but its chief modern interest is the aid which it affords in tracing the *courbes de raccordement*, according to the methods of Prony and Brianchon.

PROP. XVIII.

The area of any parabolic segment is two-thirds of the area of its conjugate parallelogram.



Let SAE be a segment of the parabola, GH the tangent at A parallel to SE, and AD the diameter through A; and draw SH, EG parallel to AD: then the area of the segment SAE is two-thirds of the parallelogram SHGE.

For, draw another ordinate E'D'S' to the diameter AD, and the tangents at E and E' meeting the diameter in T and T', and each other in R; through R draw RK parallel to AD, cutting EE' in I, and DE in K, and denote by KN the distance of the lines AD, KI.

Then, since ER, RE' are tangents at E and E' meeting in R, and RK is a diameter, the ordinate EE' is bisected in I by RK (*prop. vii.*); wherefore the trapezium $D'DEE' = DD'.KN$.

Again, since $AD = AT$, and $AD' = AT'$ (*prop. xi. cor.* 1), we have $DD' = TT'$; and hence the triangle $TRT' = \frac{1}{2}TT'.KN = \frac{1}{2}DD'.KN$. Wherefore trapezium $D'DEE' = 2$ triangle TRT'.

Now this conclusion is accurately true, however small the distance DD' may be, and applicable to every one of such inscribed trapezia, how many soever be taken in succession. If, then, we take DD' indefinitely small, and the trapezia so formed indefinitely numerous, so as to form by their assemblage a polygon inscribed in the parabola SAE; there will be formed a corresponding number of triangles, each of which is the half of the area of the corresponding trapezium, and hence the whole of which triangles will together be half the area of the corresponding trapezia.

Again, the limit towards which the sum of all the trapezia approximates, and the more closely as the distances DD' are diminished, and the chord EE' approximates to the arc of the curve EE', is the area of the segment of the parabola DAE; and the limit to which the sum of all the triangles approximates under the same circumstances is, the exterior space contained by the lines AT, TE, and parabolic arc AEE'. The former space, then, when DD' is diminished *sine limite*, is double the latter. But both taken together, are equal to the triangle DTE; wherefore, the segment DAE is two-thirds of the triangle DTE. In the same way, the segment DAS is two-thirds of the triangle DTS; and hence the segment SAE is two-thirds of the triangle STE.

Lastly, the triangle STE, and parallelogram SHGE, standing on the same base, but the base DT of the triangle being double AD of the parallelogram, the triangle and parallelogram are equal. Wherefore the parabolic segment is two-thirds of the parallelogram SHGE.

Corollary 1.

A parabolic segment cut off by any chord is two-thirds of the triangle having that chord for its base, and the tangents at the extremities of the chord for its sides.

This has been proved in the proposition, where it was shown that the segment SAE = $\frac{2}{3}$ triangle STE.

Corollary 2.

(*See fig. pr. xvii.*)

The triangle CIE is double the area of the triangle GHD.

For, (*cor.* 1,) 2 exterior space CGI = segment on chord CI;
2 exterior space IHE = segment on chord IE;
2 exterior space CDE = segment on chord CE;

whence,

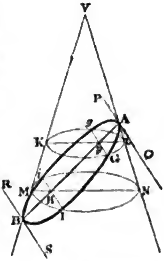
2 ext. sp. CDE—2 ext. sp. CGI—2 ext. sp. IHE=seg. CE—seg. CI—seg IE;
or 2 triangle GHD = triangle CIE.

THE ELLIPSE.

I. PROPERTIES OF LINES RELATED TO THE PRINCIPAL DIAMETERS.

PROP. I.

The squares of the ordinates of the transverse diameter are to one another as the rectangles of their abscisses.



Let AVB be the transverse plane; AGIB the elliptic sectional plane; AB the transverse axis; KGLg and MINi any two sections of the cone perpendicular to its axis, which also cut the sectional plane within the cone; and let KL, MN be the intersections of these planes with the transverse plane, and GFg, IHi those with the sectional plane.

Then, since KGLg, MINi are sections perpendicular to the axis of a right cone, they are circles; and since the transverse plane passes through the axis of the cone, it passes through the centres of these circles, and the lines KL, MN are diameters of the circles KGLg, MINi.

Again, since AVB passes through the axis of the right cone, it is perpendicular to the base and to each of its parallels, as to KGLg, MINi, *etc.*; and it is perpendicular to AGBg by definition (*def.* 5): wherefore, the two planes KGL, AGB being each perpendicular to AVB, their common section Gg is perpendicular to AVB, and hence to every line, as AB, KL situated in it.

Now KL being a diameter of the circle KGLg and Gg perpendicular to it, this line Gg will be bisected in F. In the same manner, Ii is perpendicular both to AB and MN, and is therefore also bisected in H.

The tangent planes to the cone at AV, BV, and likewise the sectional plane AGB, being perpendicular to the transverse plane AVB, their intersections PQ, RS are also perpendicular to AVB; and hence also to the line AB in it. Wherefore, in the sectional plane AGB, the lines PQ, RS, Gg, Ii being perpendicular to AB in it, are all parallel to one another.

But QVP, being a tangent plane to the cone, touches it only in the straight line AV; and hence the straight line PAQ meets the cone only in the point A, and therefore also meets the section only in that point. It is, therefore, the tangent at A, the vertex of the transverse diameter; and hence, again, G*g*, I*i* are ordinal chords to that diameter, and FG, HI are ordinates.

Now by similar triangles, AFL, AHN, and BFK, BHM, we have

$$AF : AH :: FL : HN,$$
$$FB : HB :: KF : MH;$$

hence, compounding, $AF \, . \, FB : AH \, . \, HB :: KF \, . \, FL : MH \, . \, HN$.

But by the properties of the circles KGL, MIN, we have

$$KF \, . \, FL = FG^2, \text{ and } MH \, . \, HN = HI^2;$$

wherefore, $AF \, . \, FB : AH \, . \, HB :: FG^2 : HI^2$.

Corollary 1.

The transverse diameter AB bisects its conjugate chords G*g*, I*i*; and the curve is composed of two equal branches symmetrically situated on each side of the transverse diameter. The term diameter is, therefore, properly applied to the line AB.

Corollary 2.

The tangents at the vertices of the transverse diameter, the ordinates of that diameter, and its conjugate diameter, are all perpendicular to the transverse axis.

Corollary 3.

The ellipse is entirely situated upon a single sheet of the cone, and composed of one single and continuous curve.

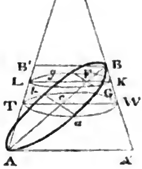
For, since the directing plane separates the two sheets, and every plane parallel to the directing plane must lie wholly on one side of that plane, the sectional plane cannot cut the sheet of cone situated on the other side of the directing plane. The sectional plane, therefore, cuts but one sheet of the cone, and the curve is situated entirely upon it.

Also, since the generatrices of the cone take continuous positions, and there is no one of them not cut by the directing plane in the vertex of the cone, they must all be cut again by the sectional plane, and the points in which it cuts them must form a continuous curve.

PROP. II.

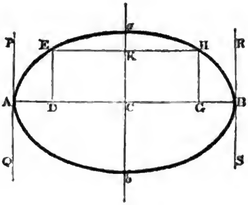
The square of the transverse diameter is to the square of its conjugate as the rectangle of the abscisses to the square of their ordinate.

CONCEIVE AB of the preceding figure to be bisected in C, as in that annexed, and a circular section to be drawn cutting the transverse plane in TW, and the sectional plane in *ab*. Also, let AA′, BB′ be the diameters of the circular sections at A, B. Then B′B = 2TC, and AA′ = 2CW. Wherefore *ab* is a mean proportional between AA′ and B′B, and is therefore the conjugate diameter of the ellipse (*def.* 8). Wherefore, as in the former proposition, and alternately,



$$\begin{aligned} AC \,.\, CB : TC \,.\, CW &:: AF \,.\, FB : KF \,.\, FL; \text{ or}\\ AC^2 : \quad Ca^2 &:: AF \,.\, FB : FG^2; \text{ or again,}\\ AB^2 : \quad ab^2 &:: AF \,.\, FB : FG^2. \end{aligned}$$

Corollary 1 [*].



Ordinates equidistant from the centre of the ellipse are equal: that is, if $CD = CG$, then $DE = GH$.

For $AD \,.\, DB : AG \,.\, GB :: DE^2 : GH^2$; also $AD = AC - CD = CB - CG = GB$; and $DB = DC + CB = GC + CA = AG$; wherefore $AD \,.\, DB = AG \,.\, GB$; and hence $DE^2 = GH^2$, or $DE = GH$.

Corollary 2.

Join EH: then it is parallel to the transverse AB, and perpendicular to the conjugate ab, and bisected by the latter in K. Whence also, the ellipse is divided by ab into two branches symmetrical with respect to ab; and ab is a diameter of the ellipse.

Corollary 3.

Ca is greater than any ordinate on either side of it. For by the proposition,

$$AC^2 : Ca^2 :: AC^2 - CD^2 \; (= AD \,.\, DB) : DE^2;$$

and since the first term AC^2 is greater than the third, $AC^2 - CD^2$, the second Ca^2 is greater than the fourth DE^2. Wherefore Ca is greater than DE.

Corollary 4.

Draw a line through a, parallel to the transverse AB, it will be a tangent to the ellipse at a.

For if not, let it cut the ellipse, as at H. Then, by the preceding corollary aC is greater than GH, and hence aH is not parallel to AB; but it is parallel by construction, which is impossible. The parallel through a does not, therefore, cut the curve in any other point H, and hence it is a tangent at a. In the same manner, a line through b parallel to AB will be a tangent to the ellipse.

Corollary 5.

The tangents at the vertices a, b, of the conjugate diameter being parallel to AB, and ab being perpendicular to AB and to EH; the line EH is an ordinal chord to ab, and being bisected in K, EK, and KH, are ordinates to the diameter ab.

Corollary 6.

The axes AB, ab, are conjugate diameters to each other; and the tangents at their vertices constitute the conjugate rectangle (*def.* 21).

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[*] A number of curious investigations connected with this part of the subject, demonstrated with the rigour of the ancients, but on a principle peculiar to the author, are given by the Duke of Somerset, in his *Treatise on the Elementary Properties of the Ellipse*, 1842. The same properties may, again, also be deduced from the corresponding properties of the circle, by means of an oblique section of the right cylinder standing upon the circle.

PROP. III.

The transverse diameter is to its parameter as the rectangle of the abscisses to the square of their ordinate.

(See fig. to corollary 1, *prop. ii.)*

For, by the definition (17), p denoting the parameter,

$$AB : ab :: ab : p;$$

and hence AB has to p the duplicate ratio of AB to ab; hence, and (*prop. ii.*)

$$AB : p :: AB^2 : ab^2 :: AD \, . \, DB : DE^2.$$

PROP. IV.

The square of the semi-conjugate diameter is to the square of the transverse as the rectangle of the abscisses of the conjugate is to the square of their ordinate.

(See fig. to corollary 1, *prop. ii.)*

For let EK be the ordinate to the point E referred to the conjugate axis ab, and let C be the centre of the ellipse; and draw ED an ordinate to the axis AB. Then (*prop. ii.*)

$$Ca^2 : CA^2 :: DE^2 \; (= CK^2) : AD \, . \, DB \; (= CA^2 - CD^2).$$

Whence $Ca^2 : CA^2 :: Ca^2 - CK^2 \; (= aK \, . \, Kb) : CD^2 \; (= KE^2)$;

which, converted into words, is the proposition enunciated.

Corollary 1.

The square of the semi-conjugate diameter is to the square of the transverse as the difference between the squares of the semi-conjugate and the distance of the ordinate from the centre, to the square of the ordinate.

This form of enunciation preserves the analogy to the hyperbola unbroken as to general properties.

Corollary 2.

If the ellipse and circle be conceived to be generated by the motion of a line perpendicular to the transverse axis, and by points whose distances from that axis are always in the given ratio of the diameters of the ellipse, the spaces which are simultaneously described by the lines contained between the transverse diameter and those points, will also be in the same constant ratio. Whence

The area of the whole ellipse is to the area of the circle as the conjugate to the transverse diameter.

The areas of any two corresponding portions of the ellipse and hyperbola are in the same ratio.

The same entire or partial areas are as the square of the diameter of the circle to the rectangle under the two axes of the ellipse.

The areas of the two circles described as in the proposition are as the squares of the axes of the ellipse.

The area of the ellipse is a mean proportional between those of the two circles.

Corollary 3.

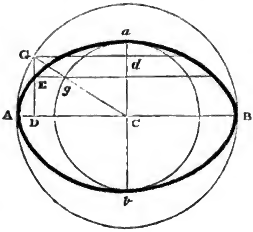
If the two axes become equal, the ellipse is reduced to a circle, showing that the circle is a simplified variety of the ellipse.

Corollary 4.

If two ellipses have the same line for their transverse or for their conjugate diameters, their ordinates to the same abscisses are always in the same ratio, viz. that of the other axes of the ellipses.

PROP. V.

1. *If two circles be described, one on each diameter of an ellipse, the one will lie wholly without, and the other wholly within, the ellipse.*
2. *An ordinate in either circle is to the corresponding ordinate in the ellipse, as the diameter to which this ordinate is drawn is to the other diameter.*



First. Let AB, *ab*, be the principal diameters, upon which, as diameters, let the circles AGB, *agb*, be described; these will lie, in the first without, and the second within, the ellipse.

For, draw the ordinates ED, E*d*, from any point E in the ellipse to cut the circles in G and *g*, and the axes in D and *d*.

Then $AD . DB : DE^2 :: AC^2 : Ca^2$, and hence AD . DB is greater than DE^2 in the ellipse: but $AD . DB = DG^2$ in the circle; wherefore DG is greater than DE. In the same manner, it may be proved that any other ordinate in the circle AGB is greater than the corresponding ordinate of the ellipse AEB: wherefore, all the points of the circle AGB lie farther from the axis AB than those of the ellipse AEB (except A and B); and hence the circle AGB lies wholly without the ellipse.

Again, $ad . db : dE^2 :: Ca^2 : AC^2$, and hence *ad* . *db* is less than dE^2 in the ellipse: but $ad . db = dg^2$ in the circle; wherefore *dg* is less than *de*. And by reasoning analogously to the preceding paragraph, the circle *agb* lies wholly within the ellipse, the points *a* and *b* excepted.

Secondly. The ordinates of the ellipse are to those of either circle as the diameters of the ellipse themselves; that is,

$$AC : Ca :: DG : DE, \text{ and } aC : CA :: dg : dE.$$

For by the circle, $AD . DB = DG^2$: whence, by the ellipse, we have

$$AC^2 : Ca^2 :: AD . DB \; (= DG^2) : DE^2, \text{ or } AC : Ca :: DG : DE.$$

And, in a similar manner, we get $aC : CA :: dg : dE$.

Corollary.

The points G, *g*, C are in one straight line.

For draw GC, *g*C, and G*g*; then,

$$\text{since } AC : Ca :: DG : DE, \text{ and } aC : CA :: dg : dE, \text{ we get}$$

$$DG : DE :: dE : dg, \text{ or } DG : DC :: Cd : dg.$$

Now (*constr.*) E*gd* is a straight line parallel to CD, and hence the triangles GDC, *gd*C, having the angles at D and *d* equal, and the sides about these proportional, are similar. Hence the angles DGC, *d*C*g* are equal.

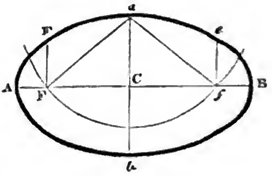
Again, $DG : DC :: DG - Cd : DC - dg :: GE : Eg$, and the angles GE*g*, GDC, are equal, by the parallels E*gd* and DC; therefore GE*g*, GDC, are similar triangles, and the angles DCG, E*g*G, are equal.

Whence also the angles E*g*G, *dg*C, made by G*g*, C*g*, at the point *g* in the straight line E*gd* are equal, and hence G, *g*, C, are in one line.

PROP. VI.

The square of the distance of the focus from the centre is equal to the difference of the squares of the semi-diameters; or, the square of the distance of the foci is equal to the difference of the squares of the two diameters.

That is, in reference to CF and fF,
$\mathrm{CF}^2 = \mathrm{CA}^2 - \mathrm{C}a^2$; or $\mathrm{F}f^2 = \mathrm{AB}^2 - ab^2$.



For to the focus F draw the ordinate FE, which will be the semi-parameter. Then, (*prop. ii.*,)
$\mathrm{CA}^2 : \mathrm{C}a^2 :: \mathrm{CA}^2 - \mathrm{CF}^2 : \mathrm{FE}^2$; and (*def.* 17)
$\mathrm{CA}^2 : \mathrm{C}a^2 :: \mathrm{C}a^2 \quad : \mathrm{FE}^2$. Whence,
$\mathrm{C}a^2 = \mathrm{CA}^2 - \mathrm{CF}^2$, or $\mathrm{CF}^2 = \mathrm{CA}^2 - \mathrm{C}a^2$.
Also, doubling the lines, $\mathrm{F}f^2 = \mathrm{AB}^2 - ab^2$.

Corollary 1.

If the two semi-axes and the focal distance be made to constitute a triangle, it will be rectangular.

Corollary 2.

The distance of either vertex of the conjugate diameter from the foci is equal to the semi-transverse diameter.

Corollary 3.

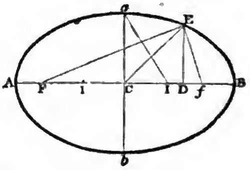
The semi-conjugate Ca is a mean proportional between AF, FB, or between Af, fB, the distances of either focus, F or f, from the vertices A and B of the transverse.

$$\text{For } \mathrm{C}a^2 = \mathrm{CA}^2 - \mathrm{CF}^2 = (\mathrm{CA} + \mathrm{CF})\,(\mathrm{CA} - \mathrm{CF}) = \mathrm{AF}\,.\,\mathrm{FB},$$
$$\text{and } \mathrm{C}a^2 = \mathrm{CB}^2 - \mathrm{C}f^2 = (\mathrm{CB} + \mathrm{C}f)\,(\mathrm{CB} - \mathrm{C}f) = \mathrm{A}f\,.\,f\mathrm{B}.$$

PROP. VII.

If lines be drawn from the foci and centre of an ellipse to any point in the curve:
(1). *That drawn to the centre is equal to the distance of the vertex of the conjugate diameter from the dividing point.*
(2, 3). *Those drawn from the foci are equal to the distances of the corresponding vertices of the transverse from the dividing point.*
(4, 5). *The sum of those drawn from the foci is equal to the transverse diameter, and their difference to the difference of the segments of the transverse, made by the dividing point.*

That is, if I be the dividing point correspondent to E in the curve, and make CI′ = CI: then



(1). $a\mathrm{I} = \mathrm{CE}$
(2). $\mathrm{EF} = \mathrm{AI}$
(3). $\mathrm{E}f = \mathrm{BI}$.
(4). $\mathrm{FE} + \mathrm{E}f = \mathrm{AB}$
(5). $\mathrm{FE} - \mathrm{E}f = \mathrm{II}'$.

(1). For (*def.* 23), $\mathrm{CB}^2 : \mathrm{C}f^2 :: \mathrm{CD}^2 : \mathrm{CI}^2$, or

$\mathrm{CB}^2 : \mathrm{CB}^2 - \mathrm{C}f^2\ (= \mathrm{C}a^2) :: \mathrm{CD}^2 : \mathrm{CD}^2 - \mathrm{CI}^2$; but (*pr. ii.*)
$\mathrm{CB}^2 : \mathrm{C}a^2 :: \mathrm{CB}^2 - \mathrm{CD}^2 : \mathrm{DE}^2$; whence, *ex eq.* and *comp.*

$$\mathrm{CB}^2 : \mathrm{C}a^2 :: \mathrm{CB}^2 : \mathrm{CD}^2 - \mathrm{CI}^2 + \mathrm{DE}^2$$
$$:: \mathrm{CB}^2 : \mathrm{CE}^2 - \mathrm{CI}^2.$$

Whence $\mathrm{C}a^2 = \mathrm{CE}^2 - \mathrm{CI}^2$, or $\mathrm{CE}^2 = \mathrm{CI}^2 + \mathrm{C}a^2 = a\mathrm{I}^2$;
and, therefore, $\mathrm{CE} = a\mathrm{I}$.

(2). Again, (*Euc. ii.* 12,) $\mathrm{FE}^2 = \mathrm{FC}^2 + \mathrm{CE}^2 + 2\mathrm{FC}\,.\,\mathrm{CD}$.

But $\mathrm{CE}^2 = a\mathrm{C}^2 + \mathrm{CI}^2$ (*preceding case*), and $2\mathrm{FC}\,.\,\mathrm{CD} = 2\mathrm{CB}\,.\,\mathrm{CI} = 2\mathrm{AC}\,.\,\mathrm{CI}$ (*def.* 23); whence we have

$$\mathrm{FE}^2 = \mathrm{FC}^2 + \mathrm{C}a^2 + \mathrm{CI}^2 + 2\mathrm{AC}\,.\,\mathrm{CI}, \text{ or } (\textit{pr. vi.})$$
$$= \mathrm{AC}^2 + 2\mathrm{AC}\,.\,\mathrm{CI} + \mathrm{CI}^2 = (\mathrm{AC} + \mathrm{CI})^2 = \mathrm{AI}^2.$$

Wherefore, we have $\mathrm{FE} = \mathrm{AI}$.

(3). Also, (*Euc. ii.* 13,) $f\mathrm{E}^2 = f\mathrm{C}^2 + \mathrm{CE}^2 - 2f\mathrm{C}\,.\,\mathrm{CD}$

$$= f\mathrm{C}^2 + \mathrm{CI}^2 + \mathrm{C}a^2 - 2f\mathrm{C}\,.\,\mathrm{CD}, \text{ or } (\textit{pr. vi.})$$
$$= \mathrm{CA}^2 - 2f\mathrm{C}\,.\,\mathrm{CD} + \mathrm{CI}^2, \text{ or } (\textit{def. } 23)$$
$$= \mathrm{CA}^2 - 2\mathrm{AC}\,.\,\mathrm{CI} + \mathrm{CI}^2,$$
$$= (\mathrm{CA} - \mathrm{CI})^2 = (\mathrm{CB} - \mathrm{CI})^2 = \mathrm{BI}^2.$$

Wherefore, $f\mathrm{E}^2 = \mathrm{BI}^2$; or $\mathrm{E}f = \mathrm{BI}$.

(4). From (2, 3) we have

$$\mathrm{FE} + \mathrm{E}f = \mathrm{AI} + \mathrm{IB} = \mathrm{AB}.$$

(5). Make $\mathrm{CI}' = \mathrm{CI}$: then we have

$$\mathrm{FE} - \mathrm{E}f = \mathrm{AI} - \mathrm{IB} = 2\mathrm{IC} = \mathrm{II}'.$$

Corollary 1.

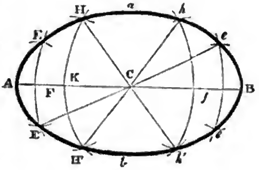
The square of the line drawn from the centre of an ellipse to any point in the curve is equal to the sum of the squares of the semi-conjugate diameter and distance of the dividing point from the centre; that is, $\mathrm{CI}^2 = \mathrm{CE}^2 - \mathrm{C}a^2$.

In this form the analogy with the corresponding property of the hyperbola is maintained.

Corollary 2.

From (2, 3) is derived the common method of tracing an ellipse, when small, and upon paper, by points; and when larger, as in the field or garden, by means of a string or cord.

(1). In the transverse diameter AB, take the foci F, *f*, and any convenient number of points in AB, of which I is one: with both centres F, *f*, and both radii AI, IB, describe arcs intersecting in E, E′, *e*, *e*′: these will be four points in the ellipse. Proceed similarly with a new dividing point K, and obtain the four points H, *h*, H′, *h*: and when a sufficient number of points have been thus found, trace the curve carefully through them by hand, or by an elastic ruler, capable of being so bent as to pass through each succeeding three of the points.



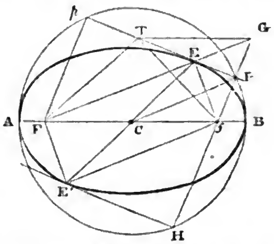
(2). With a cord whose length is equal to the transverse axis, having its ends fastened in F and *f*, kept always stretched, in passing a marker along, the point E will trace the curve.

Corollary 3.

Suppose a point H′, to be taken *below* AB, at the same distance from F that *h* is from *f*, and it will follow that $\mathrm{H}'f = h\mathrm{F}$, and the figure H′F *hf* will be a parallelogram; and hence the diagonals Hh' and $H'h$ would bisect each other. That is, Hh' is a diameter, it passing through the middle C of Ff, or of the diameter AB, and is bisected in that point.

PROP. VIII.

If the exterior angle at a point in the curve made by producing one of the focal lines be bisected, the bisecting line will be a tangent to the ellipse at the point.



Let the focal line FE be produced to G, and the angle GEf be bisected by EP: then EP will be a tangent at E.

For if not, let it meet the curve again in some point T: take $EG = Ef$, and join FT, fT, TG, and fG, meeting ET in P.

Then since the vertical angle fEG of the isosceles triangle fEG is bisected, the bisecting line EP bisects the base fG in P at right angles. Hence the triangle fTG is also isosceles, having $fT = TG$, and $FT + Tf = FT + TG$. It follows, therefore, that $FT + Tf$, which is equal to $FT + TG$, is greater than FG, that is, than $FE + EG$, or again, than $FE + Ef$. But since T is a point in the curve, $FT + Tf = FE + Ef$: which is impossible. The point T is, therefore, not in the ellipse: and it may be shown in the same way, that no other point besides E is in the curve; and therefore TP is a tangent at E.

Corollary 1.

The tangents EP, E′H, at the opposite extremities E, E′, of any diameter EE′, are parallel to one another: and parallel tangents are at the opposite extremities of the same diameter.

For the diameter EE′ is bisected in C (*pr. vii., cor.* 3), and likewise Ff is bisected in C: hence $FEfE'$ is a parallelogram, and the angles FEf, $FE'f$ are equal. Wherefore, $TEE' = TEF + FEE' = HE'f + fE'E = HE'E$; and the alternate angles TEE′, HE′E, being equal, TE is parallel to HE′. The converse follows, *ex absurdo*.

Corollary 2.

The perpendiculars fP, FP′, from the foci, intersect the tangents drawn to all points of the curve, in a circle on the transverse diameter AB.

For, draw CP. Then, since Ff and fG are bisected in C and P respectively, CP is parallel to FG, and FfG, CfP are similar triangles. Wherefore also

$$Ff : fC :: FG\ (= AB = 2CB) : CP :: 2 : 1.$$

$$\text{Whence } CP = \tfrac{1}{2}AB = CB.$$

Now C being a given point, and CB a given line, the point P is in a circle on AB.

In the same way we may show that P′ is in the same circle.

Corollary 3.

The rectangle under the perpendiculars from the foci upon any tangent is equal to the square of the semi-conjugate diameter.

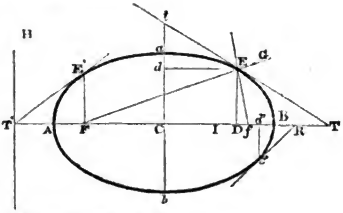
For let FP′ and *f*P be those perpendiculars upon the tangent at E; and produce P*f* to meet the circle on AB in H.

Then $Pf . fH = Af . fB$.

Again, since the tangent at E′ is parallel to P′P, the perpendicular *f*H from *f* upon it will coincide with *f*P; and the intersection of the tangent at E′ with the perpendicular from *f* will be in the circle on AB. The point H, is, therefore, that intersection. Hence the two triangles FEP′, *f*E′H are equal in all respects, and $fH = FP'$. Hence the condition $Pf . fH = Af . fB$ becomes $Pf . P'F = Af . fB = Ca^2$.

PROP. IX.

If a tangent and an ordinate be drawn from any point in the curve to meet the transverse axis, the semi-transverse will be a mean proportional between the distances of the two intersections from the centre.



THAT is, if E be the point from which the tangent ET and ordinate ED be drawn to the transverse diameter,

CT : CB :: CB : CD.

For draw FE, and produce it to G, and draw *f*E: also let I be the dividing point.

Then, (*Euc. vi. A*,) FT : T*f* :: FE : E*f*; that is,

TC + C*f* : TC − C*f* :: FE : E*f*; and *comp. et divid.*

CT : C*f* :: BC : CI, or CT : CB :: C*f* : CI, or (*def.* 23),
:: CB : CD.

Corollary 1.

If follows, that if *any* ellipse have AB for its transverse diameter, and D for the intersecting point of the ordinate, then the tangent at the point in which the ordinate from D cuts the ellipse, will always pass through T.

For in all cases, two of the lines CD, CB, being given, the third proportional CT is fixed in magnitude.

Whence, the circle being a variety of the ellipse, (the foci coalescing in the centre, and the conjugate diameters equal) we are enabled to construct a tangent to the ellipse from a point T in its transverse axis.

For describe the circle on AB, and from T draw the tangent TE′: and then the ordinate at E′ to cut the ellipse in E. The line TE is a tangent to the ellipse.

Corollary 2.

The transverse diameter AB is harmonically divided (*vol.* i. *p.* 340) in C and T.

For since CT : CB :: CB : CD, we have, *comp. et divid.*

CT + CB : CT − CB :: CB + CD : CB − CD, that is
AD : DB :: AT : TB.

Corollary 3.

$AD . DB = CD . DT$.

For each is equal to $CB^2 - CD^2$, the former by *Euc. ii.* 5, and the latter follows from the proposition.

Corollary 4.

Let another tangent e'R and ordinate $e'd'$ be drawn: then

$$\mathrm{C}d' : \mathrm{CD} :: \mathrm{CT} : \mathrm{CR}.$$

For as before, $\mathrm{C}d' . \mathrm{CR} = \mathrm{CB}^2 = \mathrm{CD} . \mathrm{CT}$; wherefore, *etc.*

Corollary 5.

$$\mathrm{BD}^2 : \mathrm{BT}^2 :: \mathrm{CD} : \mathrm{CT}.$$

For by the proposition CD : CB :: CB : CT, and hence, *comp.*

$$\mathrm{CB} - \mathrm{CD} : \mathrm{CT} - \mathrm{CB} :: \mathrm{CD} : \mathrm{CB};\ \text{that is, } \mathrm{BD} : \mathrm{BT} :: \mathrm{CD} : \mathrm{CB}, \text{ or}$$

$$\mathrm{BD}^2 : \mathrm{BT}^2 :: \mathrm{CD}^2 : \mathrm{CB}^2 :: \mathrm{CD}^2 : \mathrm{CD} . \mathrm{CT} :: \mathrm{CD} : \mathrm{CT}.$$

Corollary 6.

If F be the focus, FE′ the ordinate, and E′T′ the tangent at E′, and if T′H be drawn perpendicular to the transverse axis, it will be the directrix (*def.* 24): whence by the proposition, the semi-transverse is a mean proportional between the distances of the centre from the focus and the directrix.

Corollary 7.

Let the tangent meet the conjugate axis at t, and an ordinate also be drawn to meet it at d. Then

1. $\mathrm{C}d : \mathrm{C}a :: \mathrm{C}a : \mathrm{C}t.$
2. $da^2 : at^2 :: \mathrm{C}d : \mathrm{C}t.$
3. $bd : da :: bt : ta.$
4. $bd . da = \mathrm{C}d . dt.$

For these are related to the conjugate diameter as the other magnitudes are to the transverse, and similar reasoning evidently applies to both cases.

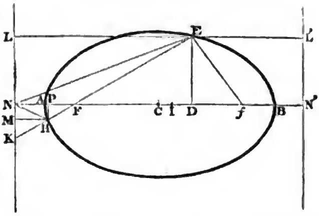
PROP. X.

*The distance of the directrix from a point in the ellipse has to the distance of the focus from the same point the same ratio, wherever in the curve that point be taken**.

That is, if NL be the directrix with respect to the focus, and N′L′ that with respect to f, we shall have

$$\mathrm{FE} : \mathrm{EL} :: f\mathrm{E} : \mathrm{EL}' :: \mathrm{CF} : \mathrm{CA};$$

that is, a constant ratio wherever E be taken in the curve.



For let I be the dividing point: then by the property of the directrix (*pr.* ix. *cor.* 6, and *def.* 23),

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* This constant ratio is that defined at page 107 as the *determining ratio*. The property itself was known to the ancients, and was probably employed in discussing the conic sections; as it is accidentally mentioned by Pappus, but not made further use of, his *Collections* not including any systematic exposition of the properties of conics.

Boscovich, about a century ago (1744) made it the basis of a treatise on these curves; and about half a century later, the Rev. Thomas Newton, of Cambridge, published a very elegant little treatise founded on the same principle. Several geometrical treatises, of greater or less merit, have since been founded upon it; the principal, and by far the most profound, is that of Sir John Leslie. The property has, also, been often used as the basis of the *co-ordinate methods* of treating the subject.

More recently, (since 1820,) the researches of Quetelet and Dandelin have given new interest to this subject, by deducing the property from the cone itself by the aid of the inscribed sphere. Their theorems will be briefly adverted to in a future page.

CF : CA :: AC : NC, and AC : NC :: CI : CD; or *comp. etc.*,
CF : CA :: AC + CI : NC + CD :: AI : ND :: FE : EL.

In like manner, C*f* : CB :: CF : CA :: *f*E : EL′.

Corollary 1.

If a line HE be drawn through the focus F, and the lines HN, EN be drawn, they will make equal angles with CN.

For, draw the ordinates ED, HP, to meet the axis in D and P, and draw HM, EL parallel to the axis meeting the directrix in M and L, and produce EH to K.

Then by the proposition and the sim. trians. PHF, DEF,

DN : PN :: EL : MH :: FE : FH :: ED : HP.

Hence the right-angled triangles DEN, PHN have the sides about the right angles proportional; and they are therefore equiangular, having END = HNP.

Corollary 2.

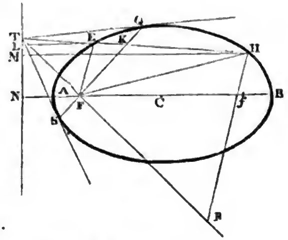
The chord HE is divided harmonically by the focus in F and K.

For by the similar triangles KHM, KEL, and by the *prop.*

KH : KE :: MH : EL :: HF : FE.

PROP. XI.

If two lines be drawn from any point in the directrix of an ellipse to cut the curve, one of which passes through the focus, and lines be drawn from that focus to the points where the other line cuts the curve: then these will make equal angles with the line drawn through the focus.



THAT is, if from T in the directrix NL a line TR be drawn through the focus F corresponding to the directrix NL, and any other line through T cut the curve in E and H; then the angles EFT, HFR will be equal.

For, draw EL, HM parallel to the axis, and HR parallel to EF meeting TF in R.

Then, by parallels EF, HR, and EL, HM, and *prop. x.*, we have

EF : HR :: TE : TH :: LE : MH :: EF : FH·

Whence HR = HF, and HFR = HRF = EFT.

Corollary 1.

If the secant TEH by revolving so as to bring the points E and H to coalescence, become a tangent at Q, the property still holds good, and the angles QFT, QFR become right angles. Whence,

If from a point T in the directrix a tangent TQ be drawn, and a line TF to the corresponding focus, then QF will be always perpendicular to FT.

In like manner, if the tangent TS be drawn, TFS will be a right angle.

Corollary 2.

If tangents TS, TQ be drawn from a point T in the directrix, the chord of contact SQ will pass through the focus.

For, (*cor.* 1,) SFT and TFQ are right angles; whence the points S, F, Q are in one line, or the chord SQ passes through F.

Corollary 3.

The angle EFH is bisected by the line QF; as is at once obvious.

Corollary 4.

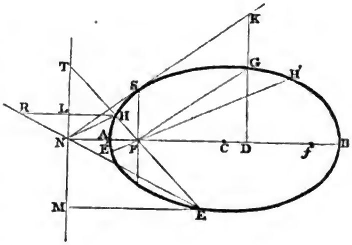
The chord EH is harmonically divided in K and T, by the chord QF and directrix NT.

For the interior angle EFH being bisected by FQ, the exterior one is bisected by FT, the perpendicular to FQ. Hence

$$HK : KE :: HF : FE :: HT : TE.$$

PROP. XII.

A chord drawn through the focus of an ellipse is so divided in that point, that four times the rectangle contained by its segments is equal to the rectangle contained by the chord and the parameter.



That is, $4HF \cdot FE = p \cdot HE$.

For, draw the lines HL, EM perpendicular to the directrix, and join EN, which produce to cut HL in R, and draw FS the focal ordinate, or semi-parameter.

Then, (*pr. x. cor.* 1,) $HNL = ENM = RNL$; hence $RL = LH$, or $RH = 2HL$.

By similar triangles RHE and NFE, we have

$$EH : EF :: RH : NF, \text{ or } NF \cdot EH = EF \cdot RH = 2HL \cdot EF.$$

And (*pr. x.*) $HL : HF :: NF : FS$, or $2HL \cdot FE : 2HF \cdot FE :: NF \cdot HE : FS \cdot HE$.

But since $2HL \cdot EF = NF \cdot HE$, we have $2HF \cdot FE = FS \cdot HE$, or doubling both, $4HF \cdot FE = 2FS \cdot HE = p \cdot HE$.

Corollary 1.

If another line H′E′ pass through the focus; then we shall have

$$HF \cdot FE : H'F \cdot FE' :: p \cdot HE :: H'E' : H'E'.$$

Corollary 2.

Draw the focal tangent NS to meet any ordinate DG in H, and join FG; then $FG = DK$.

For (*prop. xi.*) SF is perpendicular to FN, and therefore parallel to DK. Wherefore NFS, NDK are similar triangles, and we have

$$NF : FS :: ND : DK;$$

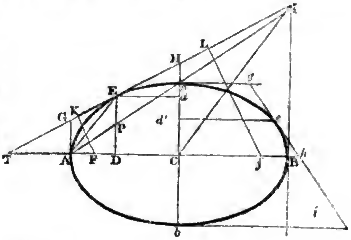
also (*prop. x.*) $NF : FS :: ND : FG$.

Hence, since the first three terms of each proportion are identical, the fourth terms of each must be equal; that is, $FG = DK$.

PROP. XIII.

If there be any tangent meeting four perpendiculars to the axis drawn from these four points, namely, the centre, the two extremities of the axis, and the point of contact; those four perpendiculars will be proportionals.

That is, AG : DE :: CH : BI.
For (*pr. ix.*), TC:CA::CA:CD,
or, *div.* TA:AD::TC:CA,
::TC:CB;
and *comp.* TA:TD::TC:TB;
and hence, by the similar triangles TAG, TDE, TCH, TBI,
AG : DE :: CH : BI.



Corollary 1.

TA : TD :: TC : TB, and TG : TE :: TH : TI.

They being homologous sides of the four triangles mentioned in the demonstration.

Corollary 2.

Draw AI to intersect DE in P: then DE is bisected in P.

For, by the *prop.* and preceding *corollary*,

TA : TC :: TD : TB :: TE : TI;

wherefore the triangles TAE, TCI are similar, and AE parallel to CI. Hence, again, the triangles AED, APD are similar, respectively, to CIB, AIB; and, consequently,

AD : DE :: CB : BI, and AD : DP :: AB : BI; and, therefore,
DE,: DP :: AB : CB :: 2 : 1, or DE = 2DP.

Corollary 3.

$$AG \,.\, BI = Ca^2.$$

For draw E*d* parallel to AB: then C*d* = DE. Also, by the *pr.* and *Euc. vi.* 16, ED . HC = C*d* . CH = Ca^2, (*prop. ix. cor.* 4.)

Whence we derive an elegant method of drawing a tangent at any point, E, in the curve of the ellipse: viz.

Bisect the ordinate ED in P; draw AP to meet the tangent BI in I; and, the line IE being drawn, it will be the tangent required.

Corollary 4.

Draw the perpendiculars FK, *f*L from the foci, then FK . *f*L = AG . BI.
For (*preceding cor.*), AG . BI = Ca^2 = FK . *f*L (*prop. viii. cor.* 3).

Corollary 5.

Let *ag*, *d'e*, C*h*, *bi*, be drawn correspondingly to the conjugate axes: then will the same mode of proof apply to establish the following conditions:

$ag \,.\, bi = Cb^2$; *ae* parallel to C*i*; and *d'e* bisected by *ai*.

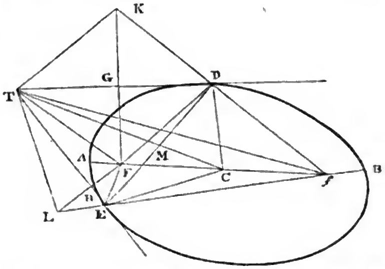
SECTION II.

THE PROPERTIES OF THE ELLIPSE DEPENDING UPON OBLIQUE CONJUGATE DIAMETERS.

PROP. XIV.

If from the foci and centre of an ellipse lines be drawn to the points of contact, and also the intersection of two tangents of the ellipse, then:

1. *The angle contained by the lines from the points of contact to either focus, is bisected by that drawn from the intersection of the tangents to the same focus.*
2. *The lines drawn from the intersection of the tangents to the two foci, make equal angles with those tangents.*
3. *The line drawn from the intersection of the tangents to the centre of the ellipse, bisects the chord which joins the points of contact.*



Let AB be the transverse diameter, F and *f* the foci, C the centre; and let DT, ET be the tangents meeting in T and touching the ellipse at D and E; draw EF, FD, FT, F*f*, TC, and ED, meeting TC in M: then

1. $\left\{\begin{array}{l} DFT = EFT \\ DfT = EfT \end{array}\right\}$
2. $\left\{\begin{array}{l} DTf = ETF \\ DTF = ETf \end{array}\right\}$
3. $DM = ME.$

First. Draw FG, FH perpendicular to the tangents DT, ET, and meeting *f*D, *f*E in K and L, and join TK, TL.

Then (*prop. viii.*), $FDG = KDG$, and the angles at G are equal (*constr.*), and the side DG common; hence $KD = DF$, and $Kf = Df + KD = Df + FD = AB$ (*pr. vii.*); and in the same way $Lf = AB$. Whence $fK = fL$.

Again, by the triangles FDG, KDG, we have $FG = GK$; and (*constr.*) the angles FGT, KGT are equal, and the side TG common to the triangles FGT, KGT. Therefore $KT = TF$; and similarly, $LT = TF$. Whence $LT = TK$.

In the two triangles T*Kf*, T*Lf*, we therefore have the sides TK, K*f*, equal to the sides TL, L*f*, each to each, and T*f* common to the two triangles; whence the angles at *f* are equal; that is, $KfT = LfT$, or $DfT = EfT$. In the same way, by drawing perpendiculars from *f* upon the tangents at D and E to meet FD, FE produced, it may be shown that $DFT = EFT$.

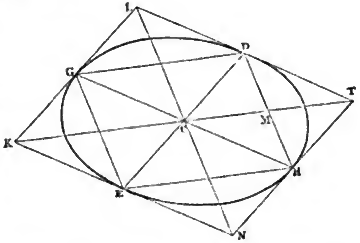
Second. Since the triangles TK*f*, TL*f* have been shown to be equal, we have the angles LT*f*, KT*f* equal; that is,

$$LTf = KTf, \text{ or } 2ETF + FTf = 2DTF - FTf; \text{ or}$$

$$DTF = FTE + FTf = ETf, \text{ and } FTE = DTF - FTf = DTf.$$

Third. Draw CD, CE; then the triangle TEC is composed of the three triangles TEF, FTC, and CEF, which are respectively equal to the three TEL, TC*f*, and *f*EC; and the whole six constitute the triangle TL*f*. Wherefore TEC is half the triangle TL*f*; and similarly, TCD is half the triangle TK*f*. But the triangles TK*f*, TL*f* have been already proved to be equal, and hence their halves TEC, TDC are also equal. Whence the triangles TEC, TDC standing on the same base and on opposite sides of it, the line DE, joining their vertices, is bisected in M by the base, or EM = MD.

Corollary 1.



If one diameter of an ellipse (*def.* 23) be conjugate to another, the second is conjugate to the first: that is, if GH be parallel to the tangent DT, at the extremity of DE; then DE will be parallel to the tangent at H, the extremity of GH.

For, join EH, HD, and CT cutting HD in M. Then DH is bisected in M, and DE in C. Hence CT is parallel to EH. Consequently, CEH = DCT, and because by hypothesis, CH is parallel to DT, TDC = HCE, and DC = CE, the triangles DCT, ECH are in all respects equal, and CT = EH and parallel to it. Hence (*Geom. theor.* 24, or *Euc. i.* 33), HT is equal and parallel to CD.

Corollary 2.

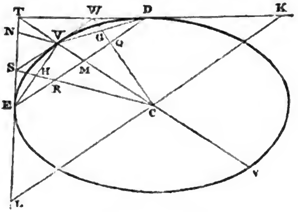
Hence any circumscribed *conjugate parallelogram* (*def.* 31) can be drawn when one of the points of contact is given: and it will be double the corresponding *inscribed conjugate parallelogram.* That the inscribed figure is a parallelogram, is evident, since GH, DE are bisected at their intersection C, (*prop. vi. cor.* 3.)

Corollary 3.

The diagonals of the conjugate parallelograms are conjugate to one another. Thus LN and KT are conjugate; they being each parallel to the tangents at the other's extremities.

PROP. XV.

1. *Every ordinal chord is bisected by its conjugate diameter.*
2. *The tangents to every ordinal chord meet in that diameter produced.*
3. *The semi-diameter is a mean proportional between the segments, estimated from the centre, cut off by the chord and tangent.*



Let WS be a tangent, and VA a diameter to the ellipse through the point V situated in the curve; and let any chord ED be parallel to WS, which, (*def.* 14,) is an ordinal chord. Then,

1. DE is bisected at M by the diameter VA.

2. The tangents at D and E meet at T in CV.

3. CM : CV :: CV : CT.

Let the tangents at D and E meet that at V in W and S; join SC, WC, cutting the chords DV, EV, in G and H, and the chord DE in Q and R. Draw, also, through C the chord LK to meet the tangents in K and L, and VN parallel to CS meeting ET in N.

First. The three lines WS, DE, LK, are parallel (*def.* 14, and *constr.*), and hence the triangles VWG, SVH, are respectively similar to DGQ and REH: and (*prop. xiv.*) VG = GD and VH = HE. Hence the triangles are also respectively equal in all respects, or VW = DQ, and SV = ER.

Again, the similar triangles LCS, RES, and the similar triangles CWK, QWD, respectively give,

$$CL : RE :: CS : SR :: CW : WQ :: CK : QD.$$

But (by *parallels* DE, LK, and *prop. xiv.*) we have LC = CK, and hence RE = QD; and, again, since RE = SV, and QD = VW, we have SV = VW.

Again, by similar triangles SCV, RCM, and VCW, MCQ, we have

$$SV : RM :: VC : CM :: VW : MQ.$$

Also, SV = VW; wherefore RM = MQ, and hence again, EM = MD.

Secondly. It was proved (*prop. xiv.*) that TC passes through M, the middle of ED, or that T, M, C, are in one line; and (*former case of xv.*) that C, M, V, are in one line: hence T and M are in CV: that is, the tangents at D and E meet in the same point T of the diameter DV produced.

Thirdly. Because SH is parallel to NV, and EH = HV, we have ES = SN. Then by the similar triangles TNV, TSC, and MET, VST, we have

$$CV : CT :: SN : ST :: SE : ST :: MV : VT, \text{ or } \textit{alt. et div.}$$

$$CV - MV : CV :: CT - VT : CT; \text{ that is, } CM : CV :: CV : CT.$$

Corollary 1.

The diameter AV is harmonically divided in M and T.

For (*com. et div.*), CV + CM : CV − CM :: CT + CV : CT − CV; that is, AM : MV :: AT : TV.

Corollary 2.

$$AM . MV = CM . MT.$$

For, $CV^2 - CM^2 = AM . MV$, and $CV^2 - CM^2 = CT . CM - CM^2 = CM\,(CT - CM) = CM . MT$. Whence, by equality, the result follows.

Corollary 3.

$$AM . MV : AT . TV :: CM^2 : CV^2.$$

For (*prop. xv.*) $CM^2 : CV^2 :: CV^2 : CT^2$; hence, *div. et alt.*,

$$CV^2 - CM^2 : CT^2 - CV^2 :: CM^2 : CV^2; \text{ or, as above stated.}$$

Corollary 4.

Every *ordinate* to a diameter is parallel to the tangent at its vertex.

Corollary 5.

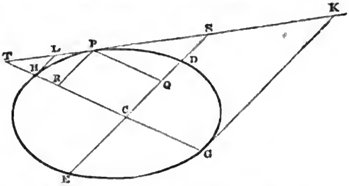
All ordinates to the same diameter are parallel to one another.

Corollary 6.

Any straight line which bisects two parallel chords is a diameter, and is conjugate to those chords, and any diameter which bisects a chord is conjugate to it.

PROP. XVI.

The square of any diameter has to the square of its conjugate the same ratio that the rectangle of the abscisses has to the square of the ordinate.



Let DE, GH be two conjugate diameters, and PQ an ordinate to the former, corresponding to the point P in the curve: then

$ED^2 : GH^2 :: EQ \cdot QD : PQ^2$.

Draw the tangent at P to meet the conjugate diameters in S and T. Then (*prop. xv.* and *Euc. vi.* 20, *cor.* 2), we have

$CQ : CD :: CD : CS$, or $CQ^2 : CD^2 :: CQ : CS$; and hence, *div.*,

$CD^2 - CQ^2 : CD^2 :: CS - CQ : CS$, or $CD^2 - CQ^2 : CD^2 :: QS : SC$.

Again (by *parallels* and *pr. xv.*) we have

$CD^2 - CQ^2 : CD^2 :: QS : SC :: PQ\ (= CR) : CT :: PQ^2 : CH^2$.

Hence $CD^2 : CH^2 :: CD^2 - CQ^2 : PQ^2$, or $ED^2 : GH^2 :: EQ \cdot QD : PQ^2$.

In the same manner, $GH^2 : ED^2 :: HR \cdot RG : RP^2$.

Corollary 1.

The rectangles of the abscisses of any diameter are as the squares of the corresponding ordinates: for each ratio is the same as that of the squares of the conjugate diameters.

Corollary 2.

Any diameter is to its parameter, as the rectangle of its abscisses to the square of the corresponding ordinate. It is shown as in *cor.* 1, *prop. i.*

Corollary 3.

Draw the tangents GK, HL at the extremities of the diameter GH; then $HL : PR :: CS : GK$ (see *dem.* of *prop. xi.*)

Corollary 4.

If *any* tangent LK be drawn to intersect two parallel tangents GK, HL, it will cut off segments HL, GK, whose rectangle is equal to the square of the semi-diameter CD parallel to them.

Corollary 5.

The tangent KL is harmonically divided in P and T.

Corollary 6.

In the same way as in *cor.* 1, *prop. xi.* it may be proved that HK and LG will bisect PR, and hence that the three lines pass through the same point.

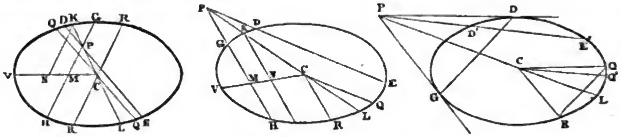
Corollary 7.

If, as in *prop. xiii. cor.* 3, the line HK were drawn, it would bisect PR at the point of intersection.

We are thus furnished with a more general method of drawing a tangent to the ellipse at the point P than that given in *prop. xiii.*, *cor.* 3.

PROP. XVII.

If any two straight lines which intersect each other and meet the ellipse be parallel to two fixed diameters, then the rectangles under the segments of the one, intercepted between the point of section of the lines and the two sections with the curve, will be to the rectangle under the segments of the other similarly estimated, as the square of the semi-diameter parallel to the first is to the square of that parallel to the second line.



Let the two lines DE, GH, parallel to the semi-diameters CQ, CR, intersect in the point P: then

$$DP.PE : GP.PH :: CQ^2 : CR^2.$$

For, draw the semi-diameter CV to bisect GH in M; also draw the diameter KL through P, and KN parallel to GH. Then (*pr. xvi. cor.* 6) the diameter CV is conjugate to the chord GH, and therefore to the diameter CR. Then,

$$CR^2 - KN^2 : CR^2 - GM^2 :: CN^2 : CM^2 \ (\textit{pr. xvi.})$$
$$GM^2 - KN^2 : CR^2 - KN^2 :: CN^2 - CM^2 : CN^2, \ \textit{dividendo},$$
$$:: KN^2 - PM^2 : KN^2, \text{ by sim. trians.}$$
$$GM^2 - PM^2 : CR^2 :: KN^2 - PM^2 : KN^2, \ \textit{alt. et comp.}$$
$$:: CK^2 - CP^2 : CK^2, \text{ sim. trians.}$$

But GH and KL are respectively bisected in M and C; and hence (*Euc. ii.* 5, *cor.*) $GM^2 - MP^2 = GP.PH$, and $KC^2 - CP^2 = KP.PL$. Wherefore,

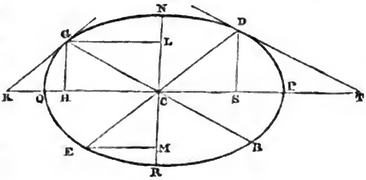
$$GP.PH : CR^2 :: KP.PL : CK^2.$$
$$\text{Similarly, } EP.PD : CQ^2 :: KP.PL : CK^2.$$
$$\text{Whence, } DP.PE : GP.PH :: CQ^2 : CR^2.$$

Corollary.

If one of the lines be a tangent, the rectangle of its segments becomes the square of the tangent; and if both be tangents, (*fig.* 3,) then these tangents have the same ratio as their parallel diameters, and GD is parallel to QR.

PROP. XVIII.

If from the extremities of any two conjugate diameters of an ellipse ordinates be drawn to any third diameter, the rectangle of the abscisses made by one of the ordinates is equal to the square of the distance from the centre (estimated on that diameter) of the other ordinate.



Let DE, GB be the conjugate diameters, and from the extremities D and G to any other diameter PQ let there be drawn the ordinates DS, GH: then $QH.HP = CS^2$, and $QS.SP = CH^2$.

For, let the tangents at D and G meet the diameter PQ in T and

K; then, by the pairs of similar triangles KGH, CDS, and GHC, SDT, we have

$$KH : CS :: KG : DC :: KC : CT.$$

Again, (*pr. xv.*) $KC . CH = CQ^2 = CS . CT$; and hence,

$$CS : CH :: KC : CT :: KH : CS, \text{ or } (\textit{pr. xv. cor. } 2),$$
$$CS^2 = KH . HC = QH . HP.$$

In precisely the same way the other case may be proved.

Corollary 1.

$$KC . CH = TC . CS.$$

Corollary 2.

$$CH . HK = CS^2, \text{ and } CS . ST = CH^2.$$

Corollary 3.

Let NR be the diameter conjugate to PQ, and draw the ordinates EM, GL to it; then

$$CQ^2 = CS^2 + CH^2, \text{ and } CR^2 = CL^2 + CM^2.$$

For, $CS^2 + CH^2 = CH . HK + CH^2 = CH (CH + HK) = CH . CK = CQ^2$, and similarly for the other case.

Corollary 4.

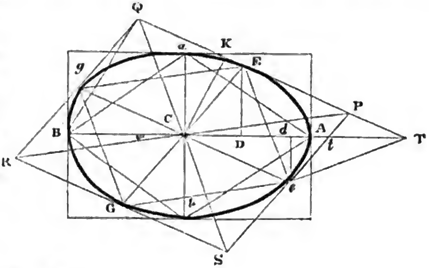
$$QC : CN :: CS : GH.$$

For (*props. xvii.* and *xviii.*) we have

$$QC^2 : CN^2 :: QH . HP (= CS^2) : GH^2; \text{ and hence } QC : CN :: CS : GH.$$

PROP. XIX.

1. *The sum of the squares of the pairs of conjugate diameters is constantly the same.*
2. *The areas of the conjugate parallelograms is constantly the same, whether circumscribed or inscribed.*



Let PQRS be a conjugate circumscribed parallelogram, and $EeGg$ the corresponding inscribed one, the diameters EG and eg being conjugate to one another; and let AB, ab be the transverse and conjugate diameters intersecting in the centre C. Also draw eT. Then,

1. $EG^2 + eg^2 = AB^2 + ab^2$.

2. $\begin{cases} \text{par}^{\text{m}}\ PQRS = AB . ab. \\ \text{par}^{\text{m}}\ EeGg\ = 2AC . Cb. \end{cases}$

For draw the ordinates ED, ed to the transverse, and CK perpendicular to one of the sides PQ of the circumscribed parallelogram; and let CA meet the sides QP, PS in T and t.

First, (*prop. xviii.*) $CA^2 = CD^2 + Cd^2$, and $Ca^2 = DE^2 + de^2$.

$$CA^2 + Ca^2 = CD^2 + DE^2 + Cd^2 + de^2 = CE^2 + Ce^2; \text{ or}$$
$$EG^2 + eg^2 = AB^2 + ab^2.$$

Whence, as above.

Secondly, (*prop. xviii. cor.* 5, and *prop. xv.*) we have

$$CA : Ca :: CD : de, \text{ and } CT : CA :: CA : CD; \text{ hence}$$
$$CA \,.\, CT : Ca \,.\, CA :: CD \,.\, CA : de \,.\, CD, \text{ or } CT \,.\, de = CA \,.\, Ca.$$

Now the rectangle contained by $CT \,.\, de$ is double of the triangle CTe, whose base is CT and altitude is de; also the parallelogram ePEC is double of the same triangle CTe which stands upon the same base Ce, and between the same parallels eC and PQ as the parallelogram does. The rectangle contained by CT and de being therefore equal both to the parallelogram ePEC and to the rectangle AC, Ca, they are equal to one another; as are also the quadruples of these, viz. the parallelogram PQRS, and the rectangle contained by the principal diameters AB and ab.

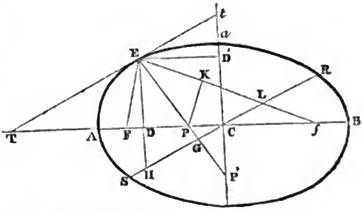
Again, the inscribed parallelograms are respectively the halves of the circumscribed ones; and hence the property is true also of them.

PROP. XX.

The central distance of the ordinate of a point on either of the principal diameters, is to the subnormal of that point upon the same diameter, as the square of that principal diameter is to the square of the other.

LET AB, ab be the principal diameters meeting in the centre C, AB being the transverse; let the normal at E cut them in P and P′, and the ordinates of E to them respectively, be ED and ED′; then DP and D′P′ are the respective subnormals, and

$$CD : DP :: AB^2 : ab^2,$$
$$\text{and } CD' : D'P' :: ab^2 : AB^2.$$



For (*prop. ii.*) $AD \,.\, DB : DE^2 :: AC^2 : Ca^2 :: AB^2 : ab^2$.

But (*prop. xviii. cor.* 2, and *right-angled triangles*) we have

$$AD \,.\, DB = CD \,.\, DT, \text{ and } DE^2 = PD \,.\, DT; \text{ wherefore,}$$
$$AB^2 : ab^2 :: CD \,.\, DT : PD \,.\, DT :: CD : DP.$$

In precisely the same way the other part of the proposition may be proved.

Corollary 1.

$$CF^2 : Ca^2 :: CP : PD.$$

For *div.* $AC^2 - Ca^2 : Ca^2 :: CD - PD : PD$; and (*prop. ii.*)

$$AC^2 - Ca^2 = CF^2, \text{ and } CD - PD = CP.$$

Corollary 2.

$AB : p :: DC : PD$, and $AB : p :: P'D' : D'C$ (*def.* 17, and *Euc. vi.* 21.)

Corollary 3.

From the preceding corollary, $PD \,.\, D'P' = DC \,.\, CD'$.

Corollary 4.

Draw the diameter SR parallel to the tangent TEt, cutting the normal in G: then PE . EG = Ca^2.

For produce ED to meet CG in H: then, since the angles at D and G are right angles, the triangles EDP, EGH are similar, and

$$\text{ED } (= \text{CD}') :: \text{EP} :: \text{EG} : \text{EH } (= \text{C}t), \text{ or EP . EG} = \text{CD}' . \text{C}t = \text{C}a^2.$$

Corollary 5.

Draw PK perpendicular to Ef (or EF); then EK $= \frac{1}{2}p$.

Let the diameter RS meet Ef in L, then EGL, EKP are similar; and

$$\text{EG} : \text{EL} :: \text{EK} : \text{EP, or LE . EK} = \text{EP . EG} = \text{C}a^2 \text{ (}cor.\text{ 4)}.$$

But EP bisects the angle FEf; whence EL $= \frac{1}{2}$(FE + Ef) = AC: whence we have

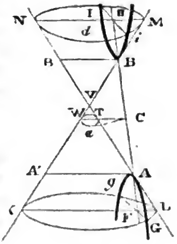
$$\text{AC . EK} = \text{C}a^2, \text{ or AC} : \text{C}a :: \text{C}a : \text{EK, and EK} = \tfrac{1}{2}p.$$

THE HYPERBOLA.

I. PROPERTIES OF LINES RELATED TO THE PRINCIPAL DIAMETERS.

PROP. I.

The squares of the ordinates of the transverse diameter are to one another as the rectangles of their abscisses.



Let AVB be the transverse plane; AGIB the hyperbolic sectional plane; AB the transverse axis; KGLg and MINi any two sections of the cone perpendicular to its axis, which also cut the sectional plane within the cone; and let KL, MN be the intersections of these planes with the transverse plane, and GFg, IHi those with the sectional plane.

Then, since KGLg, MINi are sections perpendicular to the axis of a right cone, they are circles; and since the transverse plane passes through the axis of the cone, it passes through the centres of these circles, and the lines KL, MN are diameters of the circles KGLg, MINi.

Again, since AVB passes through the axis of the right cone, it is perpendicular to the base and to each of its parallels, as to KGLg, MINi, *etc.*; and it is perpendicular to AGBg by definition (*def.* 5): wherefore, the two planes KGL, AGB being each perpendicular to AVB, their common section Gg is perpendicular to AVB, and hence to every line, as AB, KL situated in it.

Now KL being a diameter of the circle KGLg and Gg perpendicular to it, this line Gg will be bisected in F. In the same manner, Ii is perpendicular both to AB and MN, and is therefore also bisected in H.

The tangent planes to the cone at AV, BV, and likewise the sectional plane AGB, being perpendicular to the transverse plane AVB, their intersections PQ, RS are also perpendicular to AVB; and hence also to the line AB in it. Wherefore, in the sectional plane AGB, the lines PQ, RS, Gg, Ii being perpendicular to AB in it, are all parallel to one another.

But QVP, being a tangent plane to the cone, touches it only in the straight line AV; and hence the straight line PAQ meets the cone only in the point A, and therefore also meets the section only in that point. It is, therefore, the tangent at A, the vertex of the transverse diameter; and hence, again, G*g*, I*i* are ordinal chords to that diameter, and FG, HI are ordinates.

Now by similar triangles, AFL, AHN, and BFK, BHM, we have

$$AF : AH :: FL : HN,$$
$$FB : HB :: KF : MH;$$

hence, compounding, $AF . FB : AH . HB :: KF . FL : MH . HN$.

But by the properties of the circles KGL, MIN, we have

$$KF . FL = FG^2, \text{ and } MH . HN = HI^2;$$

wherefore, $AF . FB : AH . HB :: FG^2 : HI^2$.

Corollary 1.

The transverse diameter AB bisects its conjugate chords G*g*, I*i*; and the curve is composed of two equal branches symmetrically situated on each side of the transverse diameter. The term diameter, is, therefore, properly applied to the line AB.

Corollary 2.

The tangents at the vertices of the transverse diameter, the ordinates of that diameter, and its conjugate diameter, are all perpendicular to the transverse axis.

Corollary 3.

The complete hyperbola is situated upon both sheets of the cone, and is composed of two separate and infinitely extended branches.

For the directing plane cutting both sheets of cone, the sectional plane will cut the prolongations of all those generatrices of the lower sheet of the cone which lie on the opposite side of the directing plane from the sectional plane, and give the branch IB*i* in the upper sheet: whilst it will cut all those which lie on its own side of the directing plane in the lower sheet of the cone, giving the branch GA*g*. The complete curve is therefore composed of two separate branches, one upon each sheet of the cone.

Again, as the generatrix of the cone moves from the position VA on either side, as to G, I, *etc.*, the distances VG, VI, at which they are cut by the sectional plane, continually increase; till when it coincides with the edge in which the cone is cut by the directing plane, it becomes infinitely distant, the curve being continuous during this progress. The semi-branch AG is therefore infinite, and hence the entire branch GA*g* is infinite. In like manner, IB*i* is infinite in the upper sheet of the cone.

PROP. II.

The square of the transverse diameter is to the square of its conjugate as the rectangle of the abscisses is to the square of their ordinate.

CONCEIVE AB in the preceding figure to be bisected in C, and a circular section to be drawn cutting the transverse plane in TW, and draw the tangent C*t*. Then as in the former proposition, and alternately

$$AC . CB : TC . CW :: AF . FB : KF . FL; \text{ or}$$
$$AC^2 : \quad Ca^2 \quad :: AF . FB : FG^2; \text{ or again,}$$
$$AB^2 : \quad 4Ca^2 \quad :: AF . FB : FG^2.$$

Now 2C*a* is equal to the conjugate diameter, and hence $4Ca^2$ is its square: whence the proposition is established.

Scholium.

If the plane of the hyperbolas be parallel to the axis, it will cut the transverse plane AVB so that the triangle AVB will be isosceles, and hence the plane through C, the middle of its base, will pass through the vertex, and the line CV will be the conjugate semi-diameter, the circle W*a*T being then *reduced* to a point, and CT, CW, and C*a* all equal.

If now we suppose the two supplementary cones (*def.* 9) to have their axes passing through V and at right angles to the transverse plane AVB, the plane of the axes will be parallel to the plane of the hyperbolic sections.

Let us suppose, then, that a second plane parallel to the plane of the axes, and at a distance from V equal CA be made to cut the pair of conjugate cones. This will give another pair of opposite sections G′A′*g*′ and I′B′*i*′, which (by *def.* 10) are the hyperbolas conjugate to the original ones GA*g* and IB*i*: and it is necessary to show that the relation of these two pairs of hyperbolas is mutual, or that the former pair GA*g* and IB*i* are conjugate to G′A′*g*′ and I′B′*i*′.

Conceive the vertical transverse planes to be drawn to the two pairs of conjugate sections: then since these pass through the axes, they make in the conical surfaces linear sections, whose inclinations to the respective axes are equal to the generating conical angles. Also since these angles are (by *def.* 9) the complements of each other, and by construction the perpendicular CV in the right-angled triangle CVA is equal to the base C′V in the other C′V′A′, the triangles are equal, and the hypothenuse of the one VA is equal to the hypothenuse in the other VA′; and the base CA in the former equal to the perpendicular CA′ in the latter. The relation between the two diameters AB, A′B′ and the two distances CV and C′V is therefore the same, whichever be taken first. That is, if the section made through C′ be conjugate to that through C, the latter section will be in the same sense (*def.* 10) conjugate to the former one.

PROP. III.

The transverse diameter is to its parameter, as the rectangle of the abscisses to the square of their ordinate.

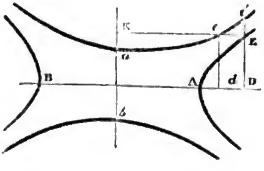
For by the definition (*def.* 17), p denoting the parameter and the notation for the diameters being the same as before,

$$AB : ab :: ab : p;$$

and hence AB has to p the duplicate ratio of AB to ab; hence *(prop. ii.)*

$$AB : p :: AB^2 : ab^2 :: AD.DB : DE^2;$$

which, in symbols, is tantamount to the enunciated proposition.



PROP. IV.

The square of the conjugate diameter is to the square of the transverse, as the sum of the square of the semi-conjugate together with the square of the distance of the ordinate from the centre, to the square of the ordinate of that point, referred to the conjugate axis. (See preceding figure.)

For let EK be the ordinate to the point E referred to the conjugate axis ab, and let C be the centre of the ellipse, and draw ED an ordinate to the axis AB. Then, *(prop. ii.)*

$$Ca^2 : CA^2 :: DE^2\ (= CK^2) : AD\,.\,DB\ (= CD^2 - CA^2),$$

$$\text{whence } Ca^2 : CA^2 :: Ca^2 + CK^2 : CD^2\ (= KE^2),$$

which, when the first and second terms are quadrupled, expresses the enunciated theorem.

Corollary 1.

If we consider *ab* to be the transverse and AB the conjugate diameter of the conjugate hyperbola *hae*; then in the same way we find that if D*e′* be an ordinate

$$CA^2 : Ca^2 :: CA^2 + CD^2 : De'^2.$$

Corollary 2.

$$DE^2 : De'^2 :: CD^2 - CA^2 : CD^2 + CA^2.$$

For *(pr. iii. iv.)* $CD^2 - CA^2 : DE^2 :: CA^2 : Ca^2 :: CA^2 + CD^2 : De'^2$, whence the conclusion follows.

Corollary 3.

Taking *ab* as the transverse diameter, and reasoning as above,

$$de^2 : dE^2 :: Cd^2 - Ca^2 : Cd^2 + Ca^2.$$

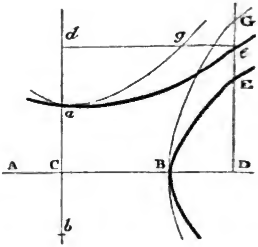
Corollary 4.

In the equilateral hyperbola the square of the ordinate is equal to the rectangle of the abscisses: for in this case the axes are equal (*def.* 10), or $AC = Ca$; and hence

$$AC^2 : Ca^2 :: AD\,.\,DB : DE^2 \text{ gives } AD\,.\,DB = DE^2.$$

PROP. V.

1. *If upon the two diameters of the hyperbola as diameters equilateral hyperbolas be described, the one pair will lie wholly without and the other wholly within the opposite hyperbolas which have these axes respectively as their transverses.*
2. *An ordinate in either equilateral hyperbola is to the corresponding ordinate of the first hyperbolas, as the diameter to which this ordinate is drawn is to the other diameter.*



First. Let AB, *ab* be the principal diameters of the given hyperbola, that is, the transverse diameters of BE, *ae*, upon which as diameters describe the equilateral hyperbolas BG, *ag*; these will lie, the first without and the second within the original hyperbolas BE and *ae*.

For draw the ordinates ED, *ed* from points E, *e* in the original hyperbolas, to cut the equilateral hyperbolas in G and *g*, and the axes in D and *d*. Suppose also AB greater than *ab*.

Then $AD\,.\,DB : DE^2 :: AC^2 : Ca^2$, and hence AD . DB is greater than DE² in the general hyperbola BE: but $AD\,.\,DB = DG^2$ in the equilateral hyperbola

BG; wherefore DG is greater than DE. In the same way it may be proved that any other ordinate in the equilateral hyperbola BG is greater than the corresponding ordinate in the general hyperbola BE: wherefore all the points of BG lie farther from the axis BD than the corresponding ones of BE (except B and A); and hence the equilateral hyperbola BG lies wholly without the general hyperbola BE.

Again, $ad \cdot db : de^2 :: Ca^2 : AC^2$; and hence $ad \cdot db$ in the general hyperbola is less than de^2: but $ad \cdot db = dg^2$ in the equilateral hyperbola; wherefore dg is less than de. And by reasoning analogously to the preceding paragraph, the equilateral hyperbola ag lies wholly within the general hyperbola ae, the points a and b excepted.

Secondly. The ordinates of the equilateral hyperbola are to those of either of the conjugate general hyperbolas as the diameters of the given hyperbolas themselves; that is

$$AC : Ca :: DG : DE, \text{ and } aC : CA :: dg : de.$$

For by the equilateral hyperbola, $AD \cdot DB = DG^2$; whence by the general hyperbola, $AC^2 : Ca^2 :: AD \cdot DB \; (= DG^2) : DE^2$, or $AC : Ca :: DG : DE$.

And in a similar manner we get $aC : CA :: dg : de$.

Corollary 1.

If the equilateral and general hyperbola be conceived to be generated by the motion of a line perpendicular to the transverse axis, and by points whose distances from that axis are always in the given ratio of the diameters of the general ellipse, the spaces which are simultaneously described by the lines contained between the transverse diameter and those points, will also be in the same constant ratio.

Corollary 2.

The two axes of the general hyperbola becoming equal reduces it to the equilateral one.

Corollary 3.

If two hyperbolas have the same line for their transverse or for their conjugate diameter, their ordinates to the same absciss are always in the same ratio, viz. that of the second principal axes of the hyperbolas.

PROP. VI.

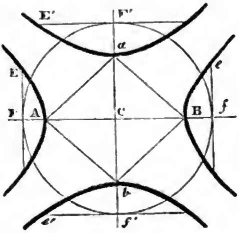
The square of the distance of either focus from the centre is equal to the sum of the squares of the semi-diameters; or, the square of the distance of the foci is equal to the sum of the squares of the two diameters.

THAT is, in reference to CF and fF, $CF^2 = CA^2 + Ca^2$; or $Ff^2 = AB^2 + ab^2$.

For to the focus F draw the ordinate FE, which (*def.* 18) will be the semi-parameter. Then (*prop. ii.*), $CA^2 : Ca^2 :: CF^2 - CA^2 : FE^2$; and (*def.* 17).

$CA^2 : Ca^2 :: Ca^2 : FE^2$. Whence, $CF^2 - CA^2 = Ca^2$, or $CF^2 = CA^2 + Ca^2$.

Also, doubling the lines, we have $Ff^2 = AB^2 + ab^2$.



Corollary 1.

If the two semi-axes and the focal distance be made to constitute a triangle, it will be rectangular.

Corollary 2.

The distance of either vertex of the conjugate diameter from either vertex of the transverse is equal to the focal distance of the centre.

Corollary 3.

The semi-conjugate Ca is a mean proportional between AF, FB, or between Af, fB, the respective distances of either focus F or f from either of the vertices A or B of the transverse.

For $Ca^2 = CF^2 - CA^2 = (CF + CA)(CF - CA) = AF \,.\, FB$,
and $Ca^2 = Cf^2 - CB^2 = (Cf + CB)(Cf - CB) = Af \,.\, fB$.

Corollary 4.

The foci of the four hyperbolas, which are composed of the conjugate pairs upon AB and ab as transverse, and ab and AB as conjugate axes, are equi-distant from the centre C.

For in both cases the proposition gives
$CF^2 = CA^2 + Ca^2$, and $CF'^2 = Ca^2 + CA^2$: whence $CF' = CF$,
and the four points F, f, F', f' are equidistant from C.

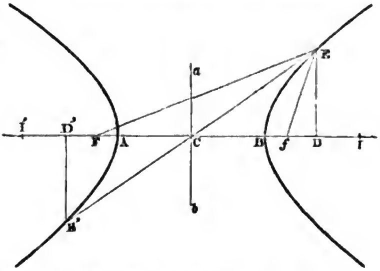
PROP. VII.

If lines be drawn from the foci and centre of an hyperbola to any point in the curve: then,

(1.) *The square of that drawn from the centre is equal to the difference of the squares of the semi-conjugate and the distance of the dividing point from the centre.*

(2, 3.) *Those drawn from the foci are equal to the distances of the corresponding vertices of the transverse from the dividing point.*

(4, 5.) *The difference of those drawn from the foci is equal to the transverse diameter, and their sum to the sum of the segments of the transverse made by the dividing point.*



That is, if I be the dividing point correspondent to E, and we make $CI' = CI$, then

(1.) $CE^2 = CI^2 - Ca^2$
(2.) $EF = AI$
(3.) $Ef = IB$
(4.) $EF - Ef = AB$
(5.) $EF + Ef = II'$.

(1.) For (*def.* 23)
$CB^2 : Cf^2 :: CD^2 : CI^2$, or
$CB^2 : Cf^2 - CB^2 (= Ca^2) :: CD^2 : CI^2 - CD^2$; but (*pr. ii.*)
$CB^2 : Ca^2 :: CD^2 - CB^2 : DE^2$; whence, *ex eq.* and *divid.*,
$CB^2 : Ca^2 :: CB^2 : CI^2 - CD^2 - DE^2$;
$:: CB^2 : CI^2 - CE^2$.

Whence $Ca^2 = CI^2 - CE^2$, or $CE^2 = CI^2 - Ca^2$.

(2). Again, (*Euc. ii.* 12,) $FE^2 = FC^2 + CE^2 + 2FC \cdot CD$.

But $CE^2 = CI^2 - Ca^2$ (*preceding case*), and $2FC \cdot CD = 2CB \cdot CI = 2CA \cdot CI$; hence, $FE^2 = FC^2 + CI^2 - Ca^2 + 2AC \cdot CI$; or (*prop. vi.*)

$$= IC^2 + CA^2 + 2AC \cdot CI = (AC + CI)^2 = AI^2.$$

Wherefore we have $FE = AI$.

(3). Also, (*Euc. ii.* 13,)

$$\begin{aligned} fE^2 &= fC^2 + CE^2 - 2fC \cdot CD, \\ &= fC^2 + CI^2 - Ca^2 - 2fC \cdot CD, \text{ or } (\textit{pr. vi. cor. } 3) \\ &= CA^2 - 2fC \cdot CD + CI^2, \text{ or } (\textit{def. } 23) \\ &= CA^2 - 2AC \cdot CI + CI^2 \\ &= (CI - CA)^2 = (CI - CB)^2 = BI^2; \end{aligned}$$

wherefore $fE^2 = BI^2$, or $fE = BI$.

(4). $FE - Ef = AI - IB = AB$.

(5). $FE + Ef = AI + IB = 2CI = II'$.

Corollary 1.

Similar properties obviously exist amongst the corresponding lines in the conjugate hyperbolas.

Corollary 2.

Let a point D′ be taken in the hyperbola opposite to EB, such that CD′=CD. Then CE′ = CE, and E′CE is one straight line bisected in C. For D′E′ = DE (as is easily proved) and CD′ = CD, and the angles at D′ and D right angles. Hence CE′ = CE, and the angles D′CE′, DCE equal. Hence, &c.

Corollary 3.

Hence every diameter is bisected by the centre of the curve.

Corollary 4.

Hence if by any contrivance (and there are several) a string can be fixed at two points F and *f* and kept stretched, so that the difference between its parts are always of the same length, a pencil moving in the intersection will trace an hyperbola.

Corollary 5.

Or, it may be done by points as in the ellipse, see *prop. vii. p.* 129.

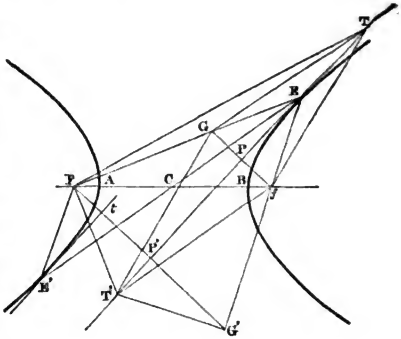
PROP. VIII.

If the interior angle made at a point in the curve by the focal lines be bisected, the bisecting line will be a tangent to the hyperbola at the point.

Let the angle FE*f* be bisected by the line TET′: then TET′ will be a tangent at E.

For if not, let TET′ cut the curve in some other point, as T. Join TF, T*f*, and from one of the foci, as *f*, draw *f*PG perpendicular to TET′, and produce it to meet the line FE at G, and join GT.

Then, since the vertical angle *f*EG of the isosceles triangle *f*EG is bisected, the bisecting line EP bisects the base *f*G at right angles. Hence the triangle *f*TG is also isosceles, having *f*T = TG, and FT − T*f* = FT − TG. It follows, therefore, FT − T*f*, which is less than FG, the third side of the triangle



FTG, is less than the difference $FE - Ef$; that is, less than AB (*prop. vii.*) But since T is a point in the hyperbola, $FT - Tf$ is equal to AB: which is impossible. The point T, therefore, is not in the hyperbola: and it may be shown in the same way, that no other point on the same side of E in TET', and in very nearly the same way that no point T', on the other side of E, is in the curve; and therefore TP is a tangent at E.

Corollary 1.

The tangents ET, E't at the opposite extremities of any diameter EE' are parallel to one another: and parallel tangents are at the opposite extremities of the same diameter.

For the diameter EE' is bisected in C, and likewise Ff is bisected in C: hence if E'f be drawn, FEfE' is a parallelogram, and the angles FEf, FE'f are equal. Wherefore, $tE'E = FE'E - FE't = fEE' - fET' = T'EE'$; and the alternate angles tE'E, T'EE' being equal, the tangents tE', T'E are parallel.

The converse follows *ex absurdo*.

Corollary 2.

The perpendiculars fP, FP' from the foci intersect the tangents drawn to all points of the curve, in a circle on the transverse diameter AB.

For draw CP. Then, since Ff and fG are bisected in C and P respectively, CP is parallel to FG, and FfG, CfP are similar triangles. Wherefore, also,

$$Ff : fC :: FG\ (= AB = 2CB) : CP :: 2 : 1.$$

$$\text{Whence } CP = \tfrac{1}{2}AB = CB.$$

Now C being a given point, and CB a given line, the point P is in a circle on AB.

In the same way it may be shown that P' is in the circle.

Corollary 3.

The rectangle under the perpendiculars from the foci upon any tangent is equal to the square of the semi-conjugate diameter.

For let FP' and fP be those perpendiculars upon the tangent at E; and let FP' meet the tangent at E' in t. Whence, the points A, t, P', B, being in the circle, $P'F \cdot Ft = BF \cdot FA = CF^2 - CA^2 = Ca^2 = (\text{semi-conjugate})^2$.

But the triangles FE't, fEP are equal in all respects, and hence $fP = Ft$; wherefore, substituting in the last result,

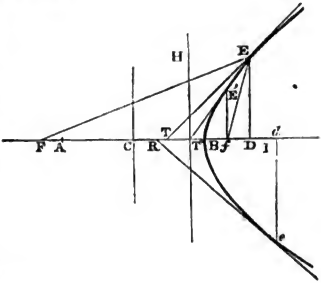
$$FP' \cdot fP = Ca^2.$$

Corollary 4.

If from the foci and the centre of an hyperbola perpendiculars be drawn, that from the centre will be half the difference of those drawn from the foci.

PROP. IX.

If a tangent and an ordinate be drawn from any point in the curve to meet the transverse axis, the semi-transverse will be a mean proportional between the distances of the two intersections from the centre.



That is, if E be the point from which the tangent ET and ordinate ED be drawn to the transverse diameter, then

$$CT : CB :: CB : CD.$$

For draw fE, EF, and let I be the dividing point.

Then (*Euc. vi.* 3), $FT : Tf :: FE : Ef$; that is,

$$fC + CT : fC - CT :: FE : Ef;$$

and *div. et comp.*

$$CT : Cf :: BC : CI,$$

$$\text{or } CT : CB :: Cf : CI, \text{ or } (\textit{def. } 23)$$

$$:: CB : CD.$$

Corollary 1.

Since CT is always a third proportional to CD and CB, if D and A remain constant, T will also remain constant. Whence every hyperbola described on AB as its transverse diameter will cut ED in a point E such that TE will be a tangent to the curve.

Corollary 2.

The transverse diameter AB is harmonically divided (*vol. i. p.* 340) in T and D.

For since $CT : CB :: CB : CD$, we have, *comp. et div.*,

$$CB + CT : CB - CT :: CB + CD : CD - CB; \text{ that is,}$$

$$AT : TB :: AD : DB.$$

Corollary 3.

$$AD . DB = CD . DT.$$

For each is equal to $CD^2 - CB^2$, the former by *Euc. ii.* 7, and the latter follows from the proposition.

Corollary 4.

Let another tangent *e*R and ordinate *ed* be drawn; then

$$Cd : CD :: CT : CR.$$

For as before, $Cd . CR = CB^2 = CD . CT$; wherefore, *etc.*

Corollary 5.

$$BD^2 : BT^2 :: CD : CT.$$

For by the proposition, $CD : CB :: CB : CT$, and hence, *divid.*,

$$CD - CB : CB - CT :: CD : CB; \text{ that is, } BD : BT :: CD : CB, \text{ or}$$

$$BD^2 : BT^2 :: CD^2 : CB^2$$

$$:: CD^2 : CD . CT$$

$$:: CD : CT.$$

Corollary 6.

If f be the focus, fE' the ordinate, and ET' the tangent at E', and if $T'H$ be drawn perpendicular to the transverse axis, it will be the directrix, (*def.* 24); whence, by the proposition, the semi-transverse is a mean proportional between the distances of the focus and directrix from the centre.

Corollary 7.

Let the tangent meet the conjugate axis at t, and an ordinate also be drawn to meet it at d. Then,

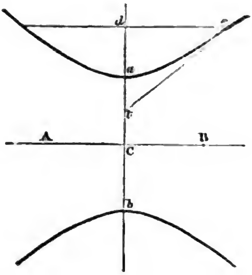
(1). $Cd : Ca :: Ca : Ct$

(2). $da^2 : at^2 :: Cd : Ct$

(3). $bd : da :: bt : ta$

(4). $bd \cdot da = Cd \cdot dt$.

For these are related to the conjugate hyperbolas in precisely the same way that the corresponding parts are to the primary hyperbolas; and the same proof applies to them.



PROP. X.

The distance of the directrix from a point in the hyperbola has to the distance of the focus, from the same point, the same ratio wherever in the curve that point be taken.

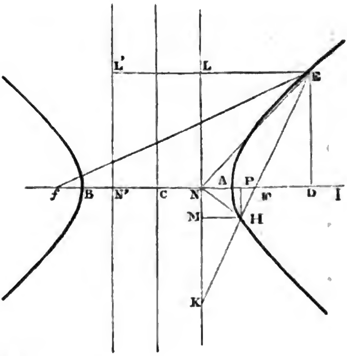
That is, if NL and N'L' be the respective directrices with respect to F and f, we shall have $FE : EL :: fE : EL' :: CF : FA$; that is, a constant ratio, wherever E be taken in the curve.

For let I be the dividing point: then by the property of the directrix (*prop. ix. cor.* 6), and *def.* 23, we have

$CF : CA :: CA : CN$, and

$AC : NC :: CI : CD$; hence

$$\begin{aligned} CF : CA &:: CI - CA : CD - CN \\ &:: \quad AI \quad : ND (= EL) \\ &:: \quad FE \quad : EL. \end{aligned}$$



In the same way, $CF : CA :: fE : EL'$.

Corollary 1.

If a line HE be drawn through the focus F, and the lines HN, EN be drawn, they will make equal angles with CN.

For, draw the ordinates ED, HP, to meet the axis in D and P, and draw

HM, EL parallel to the axis meeting the directrix in M and L, and produce EH to K.

Then by the proposition and the similar triangles PHF, DEF, we have

DN : PN :: EL : MH :: FE : FH :: ED : HP.

Hence the right-angled triangles DEN, PHN have the sides about the right angles proportional, and they are therefore equiangular, having END = HNP.

Corollary 2.

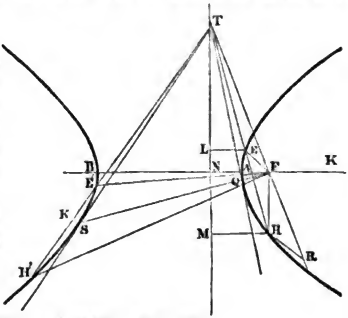
The chord HE is divided harmonically by the focus in F and K.

For by the similar triangles KHM, KEL, and by the *prop.*

KH : KE :: MH : EL :: HF : FE.

PROP. XI.

If two lines be drawn from any point in the directrix of an hyperbola to cut the curve, one of which passes through the focus, and lines be drawn from that focus to the points where the other line cuts the curve, these will make equal angles with the line drawn through the focus.



That is, if from T in the directrix NL a line TR be drawn through the focus F corresponding to the directrix NL, and any other line through T cut the curve in E and H; then the angles EFT, HFR will be equal.

For, draw EL, HM parallel to the axis, and HR parallel to EF meeting TF in R.

Then, by parallels EF, HR, and EL, HM, and *prop. x.*, we have

EF : HR :: TE : TH :: LE : MH :: EF : FH.

Whence HR = HF, and HFR = HRF = EFT.

Corollary 1.

If the secant TEH by revolving so as to bring the points E and H to coalesce, become a tangent at Q, the property still holds good, and the angles QFT, QFR become right angles. Whence,

If from a point T in the directrix a tangent TQ be drawn, and a line TF to the corresponding focus, then QF will be always perpendicular to FT.

In like manner, if the tangent TS be drawn, TFS will be a right angle.

Corollary 2.

If tangents TS, TQ be drawn from a point T in the directrix, the chord of contact SQ will pass through the focus.

For, (*cor.* 1,) SFT and TFQ are right angles; whence the points S, F, Q are in one line, or the chord SQ passes through F.

Corollary 3.

The angle EFH is bisected by the line QF, as is at once obvious.

Corollary 4.

The chord EH is harmonically divided in K and T, by the chord QF and directrix NT.

For the interior angle EFH being bisected by FQ, the exterior one is bisected by FT, the perpendicular to FQ. Hence

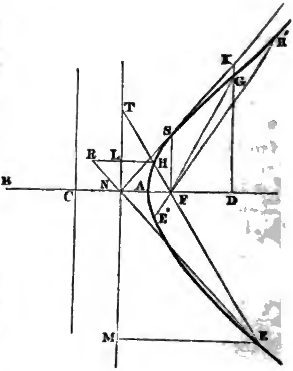
$$\text{HK} : \text{KE} :: \text{HF} : \text{FE} :: \text{HT} : \text{TE}.$$

Scholium.

This proposition involves several properties, either as cases or consequences, which it may be worth the student's while to investigate; and several more general ones will be given under the *General Properties of the Conic Sections*, a little farther on in the volume.

PROP. XII.

A chord drawn through the focus of an hyperbola is so divided in that point, that four times the rectangle contained by its segments is equal to the rectangle contained by the chord and the parameter.



That is, $4\text{HF} . \text{FE} = p . \text{HE}$.

For, draw the lines HL, EM perpendicular to the directrix, and join EN, which produce to cut HL in R, and draw FS the focal ordinate, or semi-parameter.

Then, (*pr. x. cor.* 1,) $\text{HNL} = \text{ENM} = \text{RNL}$; hence $\text{RL} = \text{LH}$, or $\text{RH} = 2\text{HL}$.

By similar triangles RHE and NFE, we have

$$\text{EH} : \text{EF} :: \text{RH} : \text{NF}, \text{ or}$$

$$\text{NF} . \text{EH} = \text{EF} . \text{RH} = 2\text{HL} . \text{EF}.$$

And (*pr. x.*)

$$\text{HL} : \text{HF} :: \text{NF} : \text{FS}, \text{ or}$$

$$2\text{HL} . \text{FE} : 2\text{HF} . \text{FE} :: \text{NF} . \text{HE} : \text{FS} . \text{HE}.$$

But since $2\text{HL} . \text{EF} = \text{NF} . \text{HE}$, we have $2\text{HF} . \text{FE} = \text{FS} . \text{HE}$, or doubling both, $4\text{HF} . \text{FE} = 2\text{FS} . \text{HE} = p . \text{HE}$.

Corollary 1.

If another line H′E′ pass through the focus; then we shall have

$$\text{HF} . \text{FE} : \text{H}'\text{F} . \text{FE}' :: p . \text{HE} : p . \text{H}'\text{E}' :: \text{HE} : \text{H}'\text{E}'.$$

Corollary 2.

Draw the focal tangent NS to meet any ordinate DG in K, and join FG; then $\text{FG} = \text{DK}$.

For (*prop. xi.*) SF is perpendicular to FN, and therefore parallel to DK. Wherefore NFS, NDK are similar triangles, and we have

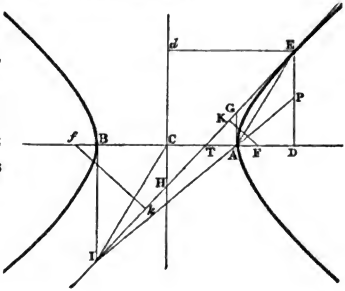
$$\text{NF} : \text{FS} :: \text{ND} : \text{DK};$$

$$\text{also } (\textit{prop. x}) \ \text{NF} : \text{FS} :: \text{ND} : \text{FG}.$$

Hence, since the first three terms of each proportion are identical, the fourth terms of each must be equal; that is, $\text{FG} = \text{DK}$.

PROP. XIII.

If there be any tangent meeting four perpendiculars to the axis drawn from these four points, namely, the centre, the two extremities of the axis, and the point of contact; these four perpendiculars will be proportionals.



THAT is, AG : DE :: CH: BI.

For (*pr. ix.*), TC:CA::CA:CD,
or, *div.* TA:AD::TC:CA,
::TC:CB;
and *comp.* TA:TD::TC:TB;
and hence, by the similar triangles TAG, TDE, TCH, TBI,
AG : DE :: CH : BI.

Corollary 1.

TA : TD :: TC : TB, and TG : TE :: TH : TI.

They being homologous sides of the four triangles mentioned in the demonstration.

Corollary 2.

Draw AI to intersect DE in P: then DE is bisected in P.

For, by the *prop.* and preceding *corollary*,

TA : TC :: TD : TB :: TE : TI;

wherefore the triangles TAE, TCI are similar, and AE parallel to CI. Hence, again, the triangles AED, APD are similar, respectively, to CIB, AIB; and, consequently,

AD : DE :: CB : BI, and AD : DP :: AB : BI; and, therefore,
DE : DP :: AB : CB :: 2 : 1, or DE = 2DP.

Whence we derive an elegant method of drawing a tangent at any point, E, in the curve of the hyperbola: viz.

Bisect the ordinate ED in P; draw AP to meet the tangent BI in I; and the line IE, being drawn, will be the tangent required.

Corollary 3.

The rectangle AG . BI = Ca^2 = $(\text{semi-conjugate})^2$.

For draw E*d* parallel to AB: then C*d* = DE. Also, by the *pr.* and *Euc. vi.* 16, ED . HC = C*d* . CH = Ca^2, (*prop. ix. cor.* 4.)

Corollary 4.

Draw the perpendiculars FK, *fk* from the foci, then FK . *fk* = AG . BI.

For (*preceding cor.*), AG . BI = Ca^2 = FK . *fk* (*prop. viii. cor.* 3).

Scholium.

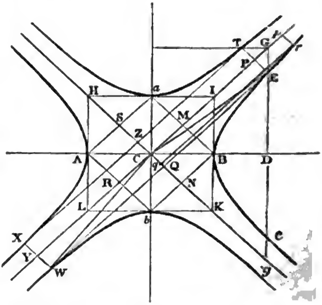
A more general property, viz. where the four lines have any, but equal, inclinations to the axis, will be found on a future page.

SECTION II.

THE ASYMPTOTES OF THE HYPERBOLA.

PROP. A.

All parallelograms formed by lines drawn from points in the hyperbola parallel to the asymptotes, and the segments of the asymptotes intercepted by them, are equal.



Let AB, *ab* be the transverse and conjugate diameters, and HIKL the conjugate rectangle (*def.* 21): then KH, LI are the asymptotes (*def.* 11). Now if from any point E in one of the branches of the curve, lines EP, EQ be drawn parallel to HK, LI, so as to form the parallelogram CQEP; this parallelogram will be of the same magnitude wherever in the curve of the hyperbola, the point E be taken, viz. equal to BMCN formed by drawing lines BM, BN from the vertex B parallel to the asymptotes.

For, draw the ordinate ED*e*, and produce it to meet the asymptotes in G and *g*. Then, by the similar triangles CBI, CDG, and *prop. ii.*,

$$CD^2 : DG^2 :: CB^2 : BI^2 :: CB^2 : Ca^2 :: AD \,.\, DB : DE^2; \text{ whence}$$

$$CD^2 : AD \,.\, DB - CD^2 :: DG^2 : DG^2 - DE^2.$$

But $AD \,.\, DB - CD^2 = CB^2$, and $DG^2 - DE^2 = GE \,.\, Eg$, whence *alt.* and by similar triangles CDG, CBI,

$$CB^2 : GE \,.\, Eg :: CD^2 : DG^2 :: CB^2 : BI^2; \text{ or } BI^2 = GE \,.\, Eg.$$

Again, by the similar triangles GPE, IMB, and EQ*g*, BNK,

$$PE : MB :: EG : IB, \text{ and } QE : BN :: Eg : BK; \text{ whence,}$$

$$PE \,.\, EQ : MB \,.\, BN :: GE \,.\, Eg : IB \,.\, BK.$$

But $IB \,.\, BK = BI^2 = GE \,.\, Eg$; and hence $PE \,.\, EQ = MB \,.\, BN$, or $PE : MB :: BN : QE$. The sides of the parallelograms EPCQ, BMCN about the common angle C are therefore reciprocally proportional, and the parallelograms themselves, therefore, equal.

Corollary 1.

If lines be drawn parallel to one asymptote of the hyperbolas, to meet those curves, these lines will be bisected by the other asymptote; as ET parallel to HK, cutting LI in P, is bisected in P.

For draw TZ parallel to LI: then the parallelogram TC is equal to SM, which is equal to MN, which is equal to PQ. The equal parallelograms TC, PQ, therefore, being between the same parallels HK, TE are upon equal bases TP, PE; or the line TE is bisected in P.

In the same way it may be shown when lines are drawn parallel to the other asymptote LI.

Corollary 2.

If TE be drawn parallel to one asymptote HK, and from T and E parallels

TX, EW be drawn to the other asymptote HK, meeting the opposite curves in X and W: then WX being drawn, it will be parallel to the first asymptote HK; and the figure TXWE will be a parallelogram.

For, (*cor.* 1,) TZ = ZX, and EQ = QW; whence, since also TZ = QE, we have ZX = QW. The lines ZQ, XW, therefore, joining the extremities of the equal and parallel lines ZX, QW are themselves equal and parallel, or XW is parallel to the asymptote HK.

Also, XW is parallel to TE, for both are parallel to HK; and since TX, EW are parallel, each being parallel to IL, the figure TXWE is a parallelogram.

Corollary 3.

If EC, CX be drawn they will be in one line, and ECX will be a diameter of the hyperbola: and the same is true of TC, CW.

For conceive the lines TC, CX (not in the figure) to be drawn. Then, PE = WY = YX, and PC = CY, and the angles EPC, XYC also equal; hence XC = CE, and the angle XCY = angle ECP. Wherefore X, C, E are in one line, and EX is bisected in C.

Corollary 4.

All parallelograms formed of lines parallel to the asymptotes, and having its angular points in the curve, will be equal to one another, and each equal to half the conjugate rectangle.

For the constituent parallelograms into which the whole parallelogram TXWE is divided by the asymptotes are severally equal to those into which the parallelogram A*b*B*a* is divided by the asymptotes; viz. TZCP to *a*SCM, EPCQ to MBNC, and so on. Hence TXWE is equal to A*b*B*a*, that is, to half HLKI, the conjugate rectangle.

Corollary 5.

If any line G*g* be drawn parallel to one of the axes, as, for instance, to *ab*, the rectangle of the segments GE, E*g*, or G*e*, *eg*, into which it is divided by the curve and its asymptotes, is always of the same magnitude, viz. to the square of the semi-axis to which it is parallel.

Corollary 6.

The segments intercepted between the curve and its asymptote are equal, viz. GE = *ge*, and G*e* = E*g*.

For DG = D*g*, and DE = D*e*: hence also GE = *ge*, and G*e* = E*g*.

Corollary 7.

The curve and its asymptotes never meet, however far produced.

For, $BI^2 = GE \cdot Eg = DG^2 - DE^2$; or $DG^2 = BI^2 + DE^2$. Wherefore DG is greater than DE, or the point E of the hyperbola always lies between the indefinite lines CB, CG. This property holding good for all positions of E in the curve, the curve can never meet the asymptote CG.

Corollary 8.

The curve and its asymptote approach more nearly than any distance which can be assigned.

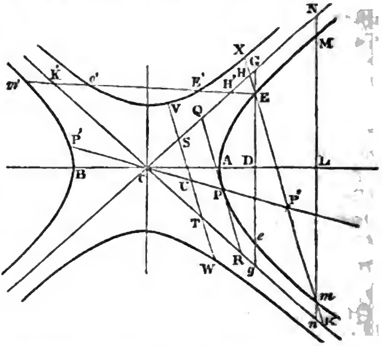
For, let PE be any distance assigned; and take C*p* in the asymptote greater than CP, and draw *pr* parallel to PE, meeting the curve in *r*; and, lastly, draw *rq* parallel to C*p*. Then the parallelograms EPCQ, *rp*C*q* being equal, and having the angle at C common, $Cp : CP :: PE : pr$. But since Cp is greater than CP, PE is greater than pr (*Euc.* v. 14). Wherefore, since this holds true wherever in the asymptote the point P is taken, it follows, that however small we take PE, it is possible to find a distance pr between the curve and its asymptote, which shall be still less than PE.

Corollary 9.

If any number of distances be taken from C on one asymptote, which are continued proportionals of greater or less inequality, the lines drawn from them parallel to the other asymptote to meet the curve will be continued proportionals of less or greater inequality, respectively.

PROP. B.

If a straight line be drawn in any way whatever to meet the hyperbola and its asymptotes, the segments intercepted between the curve and the asymptotes will be equal.



Let the line HK cut the asymptotes in H and K, and the curve in E and m: then $HE = mK$, and $Hm = EK$.

For draw the lines GEg, Nmn perpendicular to the axis AB, meeting it in D and L. Then the triangles EgK, HEG are respectively similar to Kmn, HmN, and we have

$$nm : mK :: gE : EK, \text{ and}$$
$$mN : mH :: GE : EH;$$

$$\text{wherefore, } nm \,.\, mN : Km \,.\, mH :: GE \,.\, Eg : HE \,.\, EK.$$

But, (*pr. A. cor.* 4.) $GE \,.\, Eg = nm \,.\, mN$; and hence

$$Km \,.\, mH = HE \,.\, EK, \text{ or } Km\,(mE + EH) = EH\,(Em + mK);$$
$$\text{that is, } Km \,.\, mE = HE \,.\, Em; \text{ or } HE = mK, \text{ and } Hm = EK.$$

In a similar manner it may be shown that $EH' = m'K'$, and $EK' = m'H'$.

Corollary 1.

If the line HK be moved parallel to itself till it arrive at the position QR, and touch the hyperbola at P, then $QP = PR$, or the tangent is bisected by the asymptotes.

Corollary 2.

If any line, as HK, be drawn parallel to a tangent QR, both being limited by the asymptotes, then the rectangle of its segments made by the curve will be equal to the square of the intercepted tangent: that is,

$$PQ^2 = HE \,.\, EK = Hm \,.\, mK.$$

Corollary 3.

If a straight line VW be drawn parallel to any tangent QR, then the diameter CP being drawn, it will bisect VW in U.

For, since QR is parallel to ST, and is bisected in P (*cor.* 1.), we have also

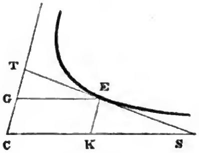
SU = UT: also, since the line VW is drawn to meet the hyperbolas conjugate to the former, VS = TW (*pr. B*); and hence we have VS + SU = WT + TU, or VU = UW.

Corollary 4.

All parallel chords are bisected by the same diameter, and that diameter is conjugate to them (*def* 14).

Corollary 5.

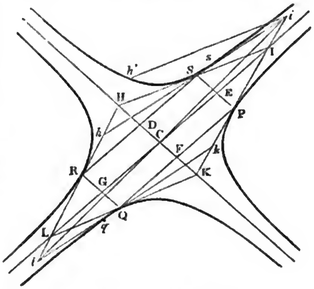
All the triangles formed by tangents and the intercepted segments of the asymptotes are each equal to half the conjugate rectangle.



For from the point of contact E draw EK, EG parallel to the asymptotes. Then it is easily shown that the triangle TCS is double of the parallelogram GK, since (*cor.* 1) the tangent TS is bisected in E. Also, (*prop. A. cor.* 4.) GK is one-fourth of the conjugate rectangle, and hence TCS is half the same rectangle.

PROP. C.

If a parallelogram be described to touch the four hyperbolas, and have one of its angular points in an asymptote, it will have all its angular points upon the two asymptotes, and will be a conjugate parallelogram (def. 21)*: and no parallelogram not having its angular points so situated can be constructed that shall be a conjugate one.*



Let any point I be taken in the asymptote CI, from which let IK, IH be drawn touching the hyperbolas in P and S, and draw HR, KQ tangents to the other two hyperbolas at R and Q: then their point of intersection L will be in the first asymptote, and the figure HIKL will be a conjugate parallelogram. And no parallelogram whose angular points are not in the asymptotes is a conjugate one.

First. Because IH, IK are bisected at the points of contact S and P, the line SP is bisected in E at its intersection with the asymptote (*prop. B. cor.* 1.): also (*Euc. vi.* 2, or *Geom. theor.* 82), SP is parallel to HK. In the same manner RQ is parallel to HK; and hence also to SP. Similarly, RS and PQ are parallel to the other asymptote IL, and the figure SPQR is a parallelogram.

Join RC, PC, SC, QC (not in the figure): and because SPQR is a parallelogram inscribed between the four hyperbolas, SP = RQ, and RS = QP, and (*prop. B. cor.* 3.) the diagonals RP and QS pass through the centre, and are diameters of the curve. But RP bisects the sides HL and IK, and SQ bisects the sides HI, KL, and hence these diameters are parallel to the sides of the parallelogram HIKL, touching the four hyperbolas: that is, each diameter is parallel to the tangents drawn to the curves from the extremities of the other diameter, or they are conjugate diameters (*def.* 14), and the figure HIKL is a conjugate parallelogram (*def.* 21).

Secondly. No parallelogram whose angular points are situated otherwise than in the asymptotes can be a conjugate parallelogram.

For, let the parallel tangents *ki*, *lh*, be drawn touching the opposite hyperbolas at P and R, and *ih*, *kl* touching their conjugate hyperbolas at *s* and *q*: and let the angular points *h*, *i*, *k*, *l*, not be in the asymptotes. Then it is to be proved that *hi* and *kl* are *not* parallel to RP.

Let *ik* and *hl* cut the asymptotes in I and L, and draw the tangents IH, LK from I and L to the conjugate hyperbolas: these will be parallel to RP by the preceding demonstration. Join S*i*. Then, since *i* is a point not in the tangent to the curve at S, and *i* lies without the curve, S*i* cuts the curve in some point. Hence the tangent from *i* to the curve lies between S*i* and *i*C, and consequently it will intersect SI in some point. It is hence not parallel to SI, and consequently not parallel to RP. Hence, the parallelogram *hikl* not having its sides parallel to the diameters drawn to the alternate points of contact, is not a conjugate parallelogram.

Corollary 1.

If one diameter of a parallelogram be conjugate to another diameter, the second is conjugate to the first.

Corollary 2.

All conjugate parallelograms described to touch the four hyperbolas are equal to one another, and to the rectangle contained by the two axes.

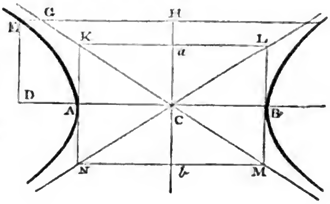
For, the parallelogram HIKL is composed of the four equal parallelograms CSIP, CPKQ, CQLR, and CRHS, which are respectively double of the four equal triangles SCP, PCQ, QCR, RCS. But these last four compose the parallelogram PQRS, which (by *prop. A. cor* 4.) is equal to half the rectangle of the axes: and hence the parallelogram HIKL is equal to the rectangle contained by the axes, and, therefore, always of the same magnitude.

Corollary 3.

If the line CP be produced to cut any chord parallel to the tangent at P in P′, that chord is bisected in P′, and P′E, P*m* are ordinates to the diameter CP (*fig. pr. B*).

PROP. D.

If from a point in the hyperbola upon the transverse diameter an ordinate be drawn to the conjugate axis to cut the asymptote; the difference between the squares of the ordinate and the portion of it intercepted between the asymptote and the axis is equal to the square of the semi-transverse diameter.



LET any ordinate EH be drawn from the point E of the hyperbola to the conjugate axis CH, cutting the asymptote CK in G: then

$$EH^2 - HG^2 = AC^2.$$

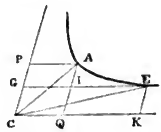
For, draw the ordinate ED to the transverse: then (*prop. ii*),

$$CD^2 - CA^2 : DE^2 :: CA^2 : Ca^2.$$

Now GH : HC :: Ka : aC :: AC : Ca; wherefore
$$CD^2 - CA^2 : DE^2 :: GH^2 : HC^2.$$
Also, $DE^2 = HC^2$, and $CD^2 = HE^2$; and hence, finally,
$$EH^2 - AC^2 = GH^2, \text{ or } EH^2 - HG^2 = AC^2.$$

PROP. E.

The three following spaces between the asymptotes and the curve are equal; namely, the sector or trilinear space contained by an arc of the curve and two radii, or lines drawn from its extremities to the centre; and each of the two quadrilaterals, contained by the said arc, and two lines drawn from its extremities parallel to one asymptote, and the intercepted part of the other asymptote.



From A, E, any two points in the curve of the hyperbola, draw AP, EG parallel to one asymptote CK, and intercepting GP upon the other; and similarly draw AQ, EK parallel to the other asymptote, intercepting QK upon the former; and also, draw AC, EC to the centre: then

quad. PAEG = quad. AQKE = sect. CAE.

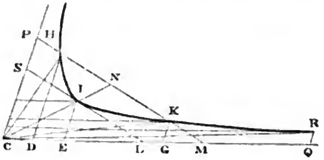
1. For (*pr.* A.) CPAQ = CGEK, hence taking away CGIQ, we have PGIA = IQKE; and adding AIE, we have

quad. PAEG = quad. AQKE.

2. Again, from the quadrilateral CAEK take separately the equal triangles CAQ, CEK, then there respectively remain the sector CAE and quadrilateral QAEK, equal to one another. Hence the stated equalities take place.

PROP. F.

If from the point of contact of any tangent, and the two intersections of the curve with a line parallel to the tangent, three parallel lines be drawn in any direction, and terminated by either asymptote, those three lines will be in continued proportion.



Let HK be any chord parallel to the tangent at I; then lines HD, IE, KG drawn parallel to either asymptote, CP, to meet the other, CM, in D, E, G, will be continued proportionals, or HD : IE :: IE : KG.

For, draw the tangent IL, and produce HK to meet the asymptote in M: then the three triangles HDM, IEL, KGM are similar, and hence

EI : IL :: DH : HM, and EI : IL :: GK : KM;

wherefore $EI^2 : IL^2$ :: DH . GK : HM . MK.

But (*pr. B. cor.* 2.), HM . MK = IL^2, and hence DH . GK = EI^2, that is, HD : IE :: IE : KG.

Corollary.

Draw the semi-diameters CH, CI, CK; then will the sectors HIC, ICK be equal.

For since CN bisects HK, the triangles HCN, NCK are equal: and since all the ordinates parallel to HK are respectively bisected by IN or CN, the segment of the hyperbola HIN is equal to the segment KIN: wherefore the differences are equal, viz. HCN — HIN = KCN — KIN, or the sectors HCI, KCI are equal.

PROP. G.

[*Figure to preceding proposition.*]

If any number of segments CD, CE, CG, CQ, etc , in one asymptote be taken continued proportionals, and DH, EI, GK, QR, etc., be drawn parallel to the other asymptote: then the spaces HDEI, IEGK, KGQR, etc., will be all equal.

First. Let CD : CE :: CE : CG; then will the sectors HCI, ICK be equal, and likewise the spaces HDEI, IEGK be equal.

For draw the tangent at I, the intermediate one of the three points H, I, K meeting the asymptotes in L and S, and draw HK meeting the asymptotes in M and P and the diameter CI in N.

Then, since (*prop.* B, *cor.* 1) SI = IL, and that (*by hypoth*) IE is parallel to CP, we have CE = EL. Also, (*prop.* B,) since PH = KM, we have CD = GM, and hence CG = DM.

Again, (*hypoth.* and *prop.* A,) we have

CE : CG :: CD : CE :: EI : DH; wherefore
EI : EC :: DH : CG, or EI : EL :: DH : DM.

Hence the triangles DHM, EIL having the angles HDM, IEL equal, and the sides about these angles proportional, are similar. Moreover, EI being parallel to DH, and EL coincident with DM, the third sides IL, HM are parallel. Wherefore, (*prop.* F. *cor.*) the sectors HCI, ICK are equal; and (*prop.* E) the spaces HDEI, IEGK are also equal.

Secondly. Let CE : CG : : CG : CQ; then will the sectors ICK, KCR be equal, and the spaces IEGK, KGQR, also be equal.

For, draw the tangent at K: then it may be proved as before that this tangent is parallel to IR, and the sectors and spaces equal as asserted.

In the same way the demonstration may be extended to any number of sectors or of quadrilateral spaces.

Corollary.

When the segments CD, CE, CG, CQ, *etc.*, are taken in geometrical progression (continued proportionals) the spaces intercepted between HD, and the successive parallels IE, KG, RQ, *etc* , will be in arithmetical progression.

Scholium.

The analogy between the property deduced in this corollary, and the characteristic or defining property of logarithms to the modulus 1, has led to the term *hyperbolic logarithms* being applied to that system which is otherwise called the *Napierean.* This will be noticed at greater length in the part of the volume devoted to the *Differential and Integral Calculus,* to which it more strictly appertains.

It will be remarked, too, that the ellipse and hyperbola, however dissimilar in visible form, have a considerable number of properties which possess a striking geometrical similarity. Of the properties, however, in this section of the work, there are scarcely any which can be discovered to have the slightest analogy to corresponding ones in the ellipse. The reason cannot be well explained in the present stage of the work; but it will be hereafter seen, in treating the subject algebraically, that the expressions which relate to corresponding lines in the ellipse and hyperbola, in all cases where the properties are not analogous, involve in the two cases the functions $\sqrt{+a^2}$ and $\sqrt{-a^2}$, the latter implying *impossibility.*

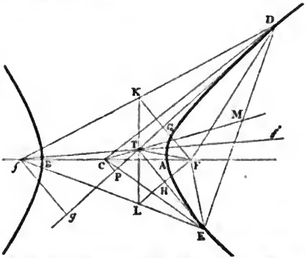
SECTION III.

THE PROPERTIES OF THE HYPERBOLA DEPENDING UPON OBLIQUE CONJUGATE DIAMETERS.

PROP. XIV.

If from the foci and centre of an hyperbola lines be drawn to the points of contact, and also the intersection of two tangents of the hyperbola, then:

1. *The angle contained by the lines from the points of contact to either focus, is bisected by that drawn from the intersection of the tangents to the same focus.*
2. *The lines drawn from the intersection of the tangents to the two foci, make equal angles with those tangents.*
3. *The line drawn from the intersection of the tangents to the centre of the hyperbola, bisects the chord which joins the points of contact.*



Let AB be the transverse diameter, F and f the foci, C the centre; and let DT, ET be tangents meeting in T and touching the hyperbola at D and E; draw EF, FD, FT, Tf, TC, and ED, meeting TC in M; also produce fT to f': then

$$1. \left\{\begin{array}{l} DFT = EFT \\ DfT = EfT \end{array}\right\}$$

$$2. \left\{\begin{array}{l} DTf' = ETF \\ DTF = ETf' \end{array}\right\}$$

$$3. \quad DM = ME.$$

First. Draw FG, FH perpendicular to the tangents DT, ET, and meeting fD, fE in K and L, and join TK, TL.

Then (*pr. viii.*) FDG = KDG, and the angles at G are equal (*constr.*), and the side DG common: hence KD = DF, and $Kf = fD - DF = AB$ (*prop. vii.*); and in the same way $Lf = AB$. Whence $fK = fL$.

Again, by the triangles FDG, KDG we have FG = GK; and (*constr.*) the angles FGT, KGT are equal, and the side TG common to the triangles FGT, KGT. Therefore, KT = TF; and similarly, LT = TF. Whence LT = TK.

In the two triangles TKf, TLf we have the sides TK, Kf equal to the two sides TL, Lf, each to each, and Tf common to the two triangles; whence the angles at f are equal; that is, $KfT = LfT$, or $DfT = EfT$.

In the same way, by drawing perpendiculars from f upon the tangents at D and E, to meet FD, FE produced, it may be shown that DFT = EFT.

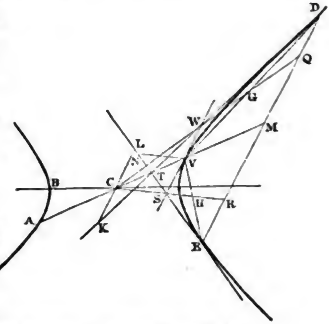
Second. Since the triangles TKf, TLf have been shown to be equal, we have the angles LTf, KTf equal; that is, $LTf = KTf$, or $LTf' = KTf'$, or $2ETF + FTf' = 2DTF - FTf'$; or $DTF = FTE + FTf' = ETf'$, and $FTE = DTF - FTf' = DTf'$.

Third. Draw CD, CE, and the perpendiculars fg and CP from f and C upon the tangent. Then tri. fKT = tri. fDT − tri. KDT = tri. fDT − tri. FDT $= \frac{1}{2}DT.fg - \frac{1}{2}DT.FG = \frac{1}{2}DT\ \{fg - FG\} = DT.CP$ (*pr. viii. cor.* 4) = 2 tri. CDT.

In the same manner, tri. fLT = 2 tri. CET; and it has been shown that the triangles fKT, fLT are equal; and hence, also, the triangles CDT, CET are equal. Moreover, the equal triangles CDT, CET lying on opposite sides of the same base, the line ED joining their vertices will be bisected by the base.

PROP. XV.

1. *Every ordinal chord is bisected by its conjugate diameter.*
2. *The tangents to every ordinal chord meet in that diameter.*
3. *The semi-diameter is a mean proportional between the segments, estimated from the centre, cut off by the chord and tangent.*



Let WS be a tangent, and VA a diameter to the hyperbola through the point V situated in the curve; and let any chord ED be parallel to WS, which, (*def.* 14,) is an ordinal chord. Then,

1. DE is bisected at M by the diameter VA.

2. The tangents at D and E meet at T in CV.

3. CM : CV :: CV : CT.

Let the tangents at D and E meet that at V in W and S; join SC, WC, cutting the chords DV EV, in G and H, and the chord DE in Q and R. Draw, also, through C the chord LK to meet the tangents in K and L, and VN parallel to CS meeting ET in N.

First. The three lines WS, DE, LK, are parallel (*def.* 14, and *constr.*), and hence the triangles VWG, SVH, are respectively similar to DGQ and REH: and (*prop. xiv.*) VG = GD and VH = HE. Hence the triangles are also respectively equal in all respects, or VW = DQ, and SV = ER.

Again, the similar triangles LCS, RES, and the similar triangles CWK, QWD, respectively give,

CL : RE :: CS : SR :: CW : WQ :: CK : QD.

But (by *parallels* DE, LK, and *prop. xiv.*) we have LC = CK, and hence RE = QD; and, again, since RE = SV, and QD = VW, we have SV = VW.

Again, by similar triangles SCV, RCM, and VCW, MCQ, we have

SV : RM :: VC : CM :: VW : MQ.

Also, SV = VW; wherefore RM = MQ, and hence again, EM = MD.

Secondly. It was proved (*prop. xiv.*) that TC passes through M, the middle of ED, or that T, M, C, are in one line; and (*former case of xv.*) that C, M, V, are in one line: hence T and M are in CV: that is, the tangents at D and E meet in the same point T of the diameter DV.

Thirdly. Because SH is parallel to NV, and EH = HV, we have ES = SN. Then by the similar triangles TNV, TSC, and MET, VST, we have

CV : CT :: SN : ST :: SE : ST :: MV : VT, or *alt. et comp.*

CV + MV : CV :: CT + VT : CT; that is, CM : CV :: CV : CT.

Corollary 1.

The diameter AV is harmonically divided in M and T.

For (*com. et div.*), $CV + CM : CM - CV :: CT + CV : CV - CT$; that is, $AM : MV :: AT : TV$.

Corollary 2.

$$AM . MV = CM . MT.$$

For, $CM^2 - CV^2 = AM . MV$, and $CM^2 - CV^2 = CM^2 - CT . CM = CM (CM - CT) = CM . MT$. Whence, by equality, the result follows.

Corollary 3.

$$AM . MV : AT . TV :: CM^2 : CV^2.$$

For (*prop. xv.*) $CM^2 : CV^2 :: CV^2 : CT^2$; hence, *div. et alt.*,

$$CM^2 - CV^2 : CV^2 - CT^2 :: CM^2 : CV^2;$$ or, as above stated.

Corollary 4.

Every *ordinate* to a diameter is parallel to the tangent at its vertex.

Corollary 5.

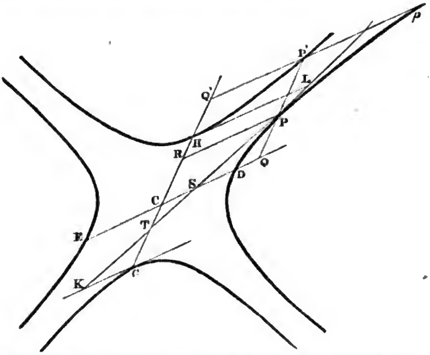
All ordinates to the same diameter are parallel to one another.

Corollary 6.

Any straight line which bisects two parallel chords is a diameter, and is conjugate to those chords, and any diameter which bisects a chord is conjugate to it.

PROP. XVI.

The square of any diameter has to the square of its conjugate the same ratio that the rectangle of the abscisses has to the square of the ordinate.



Let DE, GH be two conjugate diameters, and PQ an ordinate to the former, corresponding to the point P in the curve: then

$$ED^2 : GH^2 :: EQ . QD : PQ^2.$$

Draw the tangent at P to meet the conjugate diameters in S and T. Then (*prop. xv.* and *Euc. vi.* 20, *cor.* 2), we have

$$CQ : CD :: CD : CS, \text{ or } CQ^2 : CD^2 :: CQ : CS; \text{ and hence, } \textit{div.},$$

$$CQ^2 - CD^2 : CD^2 :: CQ - CS : CS, \text{ or } CQ^2 - CD^2 : CD^2 :: QS : SC.$$

Again (by *parallels* and *pr. xv.*) we have

$$CQ^2 - CD^2 : CD^2 :: QS : SC :: PQ \ (= CR) : CT :: PQ^2 : CH^2.$$

Hence $CD^2 : CH^2 :: CQ^2 - CD^2 : PQ^2$, or $ED^2 : GH^2 :: EQ \cdot QD : PQ^2$.

In the same manner, $GH^2 : ED^2 :: HR \cdot RG : RP^2$.

Corollary 1.

The rectangles of the abscisses of any diameter are as the squares of the corresponding ordinates: for each ratio is the same as that of the squares of the conjugate diameters.

Corollary 2.

Any diameter is to its parameter, as the rectangle of its abscisses to the square of the corresponding ordinate. It is shown as in *cor.* 1, *prop. i.*

Corollary 3.

Draw the tangents GK, HL at the extremities of the diameter GH; then $HL : PR :: CS : GK$ (see *dem.* of *prop. xi*)

Corollary 4.

If *any* tangent LK be drawn to intersect two parallel tangents GK, HL, it will cut off segments HL, GK, whose rectangle is equal to the square of the semi-diameter CD parallel to them.

Corollary 5.

The tangent KL is harmonically divided in P and T.

Corollary 6.

As the square of any diameter is to the square of its conjugate, so is the sum of the squares of that semi-conjugate and of the distance of any ordinate to it from the centre, to the square of that ordinate.

For, $CH^2 : CD^2 :: QP^2 : CQ^2 - CD^2 :: CR^2 : RP^2 - CD^2 :: CR^2 + CH^2 : RP^2$.

Corollary 7.

In the same way it may be proved that in the conjugate hyperbola,

$$CD^2 : CH^2 :: CQ^2 + CD^2 : QP'^2.$$

Corollary 8.

By reasoning similar to that in *prop. iii. cors.* 1 and 2, we have

$$QP^2 : QP'^2 :: CQ^2 - CD^2 : CQ^2 + CD^2, \text{ and}$$

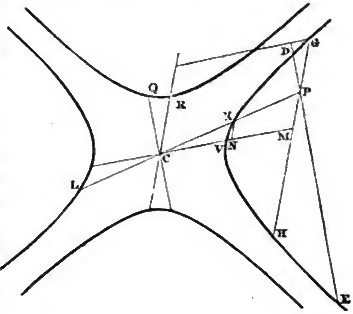
$$Q'P'^2 : Q'p^2 :: CQ'^2 - CH^2 : CQ'^2 + CH^2.$$

Scholium.

The corresponding properties of the parabola and ellipse, together with this property, are but particular cases of the succeeding proposition in each case: and those succeeding propositions admit of demonstration in a direct and simple form, as pointed out in the next *Scholium*.

PROP. XVII.

If any two straight lines which intersect each other and meet the hyperbola be parallel to two fixed diameters, then the rectangles under the segments of the one, intercepted between the point of section of the lines and the two sections with the curve, will be to the rectangle under the segments of the other similarly estimated, as the square of the semi-diameter parallel to the first is to the square of that parallel to the second line.



Let the two lines DE, GH, parallel to the semi-diameters CQ, CR, intersect in the point P: then

$$DP.PE : GP.PH :: CQ^2 : CR^2.$$

For, draw the semi-diameter CV to bisect GH in M; also draw the diameter KL through P, and KN parallel to GH. Then (*pr. xvi. cor.* 6) the diameter CV is conjugate to the chord GH, and therefore to the diameter CR. Then, (*pr. xvi. cor.* 6,)

$$CR^2 : CV^2 :: CR^2 + KN^2 : CN^2, \text{ and}$$
$$CR^2 : CV^2 :: CR^2 + GM^2 : CM^2; \text{ whence}$$
$$CR^2 + KN^2 : CR^2 + GM^2 :: CN^2 : CM^2$$
$$GM^2 - KN^2 : CR^2 + KN^2 :: CM^2 - CN^2 : CN^2, \textit{ dividendo,}$$
$$:: PM^2 - KN^2 : KN^2, \text{ by sim. trians.}$$
$$GM^2 - PM^2 : CR^2 :: PM^2 - KN^2 : KN^2, \textit{ alt. et div.}$$
$$:: CP^2 - CK^2 : CK^2, \text{ sim. trians.}$$

But GH and KL are respectively bisected in M and C; and hence (*Euc. ii.* 5, *cor.*) $GM^2 - MP^2 = GP . PH$, and $CP^2 - KC^2 = KP . PL$. Wherefore,

$$GP . PH : CR^2 :: KP . PL : CK^2.$$
$$\text{Similarly, } EP . PD : CQ^2 :: KP . PL : CK^2.$$
$$\text{Whence, } DP . PE : GP . PH :: CQ^2 :: CR^2.$$

Corollary.

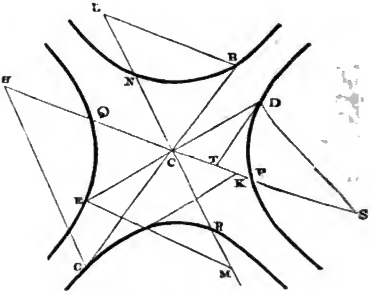
If one of the lines be a tangent, the rectangle of its segments becomes the square of the tangent; and if both be tangents, then these tangents have the same ratio as their parallel diameters, and GD is parallel to QR.

Scholium.

This general property of the conic sections has now been proved for each of the curves separately; viz. for the *parabola* at *p.* 120, the *ellipse* at *p.* 140, and for the *hyperbola* in this proposition. It might, however, have been proved for the cone itself generally, as is done hereafter at *p.* 212, and the importance and uses of it are there pointed out.

PROP. XVIII.

If from the extremities of any two conjugate diameters of an hyperbola ordinates be drawn to any third diameter, the rectangle of the abscisses made by one of the ordinates is equal to the square of the distance from the centre (estimated on that diameter) of the other ordinate.



Let DE, GB be the conjugate diameters, and from the extremities D and G to any other diameter PQ let there be drawn the ordinates DS, GH: then QH . HP = CS^2, and QS . SP = CH^2.

For, let the tangents at D and G meet the diameter PQ in T and K; then, by the pairs of similar triangles KGH, CDS, and GKC, TCD, we have

$$KH : CS :: KG : DC :: KC : CT.$$

Again, (*pr. xv.*) $KC . CH = CQ^2 = CS . CT$; and hence,

$$CS : CH :: KC : CT :: KH : CS, \text{ or } (pr.\ xv.\ cor.\ 2),$$
$$CS^2 = KH . HC = QH . HP.$$

In precisely the same way the other case may be proved.

Corollary 1.

$$KC . CH = TC . CS.$$

Corollary 2.

$$CH . HK = CS^2, \text{ and } CS . ST = CH^2.$$

Corollary 3.

Let NR be the diameter conjugate to PQ, and draw the ordinates EM, GL to it; then

$$CQ^2 = CS^2 - CH^2,$$
$$\text{and } CR^2 = CL^2 - CM^2.$$

For, $CS^2 - CH^2 = CH . HK - CH^2 = CH (CH - HK) = CH . CK = CQ^2$, and similarly for the other case.

Corollary 4.

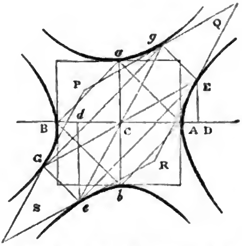
$$QC : CN :: CS : GH.$$

For (*props. xvii.* and *xviii.*) we have

$$QC^2 : CN^2 :: QH . HP (= CS^2) : GH^2;$$
$$\text{and hence } QC : CN :: CS : GH.$$

PROP. XIX.

1. *The difference of the squares of the pairs of conjugate diameters is constantly the same.*
2. *The areas of the conjugate parallelograms is constantly the same, whether circumscribed or inscribed.*



Let PQRS be a conjugate circumscribed parallelogram, and E*e* G*g* the corresponding inscribed one, the diameters EG and *eg* being conjugate to one another; and let AB, *ab* be the transverse and conjugate diameters intersecting in the centre C. Also draw *e*T. Then,

1. $\mathrm{EG}^2 - eg^2 = \mathrm{AB}^2 - ab^2$.

2. $\begin{cases} \text{par}^{\text{m}}\ \mathrm{PQRS} = \mathrm{AB} \,.\, ab. \\ \text{par}^{\text{m}}\ \mathrm{E}e\mathrm{G}g = 2\mathrm{AC} \,.\, \mathrm{C}b. \end{cases}$

For draw the ordinates ED, *ed* to the transverse, and CK perpendicular to one of the sides PQ of the circumscribed parallelogram; and let CA meet the sides QP, PS in T and *t*.

First, (*prop. xviii.*) $\mathrm{C}d^2 = \mathrm{BD} \,.\, \mathrm{DA} = \mathrm{CD}^2 - \mathrm{CA}^2$; or

$$\mathrm{CA}^2 = \mathrm{CD}^2 - \mathrm{C}d^2;\ \text{and similarly, } \mathrm{C}a^2 = de^2 - \mathrm{DE}^2;$$

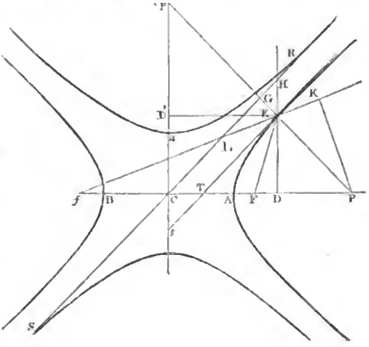
$$\text{whence, } \mathrm{CA}^2 - \mathrm{C}a^2 = \mathrm{CD}^2 + \mathrm{DE}^2 - \mathrm{C}d^2 - de^2 = \mathrm{CE}^2 - \mathrm{C}e^2;\ \text{or}$$

$$\mathrm{EG}^2 - eg^2 = \mathrm{AB}^2 - ab^2.$$

Secondly. This property has been proved in *prop.* C. *cor.* 2, in a more simple manner than it could be by the process adopted in the corresponding case of the ellipse, at p. 142.

PROP. XX.

The central distance of the ordinate of a point on either of the principal diameters, is to the subnormal of that point upon the same diameter, as the square of that principal diameter is to the square of the other.



Let AB, *ab* be the principal diameters meeting in the centre C, AB being the transverse; let the normal at E cut them in P and P', and the ordinates of E to them respectively, be ED and ED'; then DP and D'P' are the respective subnormals, and

$$\mathrm{CD} : \mathrm{DP} :: \mathrm{AB}^2 : ab^2,$$

and

$$\mathrm{CD'} : \mathrm{D'P'} :: ab^2 : \mathrm{AB}^2.$$

For (*prop. ii.*) $AD . DB : DE^2 :: AC^2 : Ca^2 :: AB^2 : ab^2$.

But (*prop. xviii. cor.* 2, and *right-angled triangles*) we have

$AD . DB = CD . DT$, and $DE^2 = PD . DT$; wherefore,

$AB^2 : ab^2 :: CD . DT : PD . DT :: CD : DP$.

In precisely the same way the other part of the proposition may be proved.

Corollary 1.

$$CF^2 : Ca^2 :: CP : PD.$$

For *comp.* $AC^2 + Ca^2 : Ca^2 :: CD + PD : PD$; and (*prop. ii.*)

$AC^2 + Ca^2 = CF^2$, and $CD + PD = CP$.

Corollary 2.

$AB : p :: DC : PD$, and $AB : p :: P'D' : D'C$ (*def.* 17, and *Euc. vi.* 21.)

Corollary 3.

From the preceding corollary, $PD . D'P' = DC . CD'$.

Corollary 4.

Draw the diameter SR parallel to the tangent TE*t*, cutting the normal in G: then $PE . EG = Ca^2$.

For produce ED to meet CG in H: then, since the angles at D and G are right angles, the triangles EDP, EGH are similar, and

$ED (= CD') :: EP :: EG : EH (= Ct)$, or $EP . EG = CD' . Ct = Ca^2$.

Corollary 5.

Draw PK perpendicular to E*f* (or EF); then $EK = \frac{1}{2} p$.

Let the diameter RS meet E*f* in L, then EGL, EKP are similar; and

$EG : EL :: EK : EP$, or $LE . EK = EP . EG = Ca^2$ (*cor.* 4).

But EP bisects the angle FEK; whence $EL = \frac{1}{2}(Ef - FE) = AC$: whence we have

$AC . EK = Ca^2$, or $AC : Ca :: Ca : EK$, and $EK = \frac{1}{2} p$.

Scholium.

The definitions of ***similar conic sections*** (indeed, of similar curves generally) may be given in various forms; but, perhaps, the best mode (and that which accords most nearly with the application of algebra to the geometry of curve lines) is, to say, that ***two curves are similar when the lines upon which their existence depends are in the same ratios, each to each.*** Under this view, the circle depends upon one line only, the diameter; and, hence, all circles are similar figures. In the same way the parabola depends on its parameter only, and, hence, all parabolas are similar figures.

An ellipse depends on its two rectangular diameters; and, hence, if these be denoted by T and C for one ellipse, and t and c for another; they will be similar when $T : C :: t : c$.

In the hyperbola, T and C are the sides of the conjugate rectangle; and $\frac{C}{T}$ is the tangent of the inclination of the asymptote to the transverse axis. Hence, if the other hyperbola be between the same asymptotes, we have $\frac{c}{t}$ for the tangent of inclination, and $\frac{C}{T} = \frac{c}{t}$, or $T : C :: t : c$, and the hyperbolas are similar.

Had any other definition been used, then this would have flowed from it as a consequence; and any other property of similar conic sections may be deduced from the one here adopted.

ELEMENTARY EXERCISES IN CONIC SECTIONS.

1. Demonstrate that if a right cylinder be cut by a plane obliquely the section will be an ellipse.

2. Show how to draw a tangent to an ellipse whose foci are F, *f*, from a given point P, situated on or without the curve.

3. Show how to draw a tangent and a normal to a given parabola from a given point P, either in, or without the curve.

4. The diameters of an ellipse are 16 and 12: required the parameter and the area, and the magnitude of the equal conjugate diameters.

5. The base and altitude of a parabola are 12 and 9: required the parameter, and the ordinates corresponding to the abscissæ 2, 3, and 4.

6. In the actual formation of arches, the voussoirs or arch-stones are so cut as to have their faces always *perpendicular* to the respective points of the curve upon which they stand. By what constructions may this be effected for the parabola and the ellipse?

7. Construct accurately on paper, a parabola whose base shall be 12 and altitude 9.

8. A cone, the diameter of whose base is 10 inches, and whose altitude is 12, is cut obliquely by a plane, which enters at 3 inches from the vertex on one slant side, and comes out at 3 inches from the base on the opposite slant side: required the dimensions of the section?

9. Suppose the same cone to be cut by a plane parallel to one of the slant sides, entering the other slant side at 4 inches from the vertex, what will be the dimensions of the section?

10. The rectangle under the transverse diameter and any focal chord is equal to the square of the diameter parallel to that chord.

11. All circles are similar figures: all parabolas are similar figures: and hyperbolas between the same asymptotes are similar figures.

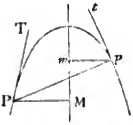
12. Similar ellipses or similar hyperbolas, have their axes in the same ratio: and if two similar figures of either kind have either their centres or a focus, coincident in position, and their transverse axes also coincident in direction, all lines drawn through the common centre or common focus, will be divided by the curves in the same ratio.

13. To describe a polygon in a conic section similar to a given polygon in a given and similar conic section.

14. Let TP, *tp* be tangents to a parabola whose axis is VM at the extremities P and *p* of the right ordinates PM, *pm*: then

1. $\tan TPM : \tan tpm :: PM : pm$, and
2. $\tan TPM - \tan tpm = 2 \tan pPM$.

Required demonstrations of these two useful theorems in practical gunnery.



15. A straight line cannot cut a conic section in more than two points.

16. More than two tangents to a conic section cannot be drawn from the same point.

17. Chords drawn through the foci of an ellipse or hyperbola are proportional to the squares of their parallel diameters; and through the focus of a parabola, proportional to the parameters of their conjugate diameters.

18. The chords which join the extremities of conjugate diameters of the ellipse and hyperbola are conjugate to one another; and the diameters which bisect these are conjugate diameters.

19. If a variable tangent to the ellipse or hyperbola cut two fixed parallel tangents, it is divided at the point of contact into segments whose rectangle is equal to the square of the semi-diameter parallel to it.

20. The segment of the normal intercepted by the principal axis of the ellipse or hyperbola is equal to the distance of the focus from the centre.

21. Two concentric ellipses or hyperbolas whose transverse axes are perpendicular to one another, if they intersect at all, intersect in four points situated in the circumference of a circle.

22. The rectangle under the focal distances FE, E*f*, is equal to the square of the semi-diameter RS, conjugate to the diameter CE.

23. If a conic section be cut by a circle in four points, the lines drawn to join the corresponding points of intersection will be equally inclined to either axis.

24. If two circles be inscribed within any conic section, and from any point in the curve a tangent be drawn to each of the circles, the sum or difference of these tangents (according as the point in the curve from which they are drawn, is situated between or beyond the two points in which the two circles touch the curve on the same side of the axis) is a constant magnitude, and is equal to the distance of the centres of the circles, multiplied by the ratio of the semi-transverse diameter to the excentricity.

25. On the same transverse and conjugate diameters describe an ellipse and hyperbola; and from any point in one of these draw tangents to the other; then the chord of contact of these tangents will be a tangent to the curve in which the point was taken.

26. In every quadrilateral circumscribed to a parabola, the straight lines drawn through the opposite extremities of two adjacent sides, and reciprocally parallel to these sides, will intersect in the diameter of the quadrilateral which connects the two other sides.

27. In every pentagon circumscribed to a parabola, the parallels drawn to the two sides of any of the angles through the opposite summits and the diagonal which joins the two other summits, intersect in one point.

28. Let ABDE be a given parallelogram, and through C the middle of one of its sides ED draw any straight line PCQ meeting the opposite sides in P and Q; and draw PB, QA to intersect in V; then V, and all points determined in the same manner, will be in the curve of a parabola, the focus and vertex it is required to assign.

29. If a circle of a given diameter roll upon another internally, of twice that diameter; then any point in the plane of the rolling circle, either within or without the circumference, will describe an ellipse, whose principal diameters it is required to find, both in magnitude and position.

30. If the generating point be taken within the circumference of the rolling circle, half the sum of the semi-axes of the generated ellipse will be equal to the radius of the circle: if the point be without the rolling circle, half the difference of the semi-axes of the ellipse will be equal to the radius; and if the point be in the circumference of the rolling circle, the ellipse becomes a straight line, diameter to the fixed circle.

31. The four radii-vertices drawn from the foci of a conic section to two points in the curve, will be tangents to one circle; and its centre is the intersection of the tangents of the points in the curve to which the radii-vertices are drawn.

32. If from a focus of a conic section perpendiculars be drawn to two tangents to the curve, and a line be drawn to join the two intersections; then, the perpendicular drawn to this line from the concourse of the two tangents, will pass through the second focus of the curve, in the ellipse and hyperbola, and be parallel to the axis in the parabola.

33. If two conic sections which cut one another have the same foci, they will be perpendicular to each other at their points of section.

34. Two fixed tangents being drawn to a conic section, the intercepted segment of a third and variable tangent, will always intercept at either focus, an angle of an invariable magnitude.

35. When the concourse of the fixed tangents in the last example is in the directrix, that invariable angle is a right angle; and when their point of contact is at the extremities of the transverse diameter, the angle is also a right angle.

36. A conic section and a fixed tangent to it being given, if about either focus, as a vertex, an angle of invariable magnitude be made to revolve, and through the point in which one of its sides meets the fixed tangent a second tangent be drawn to the curve, this second tangent will meet the second side of the angle in a line which forms another and determinable tangent to the curve.

37. If the sides of a variable right angle are tangents to a conic section, the angular point itself will always be situated in the circumference of a circle—or, in the case of the parabola, in a straight line.

38. If there be two concentric, similar, and similarly situated ellipses, having the ratio of their corresponding diameters as the diagonal to the side of a square: then, if any chord be drawn to the outer ellipse, which also meets the inner ellipse, it will be divided by the inner ellipse into segments whose rectangle is equal to the square of the semi-diameter parallel to that chord.

39. Let CF, CG, be any two conjugate semi-diameters of an ellipse, and from G draw GP perpendicular to CF; take, in contrary directions, GH and GH′, each equal to CF, and join CH, CH′: then the principal diameters bisect the angles formed by CH, CH; and the sum and difference of CH, CH′, will be the transverse and conjugate diameters of the ellipse.

40. If through the extremity of any diameter of an ellipse we draw two lines terminating at the conjugate diameter, and equal to the demi-diameter of the curve, they will pass through the foci of the ellipse.

41. If a circle be described on the minor axis *ab* of the ellipse, and from any point P in the ellipse tangents PE, PG, be drawn to the circle: also, if RS be conjugate to PQ, lines drawn for R or S, lines RF, R*f*, be drawn parallel to the tangents, they will pass through the foci F and *f*.

42. If from the angles of a quadrilateral circumscribed about a conic section lines be drawn to the focus, the angles subtended by either pair of opposite sides are together equal to two right angles.

43. Let the angle DFE, remaining constant in magnitude, revolve about the focus F of a conic section ADB, and cut the curve in D, E; and draw the chord DE, and the tangents DG, EG: then DE will always be tangent to a conic section, the point G will be in a conic section, and these conic sections will be confocal with the given conic section ADB, and similar to each other.

44. In the preceding case, the line FG will bisect the angle DFE.

45. When a quadrilateral is described about a conic section, two opposite sides subtend at either focus, angles which are the supplements of one another.

GENERAL PROPERTIES OF THE CONIC SECTIONS.

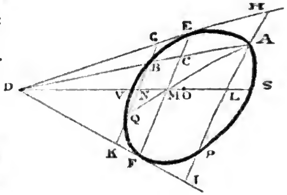
PROP. I.

If from the intersection of two tangents to a conic section, a line be drawn to meet the curve and the chord of contact, this line will be divided harmonically at the four points.

Let E, F be the points of contact of the tangents drawn from D (without the conic section), and let DB, drawn also from D, meet the section in A, B, and the chord of contact EF in C: then

$$AC : CB :: AD : DB.$$

For through A, B draw parallels HI, GK, to meet the tangents in H, G, I, K, and the curve again in P, Q; draw also the diameter through D, cutting the parallel chords AP, EF, BQ in L, M, N.



Then, since DM is conjugate to EF (*pp*. 121, 137, and 164), it bisects all the parallel chords AP, EF, BQ; and since DM bisects the chord EF, it bisects HI and GK. Wherefore, $HA = IP$, $HP = IA$, $GB = KQ$, and $GQ = KB$.

Also, by similar triangles,

$$DG : DH :: GQ\ (= KB) : HP\ (= IA),$$

$$\text{and } DG : DH :: BG : AH;$$

whence $DG^2 : DH^2 :: BG \,.\, GQ : AH \,.\, HP$.

But since HG is a tangent, we have

$$BG \,.\, GQ : AH \,.\, HP :: GE^2 : HE^2;$$

whence $GD^2 : DH^2 :: GE^2 : EH^2$, or $GD : DH :: GE : EH$.

But by parallels,

$GD : DH :: BD : DA$, and $GE : EH :: BC : CA$; whence finally

$$AC : CB :: AD : DB.$$

Corollary 1.

If the diameter be bisected in O we have $OD \,.\, OM = OV^2$, a theorem already obtained for the ellipse and hyperbola separately.

Corollary 2.

If any chord AB of a conic section be harmonically divided in C and D, C being within and D without the curve; then, if from D tangents be drawn to the curve, their chord of contact EF will pass through C.

For if it do not pass through C, let it cut AB in some other point C′. Then, by the theorem and the hypothesis respectively, we have

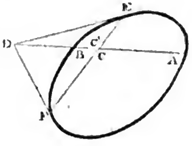
$$AC' : C'B :: AD : DB, \text{ and}$$

$$AC : CB :: AD : DB; \text{ whence}$$

$$AC' : C'B :: AC : CB, \text{ or } comp.$$

$$AB : AC' :: AB : AC, \text{ and } AC' = AC,$$

which is impossible. Whence the chord of contact passes through C.

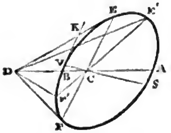


Corollary 3.

If the chord AB be divided harmonically, as in the preceding corollary, and the polar conjugate EF to D be drawn, the lines DE, DF will be tangents to the curve.

For if not, let the tangents from D touch the curve in E′F′; and to the diameter VS through C draw the tangent VK.

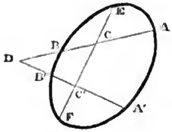
Then, since E′F′ is the chord of contact it passes through C (*cor.* 2), and is parallel to VK; also, by hypothesis EF passes through C, and is parallel to VK, which is impossible. Hence DE, DF are tangents to the curve.



Corollary 4.

If the chord AB be divided harmonically in C, D, and A′B′ in C′, D: then CC′ is the chord of contact of the tangents from D.

For (*cor.* 2), the chord of contact passes through each of the points C, C′; wherefore the line CC′ is the chord of contact.

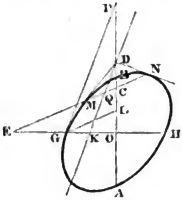


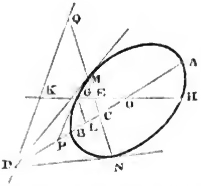
Corollary 5.

If from the points D and M in the diameter VS of a conic section, determined as in the proposition, lines DA, MA be inflected to any point A in the conic section, which also cuts the curve again in B and Q: then the line BQ will be parallel to the chord of contact EF.

PROP. II.

If lines be drawn through a given pole to cut a conic section, and to each of these lines, considered as a chord of contact, pairs of tangents be drawn: the intersection of every pair of tangents will be situated in the same straight line.





Let E be the pole without (*fig.* 1) or within (*fig.* 2) the conic section, and MN one of the lines drawn through it to cut the curve in M and N; and let the corresponding pair of tangents MD, ND meet in D: then D will be always situated in the same line, for all positions of the chord MN.

For, through E draw the diameter GH, in the ellipse and hyperbola, and let O be the centre; draw the diameter AB through D, and the tangent GP and ordinate GL at P to the diameter AB; and, lastly, draw DK parallel to GP.

Then, since MN and GL are conjugates to the diameter AB, and MD, GP are tangents at M and G, we have (*prop. i. cor.* 1.)

$$OC \,.\, OD = OB^2 = OL \,.\, OP, \text{ or}$$
$$OC : OL :: OP : OD.$$
$$\text{But } OE : OG :: OC : OL \text{ (parallels } CE, GL),$$
$$\text{and } OP : OD :: OG : OK \text{ (parallels } GP, DK);$$
$$\text{whence } OE : OG :: OG : OK.$$

But O, E, G are the same for all positions of the line MN subject to the hypothesis; and hence K is a fixed point. Also KD is parallel to a fixed line GP, and passes through a fixed point K, and hence its position is invariable. The point D is, therefore, always situated in a line drawn through K, and having a position conjugate to the diameter through E; and the point determined from $OE \,.\, OK = OG^2$.

When the conic section is a parabola we shall have DP = PC, and nearly similar reasoning will bring out the conclusion that KG = GE; thus giving also in this case K a fixed point and KD drawn through it parallel to a given line.

Corollary 1.

When the point E is *without* the conic section (as in *fig.* 1.) the line DK will be the chord of contact for the point E.

For since OK : OG :: OG : OE, the diameter GH is divided harmonically in K and E. Whence, (*prop. i. cor.* 2.) the line KD being conjugate to GH, it passes through the points of contact of tangents drawn from E.

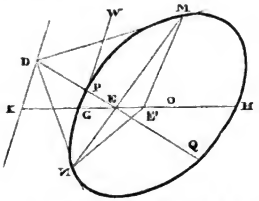
Corollary 2.

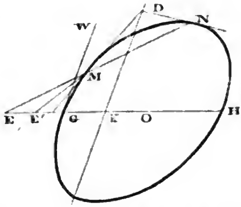
The point E being still *without* the curve (*fig.* 1.) all the chords, as MN, which pass through E will be harmonically divided by the pole E and polar DK.

For since the diameter GH is divided harmonically (*cor.* 1.) in K and E, and that KD is parallel to the tangent at G, it is the chord of contact for E (*prop. i. cor.* 2.). Whence (*prop. i.*) MN is harmonically divided in Q and E.

PROP. III.

If from points in any line which are situated without a conic section, tangents be drawn to the curve, their chords of contact will all pass through one point: and that point will be without or within the conic section, according as the line cuts or does not cut the curve.





If from points in DK, which lie *without* the conic section, tangents DM. DN be drawn to the curve: then the chords of contact will all pass through the same point E.

For parallel to DK draw a tangent GW to the curve, touching it in G, and draw the diameter HG, meeting DK in K. Draw the chord of contact for K, cutting the diameter in E: then it is to be shown that all the chords of contact pass through E.

For if the chord of contact MN, corresponding to the point D in the line DE, does not pass through E, let it cut the diameter GH in some other point E′. Then (*prop. ii.* and *cor.* 1) the diameter GH is harmonically divided in E′ and K: and, again, by the construction (and *prop. i.*) HG is harmonically divided in E and K. Whence EG : EH :: E′G : E′H, or GH : GE :: GH : GE′, or GE′ = GE, which is impossible. Wherefore the chord of contact does not pass through E′; that is, it does pass through E, as enunciated in the theorem.

Again, if the line KD (*fig.* 1.) does not cut the conic section, the tangents from all the points of K will have their contacts on opposite sides of the vertex G of the diameter GH; and the more remote from K, we take the point D, the nearer does N, and the more remote does M become from G. The chords of contact therefore cross the diameter GH between G and H, that is, within the curve. The same holds, simply reversed in order, also, when the points are taken on the other side of K from D. Whence the chords of contact pass through E situated *within* the curve.

If the line KD (*fig.* 2.) does cut the curve, it may be shown that whilst D is in the branch of KD above GH, both points of contact will be on the same side of the vertex M of the diameter GH; and hence the chords of contact pass through a point E *without* the curve.

Scholium.

This proposition and the preceding, which are the converses of one another, show that there is for every point a corresponding line, and for every line a corresponding point, which have a very remarkable relation to each other. The point itself, through which the chord of contact always passes, is called the *pole*, and the line in which the points of contact or the intersection of the varying lines is always situated, is called the *polar;* both taken in respect of the conic section. Taken together they are called *conjugate polars* *.

PROP. IV.

Any chord of a conic section drawn through a pole, is harmonically divided by that pole and its polar.

(*Figures preceding proposition.*)

Let E be the pole, and KD the corresponding polar, in reference to the conic section GMH; and let PQ be any chord drawn through E to meet the polar in D: then

QE : PE :: QD : DP.

For, draw the tangents DM, DN; then (*prop. iii.*) the chord of contact passes through E. Also, since the line PQ is drawn through the intersection D of the tangents, it is harmonically divided by the chord MN and point D (*prop. i.*): but it meets the chord of contact in E; and hence E is the conjugate harmonic point to D. Whence PQ is harmonically divided, or

QE : PE :: QD : PD.

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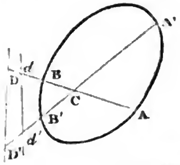
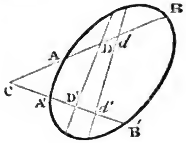
* This system of names is exceedingly appropriate. The term *pole* of a straight line in reference to a conic section was first used by Servois, and the term *polar* was proposed by Gergonne. The simplest case is that of the *focus and directrix*, which are conjugate polars according to the most general definition.

Corollary.





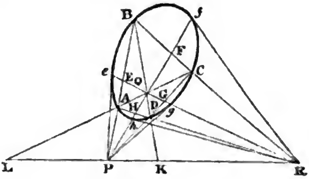
If two chords AB, A'B' intersect in C, and be harmonically divided by D and D': then DD' will be the polar to the pole C.

For if not, let *dd'* be the polar to the pole C; then by the proposition the chord AB is harmonically divided in *d* and C, and the chord A'B' is harmonically divided in *d'* and C. Wherefore, reasoning by composition of ratios, $AD = Ad$, and $A'D' = A'd'$, which is impossible. Whence *dd'* is not the polar to C, and no other line except DD' is the polar to C.

PROP. V.

The pole *and the* polar *are connected with each other in reference to a conic section in the following manner:—*

Through any point within or without a conic section draw two lines to cut the curve, and join the pairs of intersections thus made, the lines giving two other points by their mutual sections: then, any one of the three points is the pole, *and the line drawn through the other two is its* polar, *in reference to the conic section.*



If from any point P the lines PAB, PDC be drawn, meeting the curve; and if AC, BD intersect in Q, and AD, BC intersect in R; and if, finally, PQ, QR, RP be drawn: then P is *pole* to the *polar* QR; Q is the *pole* to the *polar* RP; and R is the *pole* to the *polar* PQ.

For, join PQ cutting the curve in *h*, *f*, and the sides of the quadrilateral AD, BC in H and F; and RQ cutting the curve in *g*, *e*, and the sides DC, BA in G and E; and, finally, join PR, meeting AC, BD in L and K.

Then *(property of quadrilateral)* AB, CD, are harmonically divided in E and G; and hence (*prop. i. cor.* 4) the line EG is the chord of contact in reference to the point P. In the same manner HF is the chord of contact in reference to the point R. That is, EG or RQ is the polar to the pole P; and HF or PQ is the polar to the pole R.

Again, since EG is the chord of contact to P, the chord *hf* is harmonically divided in Q and P (*prop. i.*); and for a similar reason, *eg* is harmonically divided in Q and R. Whence (*prop. iv. cor.*) PR is the polar to the pole Q. The proposition is hence established.

Corollary 1.

The system of lines and points which enter into the enunciation of this theorem will equally admit of the following general construction, giving a proposition of great elegance and utility.

If the pairs of opposite sides, AB, CD, and AD, BC, of a quadrilateral ABCD inscribed in a conic section, be produced to meet in P and R, and if the diagonals meet in Q: then PR is the polar of the pole Q, RQ the polar of the pole P, and QP the polar of the pole R.

Corollary 2.

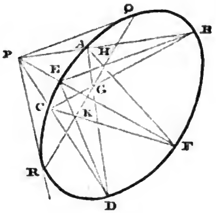
If the pairs of opposite sides AB, CD, and AD, BC of the quadrilateral ABCD inscribed in a conic section be produced to meet in P and R; and if from P and R tangents P*e*, P*g*, and R*f*, R*h* be drawn to the curve, the chords of contact *eg* and *fh* will each pass through the other point of intersection R and P respectively, and both will pass through Q the intersection of the diagonals.

For this is only a different mode of describing the general figure in the proposition; and hence the properties must be equally true.

Corollary 3.

If from a point P *without* the conic section any secants PAB, PCD, PEF, *etc.* be drawn, and the diagonals AD, BC be drawn to intersect in G, the diagonals AF, BE to intersect in H, the diagonals DE, CF to intersect in K, *etc.*: then G, H, K, *etc.*, will be in one line, viz. the chord of contact QR belonging to the tangents from P.

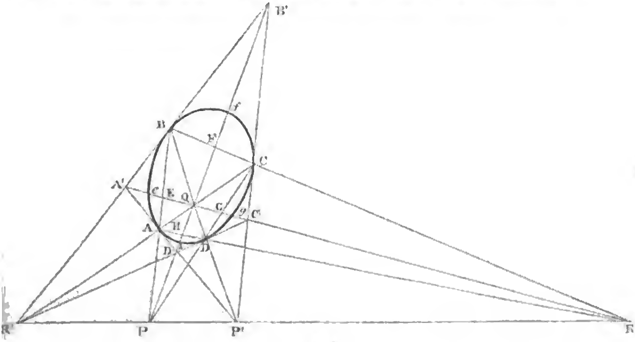
For QR is the polar to the pole P, and the conclusion is a case of the general theorem.



PROP. VI.

If a quadrilateral be inscribed within a conic section, and from its angular points, as points of contact, a quadrilateral be described about the conic section: then,

1. *If the four pairs of opposite sides be produced to meet, the four points of intersection will be situated in one line.*
2. *The four diagonals of these two quadrilaterals will pass through the same point.*
3. *The point and line will be a pole and polar in reference to the conic section.*



Let ABCD be the inscribed quadrilateral, having its opposite sides produced to meet in P and R; and let A′B′C′D′ be the circumscribed one touching the conic section in A, B, C, D, and having its opposite sides produced to meet in P′ and R′; then P′, P, R′, R, will be in one line; the diagonals AC, BD, A′C′, B′D′, all pass through the same point Q; and Q is the pole to the polar P′R.

First. For, since AC, BD are the diagonals of the quadrilateral ABCD, they are divided harmonically in P′ and R′ by the line PR and point of intersection Q; and since the chords of the conic section AC, BD are divided harmonically in Q and P′, R′, the line P′R′ is the polar to the pole Q (*cor. to prop. iv.*). But Q is also the pole to the polar PR (*prop. v. cor.* 1); and hence the lines PR and P′R′ coincide; or the four points of intersection of the four pairs of opposite sides are in one line.

Second. Again, (*prop. v. cor.* 1,) P is the pole to the polar EG or *eg*, and through this there is drawn the chord AB; wherefore (*prop. ii.*) the intersection A′ of the tangents at A, B, is in EG. In a similar manner, the tangents at C, D meet at C′ in EG. Wherefore A′C′ coincides with the polar EG of the pole P; or, in other words, the diagonal A′C′ of the circumscribed quadrilateral passes through Q the intersection of the diagonals of the inscribed one. In the same way it will follow, that the other diagonal B′D′ of the circumscribed quadrilateral also passes through Q. The second part of the proposition is, therefore, proved.

Third. It has also been proved that Q is the pole of the identical lines PR, P′R′; that is, of R′R in the figure above.

Corollary 1.

The diagonals AC, BD of the inscribed quadrilateral pass respectively through the intersections R′, P′, of the pairs of opposite sides of the circumscribing one: and

The diagonals A′C′, B′D′ of the circumscribed quadrilateral pass respectively through the intersections R, P of the pairs of opposite sides of the inscribed one.

Corollary 2.

The diagonals AC, BD, and A′C′, B′D′ of the inscribed and circumscribed quadrilaterals divide the polar PR belonging to the pole Q through which the four diagonals pass, harmonically, in the points R′, P, P′, R, or

$$R'P : PP' :: RR' : RP'.$$

For these several points, R′, P, P′, R are the limits of the segments of the diagonal PQ of the complete quadrilateral ABCD, made by the two other diagonals AC and BD.

Corollary 3.

The lines EF, GH being drawn, will pass through R′, and EH, FG will pass through P′.

Corollary 4.

If *e*, *f*, *g*, *h* be made the angular points of an inscribed quadrilateral, and the points of contact of a circumscribed one; then the intersections of the opposite sides of the former will be R′ and P′, and of the latter R and P; and the diagonals of the inscribed and circumscribed original quadrilaterals will become those of the circumscribed and inscribed secondary quadrilaterals.

For *eg* and *hf* are the chords of contact in reference to P and R; and hence, reasoning as in the general proposition, the conclusion follows at once.

Corollary 5.

If the sides A′B′, C′D′ of a trapezium circumscribed to a conic section meet in R′, and the sides A′D′, B′C′ in P′; then the chords of contact AC, BD respectively pass through P′ and R′; and if they intersect in Q, they are respectively divided in harmonical ratio in Q, P′ and Q, R′.

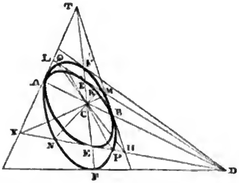
For these are the diagonals of the inscribed quadrilateral, and the conclusion follows from *cor.* 1.

Scholium.

The mutuality of the two systems of quadrilaterals suggested by the fourth corollary leads to some curious and elegant results, which, whilst they are necessarily excluded from this work, will yet furnish a useful and interesting exercise for the student's investigating powers.

PROP. VII.

If any number of conic sections touch two straight lines in the same two points, and from any point in the extension of the common chord of contact tangents to all the conic sections be drawn, their points of contact will all be in the same straight line, tending to the intersection of the first-named tangents.



Let AT, TB be two straight lines meeting in T, and are touched in A and B by any conic sections ABE, ABF, *etc.*; and from any point D in AB produced, draw tangents to all the curves: then the points of contact E, E′, F, F′, *etc.*, are all in the same straight line.

For make AC : CB :: AD : DB; then the chord of contact EE′ passes through C (*prop. i.*). Moreover, the tangents AT, BT meet at T, in the chord of contact (*prop. ii.*), and hence the fixed line CT is the chord of contact for the point D, in reference to the curve ABE. In like manner, it is the chord of contact for D in reference to the curve ABF; or to any other, and therefore to all other conic sections which touch AT and BT in A and B. All the points of contact, therefore, range in one line tending to the point T.

Corollary 1.

The point D is the pole to the polar CT, as is evident from *prop. i., cor.* 2.

Corollary 2.

The chords NP, QR are divided harmonically in E, D, and E′, D. For EE′ is the chord of contact to the point D.

Corollary 3.

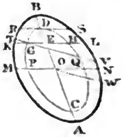
The lines LH, QP, KM, NR, all intersect in the same point; viz. that, C in which CT, AB intersect.

For FF′ is the polar to the point D in reference both to the curve and the lines TA, TB; and the conclusion is obvious.

PROP. VIII.

Let there be two concentric, similar and similarly situated conic sections, and any line be drawn to cut them both: then,

1. *The segments intercepted between the two curves will be equal.*
2. *If the chord of the exterior touch the interior, it will be bisected in the point of contact.*
3. *The rectangle under the segments into which either branch of the interior curve divides the part intercepted by the other will be constant, whilst the line preserves its parallelism.*
4. *For lines differently inclined, these rectangles will be as the squares of their parallel diameters or their parallel tangents.*



First. Let curves MBN, PDQ be similar and similarly situated, and have a common centre O; and let KL, any chord of the outer, intersect the inner one in G and H: then KG = HL, and KH = GL.

For draw the diameter conjugate to KL a chord of MBN; and since the curves are similar, it is conjugate to GH a chord of PDQ. Whence both KL and GH are bisected in E, and the conclusion follows.

Second. Draw the tangent RS at D; then it is conjugate to the diameter CD of PDQ, and hence to the diameter AB of the similar and similarly situated curve MBN. Hence it is bisected in D by the diameter AB; that is, in its point of contact with the exterior curve PDQ.

Third. The rectangle KG . GL or KH . HL is constant for the same inclination of the line KL to any fixed line.

For since the figures PDQ, MBN are similar and similarly situated, we have

$$OB^2 - OE^2 : EK^2 :: OB^2 - OD^2 : RD^2,$$

$$\text{and } OD^2 - OE^2 : EG^2 :: OB^2 - OD^2 : RD^2; \text{ whence}$$

$$OB^2 - OE^2 : OD^2 - OE^2 :: EK^2 : EG^2, \text{ and}$$

$$OB^2 - OD^2 : OD^2 - OE^2 :: EK^2 - EG^2 : EG^2.$$

$$\text{But } OB^2 - OD^2 : OD^2 - OE^2 :: RD^2 : EG^2;$$

$$\text{and hence } RD^2 = EK^2 - EG^2 = KG . GL = KH . HL.$$

Fourth. Let KL, TV be two chords intersecting at G in the interior figure, and ON, OW semi-diameters (or tangents) parallel to KL, TV: then it is at once obvious that, in virtue of *pp.* 120, 140, 167, we have

$$KG . GL : TG . GV :: ON^2 : OW^2.$$

Corollary 1.

If a series of chords to a conic section tend to mutually bisect each other, as by the law of their connexion they approximate more and more nearly to successive coincidence, they will all be tangents to another conic section, concentric, similar, and similarly situated with respect to the given one.

Corollary 2.

If through any point G in the plane of a conic section, lines KGL, TGV, *etc*, be drawn meeting the curve, as in K, L, T, V, *etc.*, in pairs; and if in these, points H, Z, *etc.*, be taken, so that HL = KG, ZV = TG, *etc.*: then H, Z, *etc.*, will all be situated in the curve of a concentric, similar, and similarly situated conic section passing through G.

The asymptotes of the hyperbola and its asymptotes are a particular case of this property.

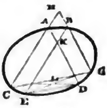
Scholium.

For the two concentric parabolas we must substitute equal parabolas, having their principal axes coincident. The same properties are true, but a slight though obvious change must be made in the mode of demonstration.

PROP. IX.

If a polygon be inscribed in a conic section, all whose sides but one are parallel to given lines; that remaining side will touch a concentric and similar conic section, or be parallel to a given line, according as the entire number of the sides is odd or even.

The demonstration involving several distinct steps, we shall take them seriatim.



1. Let two triangles ACD, BGE inscribed in a conic section, have their sides CA, AD parallel respectively to EB, BG: then the lines CG, ED will be parallel.

For let AC, BG intersect in H, and AD, BE in K. Then, (*pp.* 120, 140, 167),

$$AH . HC : GH . HB :: EK . KB : AK . KD.$$

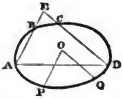
But by the parallels we have AH = BK and AK = HB; whence

$$CH : HG :: EK : KD.$$

Also the angles at H and K are equal; and hence the triangles CHG, EKD are similar: and the sides CH, HG being parallel to EK, KD, the sides CG, ED are also parallel.

2. *Let the given polygon be the triangle CAD; the third side will touch a conic section.* The lines CD, EG are generally unequal, and therefore, also, the segments into which they mutually divide each other in L; but as A and B approximate more and more closely towards coincidence, the point G approaches more and more closely towards the middle of each of these chords. At the extreme limit, or the point of absolute coincidence, they therefore bisect one another. Whence the variable side CD of the triangle CAD touches a conic section concentric, similar, and similarly situated with respect to the given one CABGDE; or in the case of the parabola, another parabola, equal and co-axial with the given one (*prop. viii. cor.*).

3. *Let the given polygon be the quadrilateral ABCD; then the fourth side AD will be parallel to a given line.*



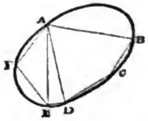
For let AB, CD meet in E: and draw the semi-diameters or tangents OP, OQ parallel to them.

Then $AE . EB : DE . EC :: OP^2 : OQ^2$, a given ratio. Also since the three sides of the triangle EBC are parallel to given lines, the ratio BE : EC is given: wherefore also the ratio AE : ED is also given.

Now the sides EA, ED of the triangle AED being parallel to given lines, and having also a given ratio, the base AD is also parallel to a given line.

4. *Let the pentagon ABCDE have four of its sides AB, BC, CD, DE parallel to given lines; the fifth EA will touch a conic section.*

For draw AD. Then by the preceding case AD will be parallel to a given line. Hence there are the sides AD, DE of the triangle ADE parallel to given lines; wherefore, the remaining side AE will touch a conic section concentric, similar, and similarly situated to the given one.



5. *Let the hexagon ABCDEF have all its sides except AF parallel to given lines; then AF will always be parallel to a given line.*

For draw AD: then, as before, the three sides AB, BC, CD of the quadrilateral ABCD being parallel to given lines, the fourth AD is also parallel to a given line. Whence three of the sides AD, DE, EF of the quadrilateral ADEF being parallel to given lines, the fourth AF (which is the sixth side of the hexagon ABCDEF) is also parallel to a given line.

6. Generally, since we may divide every polygon of an *even* number (n) of sides (into $\frac{n}{2} - 1$) quadrilaterals; we may, by reasoning in the same continuous manner, show that the last of them is parallel to a given line.

Again, a polygon of an *odd* number (n) of sides may be divided into $\frac{n-1}{2} - 1$ quadrilaterals and one triangle, we can similarly show that the last of them touches a conic section, concentric, similar, and similarly situated with the given one.

Scholium.

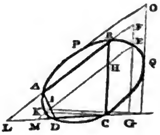
This demonstration was first given in the *Mathematical Repository, vol. vi. pp.* 45—47. There are several other connected properties, some of them more general, and some but variations of the fundamental conception involved in this. Their demonstration, however, would occupy more space than can be devoted to the subject in this work; and they are, therefore, left as exercises for the student.

PROP. X.

If from any point in the curve of a conic section lines be drawn to make given angles with two adjacent sides of a given inscribed quadrilateral; then the rectangle contained by the segments of one of these lines intercepted by either pair of opposite sides, will have a given ratio to the segments of the second line intercepted by the other pair of opposite sides of the quadrilateral.

CASE I.

Let the lines FEG, EHI be drawn from any point E, *parallel* to the two adjacent sides AB, BC of the inscribed quadrilateral ABCD; and let that parallel to AB meet the opposite sides BC, DA in H and I, and that parallel to BC meet the opposite sides AB, CD in F and G: then, wherever in the curve the point E be taken, the ratio HE.EI : FE.EG will be given and constant.



For draw the tangents PO, OQ parallel to AB, BC; and produce, where necessary, the lines AB, EH, CD, to meet in L and M; also let HI meet the curve in K, and join KC, IG.

Then, by the sim. trians. BLC, HMC, and ALD, IMD,

CL : CM :: BL : HM, and
LD : DM :: LA : MI; wherefore,
CL . LD : CM . MD :: BL . LA : HM . MI.

But (*pp.* 120, 140, 167) we have

CL . LD : CM . MD :: BL . LA : EM . MK; or
HM . MI = EM . MK.

Wherefore, HM : ME (:: CM : MG) :: MK : MI; and hence KC is parallel to GI, and the triangles KHC, IEG are similar. Hence, again,

HK : HC :: IE : EG, or
KH . HE : BH . HC :: HE . EI : BH . EG.

But KH . HE : BH . HC :: $OP^2 : OQ^2$;

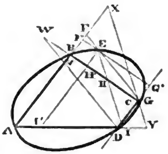
whence HE . EI : BH . EG (= FE . EG) :: $OP^2 : OQ^2$;

and PO, OQ being tangents to a given conic section parallel to given lines, are themselves given, and hence their ratio is given. Wherefore HE . EI : FE . EG is a given ratio.

CASE II.

Let FG, HI drawn through any point E in the curve be inclined to the adjacent sides AB, BC in any given angles: then the ratio HE . EI : FE . EG, will be given and constant, for all positions of E in the curve of the conic section.

For draw H′I′ and F′G′ through E parallel to the adjacent sides AB, BC, meeting the opposite pairs of sides in H′, I′, and F′, G′. Also through C draw XY parallel to EI, meeting AB, AD in X and Y; and through D draw VW parallel to FG meeting AB, BC in V and W.



Then, the triangle HEH′ is similar to BXC, and IEI′ to XAY: and we have,

H′E : EH :: BX : XC, and
EI′ : EI :: AX : XY; wherefore,
H′E . EI′ : HE . EI :: BX . XA : CX . XY.

Again, the triangles EFF′, VBW are similar, as are likewise the triangles EGG′, WDC; whence,

F′E : EF :: BW : WV, and
EG′ : GE :: WC : WD; wherefore,
F′E . EG′ : FE . EG :: BW . WC : DW . WV.

But the triangles BXC, AXY, WBV, and WDC are given in species; wherefore the two final ratios of each of the proportions above are also given, viz.—

AX . XB : CX . XY, and BW . WC : DW . WV.

Moreover, the ratio H′E . EI′ : F′E . EG′ is given by the former case of the proposition, viz. that of $PO^2 : OQ^2$; and hence the ratio

HE . EI : FE . EG

is also given, and the proposition is fully established.

Corollary 1.

The ratio of HE . EI : FE . EG is compounded of the ratios,

$H'E . EI' : F'E . EG'$, $CX . XY : AX . XB$, and $BW . WC : DW . WV$; and it may be written,

$$HE . EI : FE . EG :: \frac{CX . XY}{AX . BX} . H'E . EI' : \frac{DW . WV}{BW . WC} . F'E . EG'.$$

Corollary 2.

If there be taken any number of points E_1, E_2, E_3, *etc.*, in the plane of the conic section, such that the rectangles under the segments of the lines drawn through them parallel to given lines XDY and DVW, and intercepted by the sides of the inscribed quadrilateral, as in the proposition, have the same ratio as the segments of those drawn through E: then all these points, E_1, E_2, E_3, *etc.*, will be situated in the curve of the conic section.

This so obviously follows, *ex absurdo*, as the converse of the proposition, as to render a detailed and formal proof altogether superfluous.

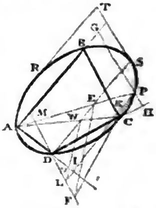
Corollary 3.

Five points are necessary, and they are sufficient for the determination of a conic section. For all the ratios are fixed by the point E, and the conclusion follows from the preceding proposition.

PROP. XI.

If there be a quadrilateral figure inscribed in a conic section, and if from one of the angular points there be drawn lines parallel to the sides of the quadrilateral about the opposite angle; if also from the two remaining angles there be drawn lines to any point in the curve to meet the parallels, the intercepted portions of the parallels, estimated from their common point, will have a given ratio, wherever in the curve the fifth point be taken.

From D, one of the angles of the inscribed quadrilateral ABCD, draw DE, DF parallel to the sides AB, BC, containing the opposite angle B; and from any point P in the curve draw PA, PC to meet DE, DM in E and F: then, DE : DF will be a constant ratio for all positions of P in the curve of the conic section.



For, through P draw GPH parallel to BC, meeting AB, CD in G and H, and PKI parallel to AB, meeting BC, AD, DM in K, I, L; and let DM meet AB in M. Then by parallels, $LM = PG$; and

$$PK : KL :: PC : CF :: PH : DF, \text{ and}$$
$$IP : DE :: AP : AE :: LM (= PG) : DM;$$

whence,

$$KP . PI : KL . DE :: HP . PG : FD . DM, \text{ or}$$
$$KP . PI : HP . PG :: KL . DE : FD . DM.$$

But $KP . PI : HP . PG$ is given (*prop.* x), and hence $KL . DE : FD . DM$ is given. Also KL and DM are both given, and hence $KL : DM$ is given; and, therefore, finally, the ratio $DE : DF$ is a given ratio, independently of the particular position of the point P.

Corollary 1.

Since if we draw the tangents RT, TS parallel to AB, BC, we have

$$KL . DE : FD . DM :: RT^2 : TS^2,$$

we shall be able to find third proportionals to KL and RT and DM and TS; then calling these m and n, we shall have

$$\mathrm{DE} : \mathrm{DF} :: m : n.$$

Corollary 2.

Since DE : DF is a given ratio and EDF a given angle, the line EF is parallel to a given line.

Corollary 3.

If any number of points, P_1, P_2, *etc.*, be taken, and the corresponding segments cut from the parallels DE, DF be always in the same ratio, the points P_1, P_2, *etc.*, will all be situated in the conic section described through A, B, C, D, P.

This follows at once, *ex absurdo*, since the value of the ratio is fixed by the point P, and the conic section depending on it, therefore, is invariable.

PROP. XII.

If about two points, as poles, two given angles revolve, in such a manner that a specified side of one angle always intersects a specified side of the other angle, in a given straight line; then the other sides of these angles will always intersect in a given conic section.

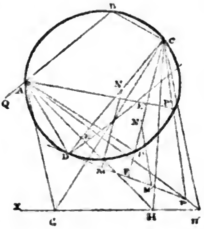
If about A and C as poles the angles DAG, DCG revolve so that the sides AG, CG always intersect in the straight line GH; then the two other sides AD, CD will always intersect in the curve of a conic section.

Let G and D, H and P, be two cotemporaneous positions of the two intersections of the sides of the angles, so as to define the line GH in which the first pair of sides constantly intersect: it is to be shown that D, P, *etc.*, are always in the curve of the same conic section.

Make DAB = AGX, and DCB = CGH; draw DE, DF parallel to AB, BC meeting AP, CP in E and F; make DM = DE; draw MN parallel to CF, and produce BA to Q.

Then, since (*hyp.*) DAG = HAP, we have GAH = DAP = DAE. Also, since (*constr.*) BAD = AGX, and DE is parallel to AB, we have ADE = QAD = AGH; and the triangles ADE, AGH having two angles of the one equal to two angles of the one, are equiangular, and therefore similar.

Again, since (*hyp.*) PCH = DCG, we have GCH = DCP = DCF. Also, since (*constr.*) DCB = CGH, and DF is parallel to BC, we have CDF = DCB = CGH; and the triangles DFC, GHC are therefore equiangular and similar.



Hence AG : GH :: AD : DE, and

CG : GH :: CD : DF :: ND : DM (= DE).

Whence AG : CG :: AD : ND; and AG : CG is given, since AG and CG are both given. Consequently, the ratio AD : DN are both given.

Moreover, the ratio AD : CD is given, since the lines themselves are both given; and hence the ratio NC : CD, or DE : DF, is given.

In the next place, conceive that a new point H′ in GH is taken as the intersection of the lines AG, CG, and that P′ is the corresponding intersection of the other sides. Then, obviously, the triangle ADE′ is similar to AGH′, and DF′C to GH′C; and as before, the ratio DE′ : DF is given, and the same as the ratio DE, DF.

If now a conic section be described through A, B, C, D, P, it will be fixed and invariable (*prop. x. cor.* 3); and since all the lines drawn to fulfil the conditions are cut by the parallels DE, DF drawn through the angle D parallel to the sides which contain the opposite angle, are divided in the same ratio, all the points P′, P″, P‴, *etc.*, in which they intersect, will be in the curve of that conic section (*prop. xi. cor.* 3).

Scholium.

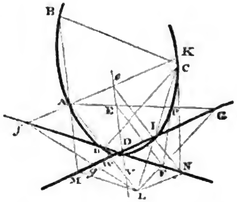
This demonstration is immediately applied to the case in which the corresponding sides revolve in the same angular direction (always giving the ellipse or its varieties, or its limiting curve, the parabola); but that for the revolution being in the opposite angular direction, will be precisely similar to it, in respect of general reasoning, and give the hyperbola.

This theorem was given by Newton, as the foundation of his method of constructing the conic sections (*Princ. lib. i. lem.* 40). Many of the details may be consulted, in a very amplified state, in Maclaurin's *Geometrica Organica*, 1719.

PROP. XIII.

If the three sides of a variable triangle pass constantly through three fixed points, and two of its angular points be always situated upon two fixed lines; then the third angle will always be situated in the curve of a conic section.

Let A, L, C, be three fixed points, and DF, DG, two fixed lines; let the side PG of the variable triangle pass through A; the side GF through L; and the side FP through C: then the point P will always be situated in the curve of a conic section.



Let us, before examining the general curve in which P moves, find such points in it as are dependent on particular relations of the variable data.

1. Suppose F to be infinitely distant, on either side of D: in which case, the lines LF, CF will become LM, CB, parallel to the given line DF. Let the parallel LM meet DG in M; then M will be the point in DG corresponding to G, when F is infinitely distant, and MA will be the corresponding position of AG or AP; and the intersection B of CB, MA will be one of the positions taken by P, or one of the points in the curve.

2. Draw in the same manner AK, LN parallel to DG, and let NC be drawn to meet AK in K; then, as in the preceding case, K will be a point in the curve traced by P.

3. The points A, C will be points in the curve. For when AG in revolving arrives at C, the line CF will cut it in C; and hence C is a point in the curve. In the same way it appears that A is another point.

4. When one of the points F, G is at D, the other will also be at D, and therefore, again, P will be at D, which is one of the points of the curve.

Now (*prop. x. cor.* 3) it has been seen that five points, as A, B, K, C, D, determine the special conic section that can be made to pass through those five points. Let it be drawn: then it is required to show that any other point P formed agreeably to the hypothesis, is in the same conic section.

General case.

Draw DE parallel to MB, meeting AG in E and AK in *e*; and join EF, *e*N, Also (though not in the figure) join AD, DC.

Then, since DE is parallel to MA, and DF to ML (*constr.*), we have

GD : GM :: DE : MA :: DF : ML; or

DE : DF :: MA : ML.

Now since MA, ML are given, their ratio is given, and hence DE : DF is a given ratio.

Again, since AM is parallel and equal to *e*D, and ML is parallel and equal to DN, it follows that A*e* is parallel and equal to LN, and therefore, again, that AL is parallel and equal to *e*N. Consequently, we have

DE : DF :: AM : ML :: D*e* : DN.

Whence (*prop. xi. cor.* 3) the point P is in the conic section described about A, B, K, C, D.

Corollary 1.

If AL, CL be drawn to cut DF, DG in H and I, these points will also be in the curve.

For in this case the point F is represented by H, and therefore CF takes the position CH; whence the intersection P of AG, CF is represented by that of AH, CH, or by H. In the same way I is a point in the *constructed curve.*

Again, these points are in the conic section ABKCD. For it has already been proved that AL is parallel to N*e*, and hence it cuts segments DV, DH from the parallels DE, DF (to AB, BC) such that

D*e* : DN :: DV : DH.

Wherefore, as before, H is in the conic section ABKCD; and in a similar way it is proved that I is in the same conic section.

Corollary 2.

Let AHDIC be any pentagon inscribed in a circle, and the adjacent sides DH, DI be indefinitely produced; and let AH, IC be produced to meet in L: then, if any line LFG be drawn to cut DH, DI, in F and G, and AG, CF be drawn; their intersection P will be in the curve of the conic section circumscribed about the pentagon AHDIC.

For draw LM, CB parallel to DH, and LN, AK parallel to DI; also draw MA to meet CB in B, and NC to meet AK in K.

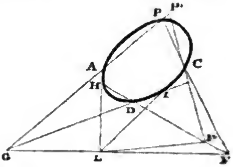
Then it has been proved in the proposition that P is in the conic section described about ABKCD; and in the first corollary, that AHDIC is the same conic section that was described about ABKCD. That is, the property alleged in the corollary is true.

Scholium.

The discovery of this theorem is claimed both by Maclaurin and Braikenridge; and it was probably the independent discovery of each. Braikenridge, however, was the first to publish it, and hence we may with propriety designate it as his theorem. Both, however, gave it in a form a little different, and subservient to the organical description of the conic sections. It is in the arrangement of this chapter the foundation of Pascal's theorem, next in the series.

PROP. XIV.

The three pairs of opposite sides of a hexagon inscribed in a conic section, being produced to meet, the three points of intersection will all be situated in one straight line.



Let AHDICP be a hexagon (all the angles salient in the figure, but not necessarily so), and let the opposite pairs of sides intersect in G, L, F: then G, L, F will be in one straight line.

For, if not, let GL meet the side HD in F', and draw $F'C$ to meet AP in P'.

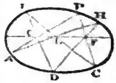
Then, (*pr. xiii. cor.* 2,) the point P' will be in the curve of the conic section AHDIC. But, by hypothesis, A and P, points in the straight line GP', are also in the curve of the same conic section; whence the straight line AP cuts the conic section in the three points A, P, P'; which (*prop. viii.*) is impossible.

Wherefore, the point P' is not in the curve of the conic section, and hence the point F' is not in the line GL.

In the same manner it may be proved that no other point but P in the line GA can be in the conic section; and hence, that no point but F in the line HD can be in the line GL. The three points are, therefore, in one straight line.

Scholium.

This theorem is one of the most remarkable and most useful properties of the conic sections yet known; and it branches out into a great number of cases, according to the relative positions of the six angular points A, H, D, I, C, P, of the hexagon inscribed in the conic section. One of these is figured in the margin, expressive of what at first sight might be considered a totally different theorem. Its enunciation may be as follows:—



Upon the opposite sides AC, HI of a quadrilateral inscribed in a conic section as bases, let triangles APC, IDH be described, having their vertices in the arcs of the sections of which IH, AC are the respective chords, and cutting each other in G and F: then FG being drawn, will pass through L, the intersection of the diagonals of the quadrilateral IHCA.

One of the converse theorems was published by Pascal, in a tract (eight pages) under the title of *Essais pour les Coniques*, in 1640. There is every reason to believe that he was acquainted with the theorem in its present form: but that from its fruitfulness in the discovery of conical theorems, and its utility in the construction of problems, he concealed it, in accordance with the practice of the time in which he lived.

The theorem attracted some attention amongst the geometers of Pascal's time. Curabelle describes it as containing the substance of the first four books of the conics of Apollonius, either as cases or consequences; and Marsennus speaks of it as a theorem with four hundred corollaries. Still, after that immediate period, the theorem attracted but little notice till comparatively recent times. Maclaurin actually describes the figure, without at all referring to the property, even though it followed as a consequence of the very method of organical description which he was then explaining. *Phil. Trans.* No. 436.

The general method of proof employed has been to establish, as can be easily done, the property for the circle, and then transfer it to the conic sections generally by means of perspective considerations. Sir John Leslie *(Geometry of Curve Lines)*, however, employed the method of *poles*, in the same manner as Maclaurin and Braikenridge; and, except as to form, the method used in this place is but a recurrence to that very elegant and effective mode of proof. Another will be found in the chapter on *Transversals*, which I first gave to the *Lady's Diary* for 1843.

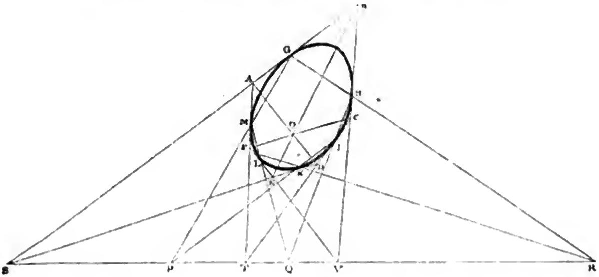
There are four very elegant demonstrations of the particular case of the conic section being a circle, given in the *Mathematical Repository*, vol. ii. NS. by Earl Stanhope, Mr. Ivory, Mr. Lowry, and Mr. R. Nicholson. That of Mr. Ivory has been copied into Bland's *Geometrical Problems;* and the late Captain Drummond, Royal Eng. and Under Secretary of State for Ireland, also gave a proof of the converse property in the *Phil. Mag.* No. 299.

The general theorem, though enunciated for the inscribed hexagon, and generally viewed as a salient figure, does, in fact, relate to the line joining six points in the conic section, *taken in any order whatever:* the only conditions being, that the lines be drawn from points, without having more than two meeting in the same point, and the point *from* which the first line is drawn being that *to* which the last is drawn.

The number of hexagonal figures thus drawn, and having the same six points for their summits, must evidently be very great; and the preceding theorem applies to them all *simultaneously*. By combining all the cases that can thus arise, in two, three, four, five, or all, at a time, many curious and interesting theorems will be found to result. The limited space to which, in a work like this, any special subject must be confined, prevents the addition of even a specimen in this place.

PROP. XV.

If a hexagon be circumscribed to a conic section, the three diagonals which join the three pairs of opposite summits will all pass through the same point.



Let ABCDEF be a hexagon circumscribed to a conic section: then the diagonals AD, BE, CF will pass through the same point O.

For let GHIKLM be the inscribed hexagon, whose angular points are the points of contact of the circumscribed hexagon. Then (*prop. xiv.*) the opposite sides have their intersections P, Q, R in one straight line.

Hence, the point R being considered as pole, EB will be the corresponding polar in reference to the conic section; and P will be the pole to the polar AD. Wherefore their intersection, O, will be the pole to the polar PR.

Again, Q is in the polar PR, and hence the line CF will pass through O, the pole of PR, *prop. iii.* The three diagonals, AD, BE, CF, therefore pass through one point O.

Corollary.

If the three diagonals of a hexagon intersect in a point, the hexagon is circumscriptible to a conic section.

This, which is the direct converse of the proposition, is deducible at once by reasoning, *ex absurdo.*

Scholium.

This remarkable theorem was discovered by Brianchon, who published it amongst other important properties of a collateral nature, in the thirteenth number of the *Journal of the Polytechnic School*, page 301. This and Pascal's theorem are so intimately connected by means of the theory of poles and polars, that either flows as an immediate consequence of the other.

EXERCISES ON THE GENERAL PROPERTIES OF THE CONIC SECTIONS.

1. If all the sides but one of a polygon inscribed in a conic section pass through given points; that one will either pass through a given point, or touch a given conic section.

2. If all the angles but one of a polygon circumscribed to a conic section be upon given lines, that one will be situated in the curve of a given conic section.

3. If each of the sides of two triangles circumscribed to a conic section intersect one and one in order, and lines be drawn to join the three pairs of opposite intersections, they will pass through the same point.

4. The six angular points of the two triangles in the last theorem are situated in one conic section.

5. If two triangles be inscribed in a conic section, they will by their mutual intersections form a hexagon, in which a conic section may be inscribed.

6. If from either angle of a pentagon described about a conic section a line be drawn to the point of contact most remote; then, if the diagonals be drawn to join the other pairs of opposite angles, they will intersect in the first-named line.

7. In a hexagon circumscribed to a conic section, if we prolong two and two the sides, separated by one side, to intersect, we obtain six points of intersection, which being considered as the summits of a new hexagon, the diagonals of this hexagon will intersect in three points situated one upon each of the three diagonals of the original circumscribed hexagon, and when the diagonals of the inscribed hexagon also meet one point, that point of intersection coincides with that in which the three diagonals of the circumscribed one intersect.

8. All the conic sections which can be inscribed in the same quadrilateral, have their centres situated on the straight line which bisects the diagonals of the quadrilateral.

9. The polars of a point in reference to all the conic sections which can be described about a given quadrilateral, pass through one point.

10. The poles of a straight line taken as polar in reference to all the conic

sections inscribable in the same quadrilateral, are situated in the same straight line.

11. When a conic section touches three sides of a triangle BCD, of which BD is touched in A, if a fourth tangent be drawn to meet BD, CD in M and N; then, whatever be the position of this fourth tangent, the ratio AM.CN : BM.DN will be constant.

12. If the straight lines BD, CD, be touched in A and C by a conic section, and any third tangent be drawn to meet these in M and N; then the ratio AM . CN : DM . DN will be constant wherever the tangent may be drawn.

13. If an indefinite number of hyperbolas, situated in the same plane, have the same focus and directrix, then the intersections of the asymptotes to the consecutive curves will be in the curve of a parabola, which will also be confocal with the hyperbolas.

14. If from a focus of a conic section we draw perpendiculars to any two tangents, the straight line which joins the feet of these perpendiculars will be perpendicular to that which is drawn from the other focus of the curve to the intersection of the tangents.

15. Let AB, CD, be two straight lines cutting one another in G, and terminating in the ellipse of which F is a focus, and draw AF, BF, CF, DF: then

$$AF . BG + BF . AG : CF . DG + CG . DF :: AB : CD.$$

16. A straight line moving parallel to a given line, touches continually a series of ellipses which have the same foci, the points of contact are situated upon an equilateral hyperbola concentric with the ellipses, and passing through the common foci of the ellipses: if it touch a series of confocal hyperbolas, the locus of the points of contact will be a circle: and if it touch confocal parabolas whose axes are coincident in position, the locus will be another parabola.

17. If two tangents EP, FP, be drawn to CS, and EF be drawn to meet the directrix in G and the line through the focus in I, the chord EF will be harmonically divided in I and G; or, EI : IF :: EG : GF.

18. Let the diameter DC be produced to meet the directrix in A, and draw AF to the focus F, and CH a tangent at C: then AF, CH, shall be perpendicular to one another.

19. Let any circle cut the ellipse in E, H, pass through the foci F, f, and cut the conjugate in G, K (G being without the ellipse), then GE, GH, will be tangents, and IE, KH, normals to the ellipse.

20. If two hexagons, situated in the plane of a conic section so that the summits of the first be respectively the poles of the sides of the second in respect of the conic section; then, reciprocally, the summits of the second will be the poles of the sides of the first hexagon.

21. If one of the hexagons in the preceding theorem be inscriptible or circumscriptible in a conic section, the other is circumscriptible or inscriptible to another conic section.

22. If any polygon in the plane of a conic section be inscriptible or circumscriptible to another conic section, then the polars to the summits of the first polygon, considered in reference to the first conic section, will form another polygon which will be circumscriptible or inscriptible to a third conic section.

23. If the poles in respect to one conic section move upon another conic section, the corresponding polars will all be tangents to a third conic section.

24. If the polars in respect of one conic section be tangents to a second, the corresponding poles will trace a third conic section.

THE CONOIDS OF REVOLUTION.

DEFINITIONS.

1. If about either of the axes of a conic section the conic section be made to revolve, it will generate a surface, called, generally, a ***conoïd of revolution***, or simply, a ***conoïd.***

2. When the generating curve is a parabola, the surface is called ***the paraboloïd.***

3. If the generating conic section be an ellipse, the surface is called ***an ellipsoïd:*** and it is distinguished as *a prolate ellipsoïd*, or an *oblate ellipsoïd*, according as the transverse or conjugate axis be made that of revolution.

The ***sphere*** is a particular case of the ellipsoïd; viz. when the elliptic generatrix is a circle. The ellipsoïds being in general form analogous to the sphere, in being bounded in all directions, led Archimedes to distinguish them by the name of ***spheroïds.***

4. When the generating curve is an hyperbola, the surface is called ***an hyperboloïd:*** and it is distinguished into ***the hyperboloïd of two sheets***, and ***the hyperboloïd of one sheet***, according as the transverse or conjugate axis be made that of revolution; or again, supposing the revolution be about the transverse axis, according as the opposite pair of hyperbolas or their conjugates be made the generatrices.

The right cone is a particular case of the hyperboloïd of two sheets; viz. when the generatrix is an angle, the bisecting line of which is the axis of revolution. The indefinite extension of the paraboloïd and the hyperboloïd of two sheets, being similar to that of the cone, led Archimedes to call those surfaces ***conoïds.*** The hyperboloïd of one sheet was not known to him. It was first observed by Sir Christopher Wren, in a form which will be presently noticed; but the properties of the general figure were mainly developed by Hachette and Chasles.

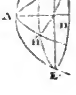
5. The ***transverse plane*** is one drawn through the axis of revolution perpendicular to any sectional plane of a conoïd.

6. Sections through the axis of revolution are called ***meridian sections;*** those parallel to the axis are called ***parallel sections;*** those perpendicular to the axis, ***perpendicular sections;*** and all others, ***oblique sections.***

7. The ***asymptotic cones*** of the hyperboloïds of two sheets and of one sheet, are those which are generated by the asymptotes of the revolving hyperbola during the generation of the hyperboloïds respectively.

PROP. I.

Meridian sections of a conoïd are identical with the generating curve; and perpendicular sections are all circles.



For the first-named section is but the generating curve in the position of the cutting plane. Thus, the plane HAH′ cuts in the conoïd generated by EAE′, a section obviously equal in all respects to the generatrix.

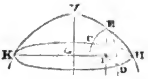
Again, the plane of the section HEH′E′ perpendicular to the axis AD, is cut by planes passing through the axis in DE, DH, *etc.*: and hence, ADE, ADH are right angles, and consequently ordinates in the equal curves EAE′, HAH′, *etc.* to the same absciss AD. They are hence equal.

In the same way it may be proved, that any other line drawn from D in the plane of section to the boundary of the section is equal to AD. The extremities of these lines are therefore in the circumference of a circle. Any section HEH′E′, therefore, perpendicular to the axis of revolution, is a circle.

PROP. II.

The section of a paraboloïd parallel to the axis of revolution, is a parabola in all respects equal to the generating one; and any oblique section of the paraboloïd is an ellipse.

First. Let DEC be a section of the paraboloïd, parallel to the axis of revolution VG, it shall be in all respects equal to the generating curve of the paraboloïd. For let KVH be a transverse section, and KDH a section perpendicular to the axis,



Then, since the plane CED is parallel to the axis VG, and a plane (the transverse KVH) is drawn through VG perpendicular to the plane CED, the section EI is parallel to VG. Wherefore the plane KDH, which is perpendicular to VG, is also perpendicular to EI; and hence the three planes KDH, KVH, CED, are perpendicular; and in consequence also, the lines of section EI, KH, CD, are perpendicular to each other. Whence, again, KH passing through the centre G of the circle KDH, and at right angles to the chord CD, that chord is bisected in I.

Now, since EI is parallel to VG, and denoting the parameter of the generating parabola KVH by p, we have (*parab. pr. i. cor.* 1,)

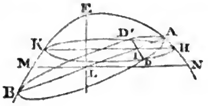
$$p \,.\, \mathrm{IE} = \mathrm{KI} \,.\, \mathrm{IH} = \mathrm{ID}^2.$$

Now this is the property of a parabola CED whose parameter is p, that is, of one equal to the generatrix.

Second. Let ADB be any oblique section, it will be an ellipse.

For let BEA be the transverse plane meeting the plane section ADB in AB, and KDH any plane perpendicular to the axis which cuts AB, and EL that diameter of the transverse section BEA which is conjugate to AB, meeting AB in L:

Then, reasoning as in the preceding case, KDH is a circle, of which KH is a diameter, and ID the right ordinate to that diameter, and hence KI.IH = ID^2. Also, since EL is the diameter conjugate to the chord AB, we have BL = LA (*parab. pr. xiii.*), and hence BL LA = LA^2; and ML.LN = p.EL (*prop. i. cor.* 1).



Now (*parab. pr. xvi.*) we have,

$$\mathrm{AI}\,.\,\mathrm{IB} : \mathrm{KI}\,.\,\mathrm{IH} :: \mathrm{AL}\,.\,\mathrm{LB} : \mathrm{ML}\,.\,\mathrm{LN}; \text{ that is,}$$
$$\mathrm{AI}\,.\,\mathrm{IB} : \mathrm{ID}^2 :: \mathrm{AL}^2 : p\,.\,\mathrm{EL};$$

which is a property of the ellipse whose transverse axis is AB, and semi-conjugate is a mean proportional between p and EL.

Corollary.

All parallel sections of the paraboloïd are similar figures. For those parallel to the axis of revolution are parabolas, and therefore similar; and those perpendicular to the axis are circles, and therefore similar.

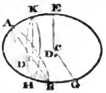
Again, for those parallel to any other plane, as the tangent plane at E, put the parameter of the diameter $EL = p_1$: then $p_1 . EL = AL^2$; wherefore we have

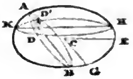
$$AI . IB : ID^2 :: p_1 . EL : p . EL :: p_1 : p;$$

that is, a constant ratio; wherefore the ellipses are similar.

PROP. III.

All the oblique and all parallel sections of the ellipsoïds are ellipses.





Let ADB be the section of an ellipsoïd (the *prolate* in fig. 1, and *oblate* in fig. 2), AEB the transverse plane, cutting the sectional plane in AB, and any one of the perpendicular planes KDH, and meeting the plane ADB within the ellipsoïd. Draw CG parallel to AB, and CE parallel to HKI.

Then, by reasoning as before, $KI . IH = ID^2$. Also, (*ellipse*, *prop. i.*,) we have

$$AI . IB : KI . IH :: CG^2 : CE^2; \text{ or}$$
$$AI . IB : \quad ID^2 \quad :: CG^2 : CE^2;$$

which is the property of an ellipse (*prop. ii.*). Hence the oblique section of an ellipsoïd is an ellipse.

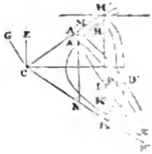
The same reasoning will apply when the section ADB is parallel to the axis of revolution, or perpendicular to the line EC, in both cases. All parallel sections of an ellipsoïd are therefore ellipses.

Corollary.

All sections of an ellipsoïd parallel to each other are similar to one another. For $CG^2 : CE^2$ is a constant ratio, the lines themselves being of fixed magnitude for any given ellipsoïd and diametral plane to which the sectional planes are parallel.

PROP. IV.

If the hyperboloïd of two sheets and its asymptotic cone be cut by a plane, the section of the hyperboloïd will be similar to that of the cone.



Let ADB, A′D′B′ be the sections made in the hyperboloïd and its asymptotic cone: these shall be similar figures. For draw H′D′K′ the perpendicular plane, cutting both these planes within the hyperboloïd; and HAK the transverse plane to the sections ADB and A′D′B′. Also, through A draw MN parallel to HK; and the semi-diameters CE, CG parallel to HK and AB.

Then, reasoning as before, we have $HI . IK = ID^2$, and $H'I . IK' = ID'^2$; also $A'A . AB' = CG^2$, and $MA . AN = CE^2$ (*hyperb. prop.* B).

Again, by parallels, we have

$$A'I : IH' :: A'A : AM, \text{ and } B'I : IK' :: AB' : AN;$$
$$\text{whence } A'I \,.\, IB' : H'I \,.\, IK' :: A'A \,.\, AB' : MA \,.\, AN, \text{ or}$$
$$A'I \,.\, IB' : \quad ID'^2 \quad :: \quad CG^2 \quad : CE^2.$$

But in the hyperbola, we have (*props. i.* and *ii.*)

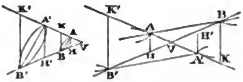
$$AI \,.\, IB : HI \,.\, IK \;(= ID^2) :: CG^2 : CE^2.$$

Wherefore, $A'I \,.\, IB' : ID'^2 :: AI \,.\, IB : ID^2$, or the sections A′D′B′, ADB are similar, as enunciated.

Corollary 1.

All parallel sections of the asymptotic cone of revolution (and hence of every cone of revolution) are similar figures.

First. In case of the directing plane being a tangent plane, the section is a parabola; and all such sections are similar figures. Also, the same is true for the perpendicular sections, they being all circles.



Second. Suppose the sections to be the elliptic or hyperbolic, and let them be cut by the transverse plane in AB, A′B′, and draw AH, BK, A′H′, B′K′ perpendicular to the axis of the cone VW.

Then by similar triangles,

$$HA : AB :: H'A' : A'B', \text{ and } KB : BA :: K'B' : B'A; \text{ hence}$$
$$HA \,.\, KB : AB^2 :: H'A' \,.\, K'B' : A'B'^2.$$

But AB^2, $A'B'^2$ are the squares of the transverse diameters, and $HA \,.\, KB$, $H'A' \,.\, K'B'$ are the squares of the conjugate diameters. Whence the parallel sections, whose transverse and conjugate diameters these are, are similar.

Corollary 2.

All parallel sections of the hyperboloïd of two sheets are similar figures; for they are similar to the sections made in the asymptotic cones by the same plane (*prop. iv. cor. i.*); and by the preceding corollary these are all similar.

Corollary 3.

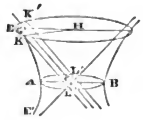
If the asymptotic cone be cut by a plane which touches the hyperboloïd of two sheets, the section will be an ellipse whose principal semi-diameters are CG and CE respectively.

PROP. V.

Every section of the hyperboloïd of one sheet by a tangent plane to the asymptotic cone is a pair of lines parallel to the line of contact of the cone and plane, and at a distance from each other equal to the transverse diameter of the hyperboloïd.

Let a plane touching the asymptotic cone in GC cut the hyperboloïd of one sheet, the section will be two straight lines, KL, K′L′, parallel to GC, and their distance, KK′, from each other will be equal to the transverse diameter AB.

For the transverse plane will in this case be GCH; CH being the conjugate axis. Let any plane KEK′, perpendicular to the axis of revolution CH, cut the



transverse plane in KGK', and the hyperboloïd in the circle KEK'. Also let the plane of the circle described by the transverse axis be cut by the transverse plane in LL', and draw the lines KL, $K'L'$ to the extremities L, L' of the diameter of the circle ALB made by the transverse plane. Also draw HK.

Then, reasoning as in former cases, the line KK' is bisected in G by EH, and KGH is a right angle. Wherefore,

$$KG^2 = KH^2 - HG^2 = EH^2 - HG^2.$$

Again, (*hyperbola, prop.* D, *p.* 159,) we have

$$LC^2 = AC^2 = HE^2 - HG^2.$$

Wherefore $KG = LC$; and as this is independent of the particular position of the point H in the axis of revolution, it is constant. Wherefore, again, the line KL is the intersection of the hyperboloïd with the tangent plane to the asymptotic cone. In the same way it may be shown that there is a second intersection $K'L'$ of the tangent plane and hyperboloïd, also parallel to CG. Moreover, KK' is perpendicular to CG, and therefore to KL, $K'L'$; and it is equal to LL' or AB.

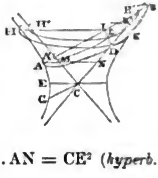
Scholium.

This figure was not known to Archimedes, and it appears to have been first noticed by Sir Christopher Wren, in connexion with the remarkable property expressed in this proposition. The surface, it is clear from this, may be generated by the revolution of a straight line KL, having an invariable connexion with its axis HC, which is not in the same plane with it. Had KL been parallel to HC, the surface would have been a cylinder, and hence Wren called it the *hyperbolic cylindroïd.*

The demonstration of the property here given is almost identical with Wren's: upon which some further information may be obtained from my *History of the Rule Surfaces, Math. Repos. vol. vi.*

PROP. VI.

If the hyperboloïd of one sheet and its asymptotic cone be cut by any plane, the sections will be similar.



Let ADB, $A'D'B'$ be the sections made in the hyperboloïd and its asymptotic cone: these will be similar figures.

For draw HDK perpendicular to the axis of revolution, and HAK the transverse plane to the sections ADB and $A'D'B'$. Also, through A draw MN parallel to HK, and the semi-diameters CE, CG parallel to HK and AB.

Then, reasoning as formerly, we have $HI \cdot IK = ID^2$, and $H'I \cdot IK' = ID'^2$; also, $A'A \cdot AB' = CG^2$, and $MA \cdot AN = CE^2$ (*hyperb. prop.* B).

Again, by parallels, we have

$$A'I : IH' :: A'A : AM, \text{ and } B'I : IK' :: B'A : AN;$$

$$\text{whence } A'I \cdot IB' : H'I \cdot IK' :: A'A \cdot AB' : MA \cdot AN,$$

$$\text{or } A'I \cdot IB' : \quad ID'^2 \quad :: \quad CG^2 \quad : \quad CE^2.$$

But in the hyperbola we have

$$AI \cdot IB : HI \cdot IK \; (= ID^2) :: CG^2 : CE^2.$$

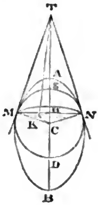
Wherefore, $A'I \cdot IB' : ID'^2 :: AI \cdot IB : ID^2$; and the sections ADB, $A'D'B'$ are similar.

Corollary.

All parallel sections of the hyperboloïd of one sheet are similar; as will be established by reasoning similar to that employed in ***prop. v. cor.*** **1.**

PROP. VII.

A sphere drawn to touch a conoïd, its centre being in the axis of revolution; and a cone drawn to touch the conoïd, its vertex being in the extended axis of revolution, will both touch the conoïd in planes, and the lines of contact will be circles.



In any conic section BMAN, one of whose principal axes is AB, let a circle DMEN be described to touch it in MN; and likewise let tangents at M and N be drawn to intersect in T. Then it is too obvious to need detailed proof that the centre C of the circle and the intersection T of the tangents will both be in the axis of revolution AB. Now by the revolution of the plane curve BMAC to produce the conoïd, the circle DMN is also made to revolve, and generate a sphere, whilst the tangent MT or NT generates a right cone.

In every point, as K, of their revolution, the cone and conoïd touch as they do at M, and likewise the sphere and conoïd touch as they do at M. The contact enunciated, therefore, takes place.

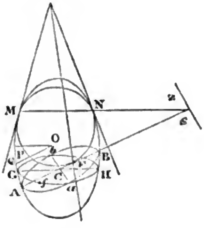
Corollary.

Any number of spheres can always be described to touch a conoïd in a circle. For by taking MN nearer to, or more remote from A, the circle DMN admits of unlimited variation.

Any number of right cones having their vertices in the extended axis of revolution may be described to touch the conoïd, or having any circular section of the conoïd for its base.

PROP. VIII.

If in each of the segments into which a conoïd is divided by a section-plane, spheres be described to touch the conoïd and the plane, their points of contact with the plane will be the foci of the section.



Let the conoïd be cut by a plane in the section A*a*B; and let spheres be described in the two segments of the conoïd touching the section-plane in F and *f*: these will be the foci of the section.

For consider one of the spheres. Draw the transverse plane, cutting the conoïd in the curve AMNB; the sphere of contact in the circle MNF; the common cone of contact to the sphere and conoïd in the linear tangents TM, TN of the circle and conic section; the section-plane in AB, in which evidently the point of contact F will be situated; the plane of contact in MN, and the plane G*a*H parallel to the plane of contact (or perpendicular to the axis of revolution) in GH. Let also the planes of section AaB and contact MLN (not traced in the cut) meet in the line SZ, cutting the transverse plane in S; and, finally, draw in the transverse plane the semi-diameters OP, OQ parallel to AB and GH respectively.

Then it may be shown, as in the preceding propositions, that the line aCb is bisected at right angles by the lines AB, GH; and hence, that $GC \cdot CH = Ca^2$, Ca being the semi-conjugate axis of the section.

Again, (*General Properties*, *prop. vii. cor.* 2,) AB is harmonically divided in F and S. Wherefore,

$$AS \cdot SB : MS \cdot SN \ (= SF^2) :: OQ^2 : OP^2, \text{ and } \textit{div.}$$

$$AS \cdot SB - SF^2 : SF^2 :: OQ^2 - OP^2 : OP^2.$$

But (by *harmonicals*) $AS \cdot SB - SF^2 = AF \cdot FB = CF \cdot FS$. Wherefore,

$$CF \cdot FS : SF^2 :: OQ^2 - OP^2 : OP^2, \text{ or}$$

$$CF : FS :: OQ^2 - OP^2 : OP^2; \text{ and again,}$$

$$CF + FS \ (= CS) : CF :: OQ^2 : OQ^2 - OP^2; \text{ or}$$

$$CS \cdot CF \ (= CB^2) : CF^2 :: OQ^2 : OQ^2 - OP^2 \quad \ldots\ldots\ldots\ldots \quad (a)$$

Again, $AC \cdot CB \ (= CB^2) : GC \cdot CH \ (= Ca^2) :: OQ^2 : OP^2$, or

$$CB^2 : \quad CB^2 - Ca^2 \quad :: OQ^2 : OQ^2 - OP^2 \ \ldots\ldots \quad (b)$$

wherefore $CB^2 - Ca^2 = CF^2$, a known property of the focus of an ellipse.

In like manner, in the case of the hyperbolic section, we should have

$$CB^2 + Ca^2 = CF^2.$$

For the parabolic section, the process becomes still further simplified; but it is left for the student to work out the detail.

A similar course of reasoning will show, that the sphere described in the other segment will touch the section-plane in the other focus f of the elliptic and hyperbolic sections. In the circular sections, the foci coincide with each other and the centre.

Corollary 1.

The line SZ is the directrix of the section AaB. For since F is the focus, and the line SZ is drawn in the section-plane perpendicular to AB from the point S so taken that

$$AF : FC :: AS : SB,$$

the definition of the directrix attaches to it.

Corollary 2.

All the conoïds which touch the cone and sphere in their circle of contact MLN, and are cut by sections through SZ, will have one of their foci in the line FT, and SZ will be the directrix of each of them.

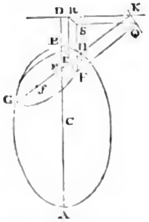
Scholium.

This remarkable theorem was first enunciated in the *Annales des Mathematiques*, and deduced as a consequence of a property of the conic sections. It was copied as a question into the *Gentleman's Diary* for 1827, and two solutions were given in the following year's *Diary*; one of which was formed out of the proof in *Annales*, and given anonymously; the other was by myself, geometrically, nearly as in the preceding pages, and which is, I think, the only purely geometrical one which has been published.

The particular case of the conoïd being an actual cone, is also separately demonstrated at p. 206.

PROP. IX.

All sections of a conoïd made by planes passing through the focus, have that focus for one of their foci, and the intersections of those planes with the directrix plane of the conoïd are the directrices of the sections.



Let GPH be any section of a conoïd, passing through its focus F, and the plane of which cuts the directrix-plane in KQ: then F will be the focus and KQ the directrix of the section GPH.

For, let the transverse plane GBHA be drawn, cutting the directrix-plane of the conoïd in DK; take any point P in the section GPH, and draw PF to the focus and PQ parallel to the transverse diameter GHK; and, lastly, draw PE perpendicular to GH, ER perpendicular to DK, PS perpendicular to the directrix-plane, and join RS.

Then, since DKF is the transverse plane, it is perpendicular to the directrix-plane and to the section-plane, and hence to their common section KQ: and therefore, again, both KF and KD are perpendicular to KQ.

Again, since PE and KQ are in the plane GPH, and perpendicular to KF, they are parallel. Also, the line PE being parallel to a line KQ in the directrix-plane, it is itself parallel to that plane. Wherefore, RE, SP are equal and parallel, and RS is equal and parallel to EP, and hence also equal and parallel to KQ; and therefore, again, SQ is equal and parallel to RK. The three sides SP, PQ, QS of the triangle PQS, are hence parallel to the three sides RE, EK, KR of the triangle EKR: and therefore the triangles are equiangular. Moreover, the triangle REK is equiangular to DFK, and hence PQS is equiangular to FKD. Whence PQ : PS :: FK : FD, a given ratio for a given section of the conoïd.

Again, since P is a point in the conoïd, of which F is the focus, and RSQK the directrix-plane, FP : PS :: FB : BD, a given ratio. Whence we have by compounding these,

$$PF : PQ :: DF . FB : BD . KF;$$

and as the latter is a given ratio, the former is also given.

The point F is therefore the focus, and KQ the corresponding directrix of the section GPH.

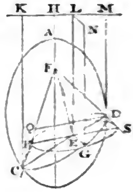
Corollary.

The other focus *f* of the variable section GPH is always in the concentric, similar, and similarly situated conoïd, which passes through F.

For since the foci F, *f*, are equi-distant from the extremities of the transverse diameters, and these diameters all pass through F, the other focus *f* will be in a conic section, similar and similarly situated to the transverse section AGBH; and as the transverse by its revolution generates the conoïd AGBH, the other will generate a corresponding one, as stated in the corollary.

PROP. X.

A cone whose vertex is at one of the foci, and whose base is any section of a conoïd, is a right cone.



The cone whose vertex is F, one of the foci of a conoïd, and whose base CGD is any plane section of the conoïd, is a right cone.

For let CAD be the transverse plane, cutting the directrix-plane in KM, and the section CGD in the transverse diameter CD. In the transverse section of the cone, CFD, draw FE to bisect the angle CFD, and ER, DQ perpendicular to it, meeting CF in R, Q. Also, through E draw the plane RGD perpendicular to EF, cutting the section-plane in EG; and join EG; and finally, draw the perpendiculars CK, EL, GN, DM, to the directrix-plane, and join LN.

Then, since FE bisects the angle CFD, and DQ is perpendicular to FE, we have $QF = FD$; and since ER is also perpendicular to FE, it is parallel to DQ, and the lines CD, CQ are similarly divided in R and E. Whence

$CF : FD :: CE : ED :: CR : RQ$; or, since $QF = FD$,

$CF : FQ\ (= FD) :: CR : RQ :: CF - FR : FR - FQ = FR - FD \ldots\ldots (a)$.

Again, as in the preceding proposition, $GN = EL$, and $GE = LN$, and they are parallel each to each; and, therefore, since KNM is the directrix-plane, we have

$$FC : FG : FD :: CK : GN : DM; \text{ and hence,}$$

$$\begin{aligned} CF - FG : FG - FD &:: CK - GN : GN - DM, \textit{ div.};\\ &:: CK - EL : EL - DM, \textit{ equal.};\\ &:: \quad CF \quad : \quad FD, \textit{ sim. trians.};\\ &:: CF - FR : FR - FD, \text{ by } (a). \end{aligned}$$

Hence $CF - FD : CF - FG :: CF - FD : CF - FR$, *comp*; and therefore, $CF - FG = CF - FR$, or $FG = FR$.

The right-angled triangles RFE, GFE, therefore, having $RE = EG$, and $FR = FG$, are equal in all respects; and the base RGD of the cone is a circle, and the vertex F being situated in the perpendicular from its centre E, it is a right cone.

Corollary.

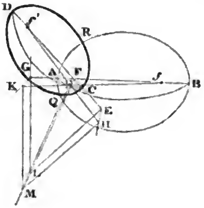
If spheres be inscribed in the cone and conoïd to touch the plane CGD, they will in pairs touch this plane in the foci of the section. This follows from *prop. viii.*

Scholium.

This proposition is but a particular case of the following one. The general type of this theorem in reference to the next, is when one of the conoïds is the hyperboloïd of one sheet; and this, by the indefinite diminution of the transverse and conjugate diameters, brings the centre, the vertices, and the foci, into coalescence, and produces the cone, as the limiting case of the general surface.

PROP. XI.

If two conoïds have a common focus, their intersection will be a plane curve.



Let two conoïds have the same focus F, and let P be any point in their curve of mutual intersection: then P will always be situated in one plane, and therefore in a conic section PQR.

For let AB, CD be their transverse axes, and let the plane ABCD cut the directrix-planes of the two surfaces in GL, EL; and let the planes GFP, EFP cut them in GK, EH: these will be directrices of the sections CPD, APB, made by meridian-planes through P. In these latter planes draw the perpendiculars PK, PH to the directrices GK, EH, and join FP.

Then, since GK is the directrix of the conic section APB, which is in all respects equal to the given one ARB, the ratio FP : PK is given. In like manner, FP : PH is given; and, therefore, the ratio of the perpendiculars PK : PH is also given.

Now, when two lines drawn from the same variable point to two given planes have a given and constant ratio, that variable point is always in one plane passing through the intersection of the other two; wherefore, the two surfaces intersect in a plane curve.

Corollary 1.

Since PH : PK :: FE : FG, by the properties of the directrix, the point F is in the plane of intersection of the two surfaces; and this section will have F as its focus.

Corollary 2.

Let f and f' be the other foci, (the figures being either *ellipsoïds* or *hyperboloïds*,) and draw fP, $f'P$: then these will be the foci of a third conoïd which has the same common section with the two given ones. For,

First, suppose the figures to be ellipsoïds: then

$$AB - CD = (fP + PF) - (f'P + PF) = fP - f'P;$$

and as AB and CD are given, $fP - f'P$ is also given. Whence the point P is in an hyperboloïd, which has thus a common section with the other two surfaces.

Second, let the figures be an ellipsoïd and hyperboloïd: then we have,

$$AB + CD = (fP \pm PF) + (Pf' \mp PF) = fP + f'P;$$

and, as before, the sum of the lines fP, $f'P$ are given. Whence the point P is in an ellipsoïd, which has thus a common section with the other two surfaces.

Third, let both be hyperboloïds: then, as before, we get

$$AB - CD = (PF - Pf') - (PF - Pf') = f'P - Pf,$$

and the third surface is an hyperboloïd.

The paraboloïd, having no second focus, has no analogous property.

Scholium.

In the right cone, the focus is at the vertex; and hence two confocal cones intersect (if at all) in a plane section, which accords with a more elementary result.

If only one of the figures be a cone, we are led to a result the converse of *prop. x.*; viz. that a right cone having its vertex in the focus of a conoïd, cuts the conoïd in a plane curve.

If the oblate ellipsoïd or hyperboloïd of one sheet be one of the surfaces, then any point in the circle described by the foci may be the focus of the other surface; but if both be surfaces whose foci are thus distributed into circles, there must be an entire coincidence of those focal circles, that the surfaces may cut in a plane curve.

The conjugate foci of the ellipsoïd, however, possess the same property as enunciated in the proposition.

The synthetic demonstration of these properties is left as an exercise.

PROPERTIES OF THE CONOÏDS FOR DEMONSTRATION.

1. Tangent planes drawn to a conoïd at the several points of intersection of a focal plane with the surface, all pass through the same point in the directrix-plane: and a line from this point to the focus is perpendicular to the focal plane.

2. The radius-vector drawn from a focus of a conoïd to the point of contact of a tangent-plane, is perpendicular to the vector-plane drawn to the line of intersection of this tangent-plane with the directrix-plane [*].

3. The radii vectores drawn to the points of contact of two tangent-planes, make equal angles with the vector-plane, drawn through the line of intersection of the two tangent-planes.

4. The radii vectores drawn to the extremities of a chord of a conoïd, make equal angles with the radius-vector drawn to the point of intersection of this chord with the directrix-plane.

5. The vector-planes drawn through the lines in which two tangent-planes meet the directrix-plane, make equal angles with the vector-plane through the line of intersection of the two tangent-planes.

6. The vector-plane through the intersection of two tangent-planes is perpendicular to the radius-vector drawn to the point where the chord of contact meets the directrix-plane.

7. If through any point two tangents be drawn to a conoïd, the vector-planes through these lines make equal angles with the vector-plane through the chord of contact.

8. Two linear tangents being drawn to a conoïd, the lines drawn from the points of contact to the focus make equal angles with the chord of contact.

9. If through a point in the line of intersection of two tangent-planes lines be drawn to the points of contact, the two vector-planes through these two tan-

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[*] The *radius-vector* is a line drawn from the focus to the specified point, and a *vector-plane* is any plane drawn through the focus. These terms have a more extended signification in modern science; but this is a sufficiently extended definition for our present purpose, and is, moreover, perfectly consistent with the general definition. See *Geometry of Co-ordinates*, in this volume.

gents will make equal angles with the vector-plane through the intersection of the two tangent-planes.

10. If a tetrahedral angle be circumscribed to a conoïd, the dihedral angle made by the vector-planes drawn to two consecutive edges is supplementary to the dihedral angle made by the vector-planes drawn to the two other edges.

11. If through a point in the line of intersection of two fixed tangent-planes a third tangent-plane revolve, the angle contained by the vector-planes through the lines of intersection of the revolving plane with the fixed planes contains a constant angle.

12. If a tetrahedron be drawn, formed of four tangent-planes to a paraboloïd, the sphere described about it will pass through the focus of the paraboloïd.

13. If a tangent cone be drawn to a conoïd, the line of contact will be a plane curve: if a cone cut a conoïd in a plane curve, it will cut it a second time in a plane curve: if one conoïd touch another it will be in a plane curve; and if one conoïd cut another in a plane curve, and likewise cut or touch it again, the section or the contact respectively will be in plane curves.

14. All the confocal conoïds which pass through a given conic section, have each their other focus situated on another conic section.

15. If a given trihedral angle so move as to always have its vertex in the focus of a conoïd, all the planes through the intersections of the three edges with the conoïd will be tangent-planes to another conoïd, having the same focus and the same directing-plane as the proposed one.

16. If through the foci of a conoïd any planes be drawn through the point of contact of a tangent-plane with the conoïd, make equal angles with the tangent-plane.

17. Through the line in which two tangent-planes to a conoïd intersect, draw planes to the foci: these will make equal angles with the tangent-planes.

18. If through any exterior point a straight line continually move, so as to touch a conoïd, the sum or difference of the angles which this straight line in each of its positions makes with lines drawn from that exterior point to the foci is constant.

19. The product of the perpendiculars from the foci of a conoïd upon any tangent-plane is constant; and the perpendiculars intersect the tangent-plane in points, all of which are upon the surface of the sphere described upon the axis of revolution. In the case of the paraboloïd, the perpendiculars are in the tangent-plane at the vertex.

20. Confocal ellipses intersect confocal hyperbolas (the axes being coincident) at right angles.

THE CONE, CYLINDER, AND SPHERE.

DEFINITIONS.

(*Continued from page* 107.)

1. The *oblique cone* is one, the vertex of which is not in the perpendicular drawn through the centre perpendicular to the plane of the base; and the line from the vertex to the centre of the base is the *generating axis* of the cone.

2. The *transverse section* of an oblique cone is that which is perpendicular to the plane of the base, and passes through the generating axis of the cone. The two edges of the cone along which it passes are the *transverse edges of the cone;*

the angle which they include is the *transverse angle of the cone;* and the triangle of which they are the sides is called the *triangle of the axis.*

3. *The directing plane* is the plane parallel to the section of a cone, and drawn through the vertex of the cone.

4. If through the vertex directing-planes be drawn, one to the base of the cone, and the other making angles with the transverse sides equal to those made by the base, but in a contrary order, are called *the cyclical planes of the cone;* and the sections of the cone corresponding to these cyclical directing-planes, are said to be *anti-parallel* or *sub-contrary sections.*

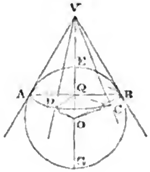
5. The line bisecting the transverse angle of the cone is the *principal axis of the cone;* the intersection of the cyclical planes is the *cyclical axis of the cone;* and that through the vertex perpendicular to the plane of these two axes (in the transverse plane) is the *conjugate axis of the cone.*

6. A sphere is said to be *inscribed* in a right cone, when its centre is in the axis of the cone, and its radius is the perpendicular from the centre upon any one of the edges.

PROP. I.

If a sphere be inscribed in a right cone, the two surfaces touch in a circle.

Let a sphere have its centre O in the axis VO of a right cone; let AO, the perpendicular upon one of the edges AV of the cone, be the radius of the sphere: then the sphere and cone will touch each other in one plane, and the figure of the contact will be a circle.



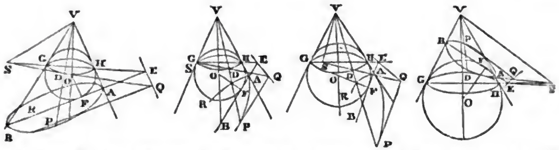
For, the cone and sphere may be conceived to be generated by the revolution of the line VA and semicircle EAG about the common axis VEOG. In all the corresponding positions of these generatrices, as VD and EDG, VC and ECG, VB and EBG, *etc.*, the lines AV, DV, CV, BV, *etc.*, will be perpendicular to the radii OA, OD, OC, OB, *etc.*; or, in other words, these edges of the cone will be tangents to the circular sections of the sphere EDG, ECG, EBG, *etc.*

Let AQ be drawn perpendicular to VO, and join DQ, CQ, BQ, *etc.* Then, during the revolution of the plane VAO through all its positions, the line AQ will be always perpendicular to VO; and hence always in one plane perpendicular to VO. All the points of contact, therefore, are in one plane, and as they are also in the surface of the sphere, they are situated in a circle.

PROP. II.

If a sphere be inscribed in a right cone, and a plane be drawn to touch the sphere in a point, and to cut the cone in a conic section and the plane of contact in a straight line; the point and line will be the corresponding focus and directrix of the conic section.

Let O be the centre of the sphere, OV the axis of the cone, GDH the plane of contact of the sphere and cone, PAR the conic section, the plane of which touches the sphere at F and meets the plane GDH in EQ: then F will be a focus of the conic section, and EQ will be the corresponding directrix.



For, draw the transverse plane cutting the section PAR in AB, the line EQ in E, the cone in the transverse edges of the cone in VH, VG, the plane of contact in HGS, and the directing plane in VS. Also, let VP be the edge of the cone passing through any point P of the conic section PAR, meeting the plane of contact in D; draw PQ perpendicular to EQ, and join SD, DQ, OF, and PF.

Then, since the transverse plane is perpendicular to the plane of contact GDH and the section-plane PAR, it is perpendicular to their common section EQ; or EQ is perpendicular to GHE and BAE, at the point E. Also, since the directing-plane is parallel to PAR (*def.* 3), the lines SV, AB are parallel; and (*constr.*) PQ is parallel to AB: whence PQ is parallel to VS (*vol. i. p.* 354); and they are, consequently, in one plane. But VDP is drawn to cut the parallels SV, PQ, and hence the point D is in the same plane SVPQ with the points S and Q: and the points S, D, Q, are likewise also in the plane of contact GDH; and hence in the line of section of the planes GDH, SVPQ, that is, in one straight line.

Again, since the transverse plane passes through the axis OV of the cone and the centre O of the sphere, and is perpendicular to the tangent-plane PAR of the sphere, the radius OF drawn to the point of contact F is in the transverse-plane; and hence, the point of contact F, being in the transverse-plane of the cone, and in the tangent-plane PAR to the sphere, is in their intersection, or in the transverse axis of the conic section PAR. Also, PF, PD, being tangents to the same sphere, are equal to each other.

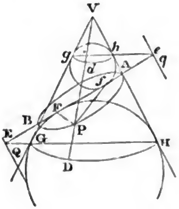
Now by similar triangles SVD, QDP, we have

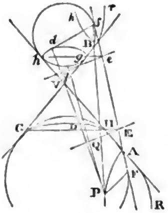
$$PD\ (= PF) : QP :: VD : SV.$$

But the line SV is given in magnitude and position, and VD is given in magnitude; whence VD : SV is a given and constant ratio, and consequently PF : QP is a given ratio. Also, the point F and the line EQ being given, together with the ratio PF : PQ, it follows (*conversely from pp.* 110, 132, and 152) that F is a focus, and EQ the corresponding directrix, of the conic section PAR.

Corollary 1.

In the case of the parabola, there is but one point of contact, F, and one line





of section, EQ, corresponding to it; since only one sphere can be described to touch the cone and the section-plane, as in *fig.* 2 of the proposition: but in the ellipse and hyperbola, as in the annexed figures, there are two spheres describable in each case, and hence there are two points of contact F, *f*, and two corresponding lines of section EQ, *eq*. This accords with the previously determined properties of the several curves.

Also in the parabola, we have $\mathrm{FP} = \mathrm{PQ}$; in the ellipse FP is less than PQ; and in the hyperbola greater.

Corollary 2.

(1). In the case of the ellipse, $\mathrm{P}f + \mathrm{FP} = \mathrm{D}d = \mathrm{AB}$,
(2). .... hyperb. $\mathrm{P}f - \mathrm{FP} = \mathrm{D}d = \mathrm{AB}$.

For let VP meet the circles of contact GDH, *gdh* in D and *d*. Then we have, in both cases,

$$\mathrm{PD} = \mathrm{PF}, \text{ and } \mathrm{P}d = \mathrm{P}f; \text{ whence,}$$
$$\mathrm{P}f + \mathrm{PF} = \mathrm{P}d + \mathrm{PD} = \mathrm{D}d \text{ in } fig.\ 1,$$
$$\mathrm{P}f - \mathrm{PF} = \mathrm{P}d - \mathrm{PD} = \mathrm{D}d \text{ in } fig.\ 2.$$

Again, $\mathrm{BF} = \mathrm{BG} = \mathrm{A}f$, and $\mathrm{B}f = \mathrm{BG}$; whence, in the two cases respectively,

$$\mathrm{A}f + \mathrm{B}f = \mathrm{BG} + \mathrm{B}g = \mathrm{G}g = \mathrm{D}d,$$
$$\mathrm{A}f - \mathrm{B}f = \mathrm{BG} - \mathrm{B}g = \mathrm{G}g = \mathrm{D}d;$$

whence both equalities are proved.

Scholium.

This theorem was first deduced, in 1822, by M. Dandelin, from a more general one given two years earlier by M. Quetelet. Seven or eight years later, it was also proposed by Mr. Morton, in the Cambridge Transactions, as the basis of a system of conic sections, in a manner nearly identical with that of Dandelin. For this purpose it is extremely well adapted, as it furnishes those two characteristic properties of the curves, from which it has, for nearly two centuries, been most usual to deduce the other properties. The theorem and corollary 1 constitute the last proposition of the seventh book of Pappus, but they are not found in Apollonius. The second corollary constitutes one of the methods of Lahire, (1679,) for deducing the properties of the curves from a plane definition. It is proper to remark, that a very near approach to this method of finding the focus and directrix in relation to the cone itself, was given in 1759 by Dr. Hugh Hamilton, in his Latin treatise on the conic sections. See page 120 of that work, or the translation of 1773.

It is remarkable, that this mode of investigation deduces both these fundamental properties of the curves, both in treating the subject geometrically and algebraically. A general theorem, of which this is a particular case, has been given at *pp.* 199, 200.

PROP. III.

All sections of a cylinder parallel to the axis are pairs of straight lines: all sections parallel or anti-parallel to the base are circles: and all other sections are ellipses, whose centres are in the axis of the cylinder.

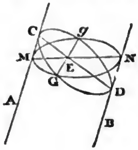
1. The sections parallel to the axis will cut any two of the planes passing through the axis in straight lines parallel to the axis; and in these the linear directrices are situated. Hence the property is obvious.

2. Sections parallel to the base are circles, as is proved in *vol. i. p.* 356.

3. Let CGD be a section of the cylinder anti-parallel to the base ARB: it will be a circle.

For draw any other section MGN to cut CGD in G*g*, and the transverse plane to cut both sections in CD and MN.

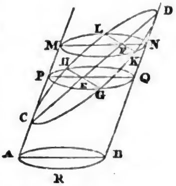
Then, by anti-parallels, MCD = MND, and the points M, C, N, D, are in a circle. Hence CE . ED = ME . EN. But MC . CN = EG^2, in the same manner as in the cone; and G*g* is a double right ordinate in the section CGD. Wherefore EG is perpendicular to the transverse axes CD, and CE . ED = EG^2: and hence CGD is a circle.



4. Any other section is an ellipse, having its centre in the axis of the cone.

Let ARB be the circular base of the cylinder, and DKGHL any other section made in the cylinder by a plane. Draw any planes PGQH, MKNL, parallel to ARB, and meeting the plane CGD in HG, LK; and through the axis of the cylinder draw any plane cutting the cylinder in the edges AM, BD, the planes ARB, PGQ, MKN, in AB, PQ, MN, and the plane CGD in CD.

Then, since PGQ, MKN, are parallel planes, PQ, MN, are parallel lines; and for the same reason, HG is parallel to LK; and PGQ, MKN, being parallel to the base, are circles. Whence PE . EQ = HE . EG, and MF . FN = LF . FK.



But by similar triangles,

MF : PE :: CF : CE, and FN : EQ :: DF : DE. Hence,

MF . FN : PE . EQ :: CF . FD : CE . ED; or from what has been just shown,

LF . FK : HE . EG :: CF . FD : CE . ED.

But by *prop. ii., Ellipse, p.* 124, this is a property of the ellipse; which, therefore, is the figure of the section.

Again, since the edges AC, BD are equidistant from the axis, the line CD drawn to cross them will be bisected by the axis; and the middle of CD is the centre of the ellipse.

Corollary 1.

The conjugate diameter of the ellipse is equal to the diameter of the base of the cylinder.

Corollary 2.

Any two sub-contrary sections of a cylinder are envelopable by a sphere; and they are likewise equally inclined to *any* edge of the cylinder.

Corollary 3.

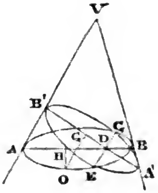
Any plane parallel to the axis of the cylinder cuts the planes of a sub-contrary section, under equal angles.

Corollary 4.

All tangent-planes to the cylinder make angles with the sub-contrary planes, whose sum is constantly the same.

PROP. IV.

The sub-contrary section of an oblique cone is a circle.



Let AVB be the transverse plane, AV, VB the longer and shorter transverse edges, AEB either the plane of the base or one similar and parallel to it; then, if the plane A'EB' be drawn also perpendicular to the transverse plane, to make the angle B'A'V equal to BAV, the section made by A'EB' will be a circle.

For, let AEB, A'EB' intersect in EDG, and let them cut the transverse plane in AB, A'B', respectively.

Then, since B'AB, B'A'B are equal, they stand in the same segment of a circle; wherefore $AD \cdot DB = B'D \cdot DA'$. Also, since both the planes AEB, A'EB' are perpendicular to the transverse plane, their intersection EDG is perpendicular to it, and ED is at right angles both to AB and A'B'. But AEB is a semicircle (*hypoth.*); whence we have $DE^2 = AD.DB = B'D.DA'$; the section A'EB', therefore, having DE at right angles to A'B', and $DE^2 = B'D.DA'$, is a circle.

Corollary 1.

Any two sub-contrary sections, AEB, B'EA', of a cone, are in the same sphere.

For a circle may be described through the four points A, A', B, B'; and its centre, O, will be in the transverse plane, or that which contains the lines AB, A'B'. The planes of the circles are, moreover, perpendicular to the plane AVB, and the perpendiculars from O upon AB, A'B', bisecting them in H and G, will be equidistant from every point in AEB, and from every point in A'EB'; and these points, being in one circle whose centre is O, are equidistant from O; and hence, every point of each of the circles is equidistant from O; that is, they are situated on the same sphere whose centre is O.

Corollary 2.

If a sphere be described about one of two sub-contrary sections of an oblique cone, and touch the other section in one point, it will include it wholly.

Corollary 3.

If a sphere include a circular section of a cone, it will cut the cone a second time in a circle.

Corollary 4.

Every sphere passing through a circular section of a cone, cuts the cone a second time in a circular section.

Corollary 5.

A sphere can be made to pass through any two sub-contrary sections of a cone.

Corollary 6.

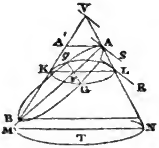
Every sphere passing through a circular section of a cone and the vertex of the cone, has the corresponding cyclic plane for a tangent at that point.

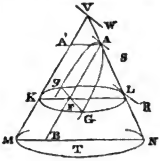
Corollary 7.

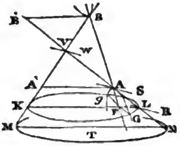
Every sphere passing through the vertex of a cone, and touching either cyclic plane, cuts the cone in a circle whose plane is parallel to the cyclic tangent plane.

PROP. V.

Any section of an oblique cone is a parabola, ellipse, or hyperbola, according as the directing plane touches the cone, does not touch or cut the cone, or cuts the cone in two edges; precisely as in the case in the right cone.







Let GA*g* be a section of an oblique cone: then, if the directing plane touch the cone, the section GA*g* will be a parabola, as in *fig.* 1: if the directing plane pass through the vertex V, and lie wholly without the cone, the section GA*g*B will be an ellipse, as in *fig.* 2: and if the directing plane cut the cone in two of its edges, the section GA*g*B will be an hyperbola, as in *fig.* 3.

Through the vertex V draw a line VW parallel to the intersection of the planes of the base and section; through VW draw a tangent-plane to the cone, touching it in VN; and through VO, the axis of the cone, draw a plane to cut the cone again in the edge VM; through A, B, and any point F in AB, draw sections parallel to the base, cutting the plane MVN in BB', KL, and AA'; and let the section KGL drawn through F cut the section-plane GAG in GF*g*, the tangent-plane at VN in LR; and, finally, let AS be the intersection of the tangent-plane with the section-plane.

Since VW is parallel to the intersection of the section-plane GA*g*, and the plane of the base MTN of the cone, it is parallel to those planes themselves, and hence to all planes which are either parallel to these or which pass through their line of section: and all planes which pass through VW intersect all the planes to which VW is parallel, in parallel lines. Wherefore, the lines AS, LR, in which the tangent-plane at VN cuts the planes GA*g*, KGL, are parallel to VW; and since KGL is parallel to the base of the cone, the section G*g* is parallel to the intersection of GA*g* and MTN, it is parallel to VW, and hence also to AS and LR.

Again, since AS and LR are in the tangent-plane to the cone, they are tangents to the sections GA*g* and KGL.

Also, since the plane MVN passes through the axis VO, of the cone, it passes through the centre of the circle KGL, and KL is a diameter of the circle, and therefore perpendicular to the tangent LR. But LR is parallel to G*g*, and hence G*g* is perpendicular to the diameter KL, and bisected by it in F, its intersection with AB. In the same manner it may be shown, that every other section made in the plane GA*g* by a plane parallel to the base, is bisected by the line AB.

This reasoning applies generally: we proceed to the particular cases.

1. *The parabola.* Here the directing-plane, being a tangent to the cone, the

plane parallel to it through A does not meet the edge of contact. The point B, and line BB′, then have no existence in this case.

By similar triangles, VA′A, AFL, we have

$$AA' : A'V :: FL :: FA,$$

$$\text{or } AA' \,.\, AF = VA' \,.\, FL = KF \,.\, FL = FG^2.$$

Now as the side AA′ is constant, the square of the ordinate FG varies as the abscissa AF: the test of the parabola (*prop. i. p.* 107).

2. *The ellipse.* In this case, AB, parallel to a plane which does not cut the cone after passing through V, lies wholly on the same side of that plane that the sheet MVN is situated, and hence cuts the line VM on the same side of the vertex that it cuts VN.

Also, by the similar triangles A′AB and KFB, and by the similar triangles AFL, ABB′, we have

$$AB : A'A :: FB : KF, \text{ and } BA :: B'B :: FA : FL; \text{ whence}$$

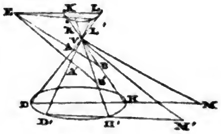
$$AB^2 : A'A \,.\, B'B :: AF \,.\, FB : KF \,.\, FL \; (= FG^2)$$

But $A'A \,.\, B'B : AB^2$ is a given ratio, since A′A, B′B, AB, are given lines; wherefore, $AF \,.\, FB : FG^2$ is a given and constant ratio. These properties are characteristic of the ellipse (*prop. i. p.* 123).

3. *The hyperbola.* In the case supposed, both sheets of cone are cut by the section-plane; and the similar triangles of the figure give, as in the ellipse, the ratio $AF \,.\, FB : FG^2$ a constant ratio. These properties characterize the hyperbola (*prop. i. p.* 145).

PROP. VI.

If any cone be cut by two lines which are parallel to two given lines, and which intersect each other, the rectangle contained by the segments intercepted from the point of mutual section of the lines upon one of them, by the conical surface, will have to the rectangle similarly contained by the segments of the other, a given and constant ratio.



LET VM, VM′ be two lines through the vertex V of the cone, parallel to which *any* two lines EAB and EA′B′ are drawn to cut the cone in A, B and A′, B′: then, wherever the point E be taken to fulfil the condition of the lines through it cutting the cone, the rectangles AE . EB and A′E . EB′ will have a constant ratio.

For let a plane through VM and the point E, cut the cone in the edges DV, VH, the base in DHM, the plane through E parallel to the base in EKL; and similarly, let the plane EVM′ cut the cone in D′V, VH′, the base of the cone in D′H′M′, and the plane parallel to the base in EK′L′.

Then VM, EB, VD, VH, EL, and DM, being in the same plane, we have, by the similar pairs of triangles, EAL, MVD, and EBK, MHV, $LE : EA :: MD : DV$, and $KE : EB :: MH : HV$; whence $LE \,.\, EK : AE \,.\, EB :: DM \,.\, MH : DV \,.\, VH$.

In the same way, we have

$$L'E \,.\, EK' : A'E \,.\, EB' :: D'M' \,.\, M'H' : D'V \,.\, VH'.$$

Now, since ELK and EL′K′ are in a plane parallel to the base of the cone, the four points L, K, L′, K′, are in a circle, and hence $LE . EK = L'E . EK'$. Also the ratios $DM . MH : DV . VH$, and $D'M' . M'H' : D'V . VH'$ are given, since all the lines entering into their expression are given. Wherefore

$$AE . EB : A'E . EB' :: \frac{DV . VH}{DM . MH} : \frac{D'V . VH'}{D'M' . M'H'};$$

that is, a given and constant ratio.

Scholium.

This property branches out into several cases, if treated according to the manner of the ancients. The point E may be within or without the cone; the lines EAB, EA′B′, may cut the same or opposite sheets of the cone; they may touch one or both sheets; or they may be parallel to a tangent-plane of the cone. These varieties are not, however, such as to modify, to any extent, the proof above given.

This theorem is perhaps the most general property of the cone in reference to the several plane sections, inasmuch as the entire system of investigation may be based upon it. Dr. Hugh Hamilton framed such a system; but his anxiety to arrive as speedily as possible at those theorems which relate to the foci, centres, and directrices, led him into a course of investigation but little calculated to show the great advantages of the basis he had chosen.

The adoption of this general property of the conic sections suffices to render the employment of projection altogether unnecessary in the higher class of researches connected with the subject. It serves to render the application of the transversal theorems applicable to the general curves as easily as in the case of the circle. I cannot but consider the extensive employment of the projective method by the continental geometers as mainly springing from not having observed how easily all necessity for it is obviated in the manner just given.

EXERCISES ON THE CONE, SPHERE, AND CYLINDER.

1. Any tangent-plane to a cone cuts the cyclic planes in two lines which make equal angles with the line of contact.

2. Every plane passing through two sides of a cone intersects the cyclic planes in two lines which make equal angles with those sides.

3. Two planes touching a cone of the second degree along any two sides, intersect the cyclic planes in four right lines, which are the generatrices of the same cone of revolution whose axis of revolution is perpendicular to the plane of the two edges of contact.

4. The sum or difference of the angles which each tangent plane makes with the cyclic planes, is constant.

5. In every cone, each tangent plane intersects the two cyclic planes in two right lines, such that the product of the tangents or the semi-angles which they make with the intersection of the cyclic planes, is constant.

6. The product of the sines of the angles which each side makes with the two cyclic planes, is constant.

7. Lines drawn from the focus of an elliptic section of the right cone, and from the vertex of the cone, to any point in the ellipse or hyperbola, will have a constant difference or sum, viz. the line from the vertex to the extremity of the conjugate diameter diminished by the semi-transverse diameter.

8. The sum or difference of the lines drawn from the vertex of a right cone to the extremities of any diameter of the elliptic or hyperbolic section, is double the line from the vertex to the extremity of the conjugate diameter.

9. The difference or sum of the lines drawn from the vertex of a right cone to the extremities of the transverse diameter of an elliptic or hyperbolic section will be equal to the distance of the foci.

10. All the right cones which have the same conic section for their bases, have their vertices upon another conic section, situated in a plane at right angles to that of the former one, the foci of one of these curves being the summits of the other, and, respectively, the summits of the one the foci of the other.

11. If a right cone be cut by planes equi-distant from the vertex, the sections will all have the same parameter.

12. If a circle of the sphere be made the base of a cone, whose vertex is in the surface of the sphere, and any plane be drawn parallel to the tangent plane at the vertex, it will cut the cone in a sub-contrary section.

13. The tangent cone to the circle of the sphere being drawn, and the vertices of the two cones being joined, this will be the axis of the oblique cone in reference to the sub-contrary base.

14. Any two great circles of the sphere passing through the vertex of the cone will have the same inclination that the sections of their planes on the sub-contrary plane make with each other.

TRANSVERSALS.

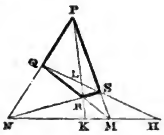
DEFINITIONS.

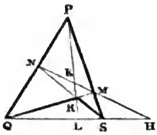
1. When a line or curve cuts a line or curve, or a system of lines or curves, the cutting line or curve is called a *transversal.*

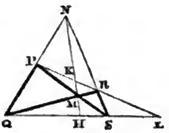
2. When a plane or curve surface cuts a line or curve, or system of lines or curves, the cutting-plane or surface is called a *transversal-plane* or *transversal surface.*

3. When a polygon is spoken of, it is understood that the lines may succeed each other in any order whatever, and that they are of unlimited extension, so far as the implied constructions require it.

4. A *simple quadrilateral* is the assemblage of four lines passing through four points in any order, and terminated at their points of intersection, though capable of unlimited extension. There are three varieties; (1) the *salient;* (2) the *re-entrant;* and (3) the *intersectant,* as in the darkened parts of the figures.

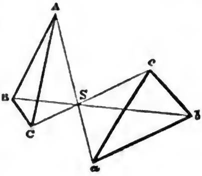






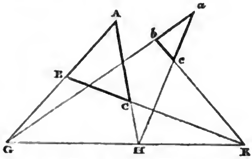
5. The *complete quadrilateral* is the figure formed by producing such of the sides of each of the three simple quadrilaterals; as in each of the figures where the simple quadrilateral is drawn darker than their prolongations to the points of section.

It thus appears, that whatever be the form of the simple quadrilateral, the general features of the complete quadrilateral are always the same.



6. When two triangles ABC, *abc*, whether in one plane or not, are so situated that the lines A*a*, B*b*, C*c*, pass through one point S, they are called *co-polar triangles*, and S * is called their *pole*.

7. If two triangles ABC, *abc*, be such that the intersection G of AB and *ab*, the intersection H of BC and *bc*, and the intersection K of CA and *ca*, be in one line: they are called *co-axial triangles*, and the line GHK is their *polar axis*, or simply their *polar* (*fig. def.* 8).



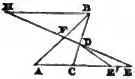
8. If one of the co-polar triangles have its angles upon the sides of the other, this one is said to be the *inscribed* or *escribed polar triangle*, and the other the *circumscribed co-polar triangle*. The *escribed* is analogous to the *escribed circles*, *vol. i. p.* 416.

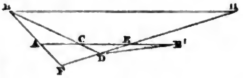
9. The points of intersection of the corresponding or homologous sides of two co-polar triangles, are called the homologous intersections of the triangle, or simply *homologous sections*.

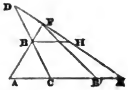
PROP. I.

If a transversal DEF cut the sides BC, CA, AB, of a triangle ABC in D, E, F, the sides will be divided in D, E, F, so that

AE . BF . CD = EC . FA . DB.







For draw BH parallel to AC, meeting the transversal in H: then,

AE . FB = BH . AF, by *sim. tr.* HBF, AFE; and
BH . CD = EC . BD .... HBD, EDC.

Wherefore, by compounding, we have

AE . BF . CD = EC . FA . BD;

that is, the solid under three alternate segments, taken in rotation, is equal to the solid under the other three.

Corollary 1.

This may be written, and often usefully, under the forms

AE : EC :: AF . BD : FB . DC;
BF : FA :: BD . EC : AE . DC;
CD : DB :: EC . AF : AE . FB.

That is, each side is divided into segments which are in the ratio compounded of the ratios of the segments of the other two.

---

* It is always to be understood in modern research, that parallelism is, in respect of properties, a case (the limiting one) of convergency; and hence, that when S is infinitely distant, the lines A*a*, B*b*, C*c*, are parallel. It is also to be understood, that these triangles may either be in the same or in different planes.

Corollary 2.

The three sides of the triangle and the transversal may be viewed as four lines, each of which cuts the other three; and hence there will arise three other equalities similar to that obtained in the proposition. The four taken collectively will be

$$AF.BD.CE = FB.DC.EA, \text{ triangle } ABC \text{ cut by } FDE;$$
$$FB.AC.ED = BA.CE.DF, \text{ triangle } AFE \text{ cut by } BDC;$$
$$AB.CD.EF = BC.DE.FA, \text{ triangle } FBD \text{ cut by } ACE;$$
$$AE.FD.BC = EF.DB.CA, \text{ triangle } CDE \text{ cut by } AFB.$$

Any three of these, however, by compounding, give the fourth; and, by compounding them in pairs, we may obtain a future theorem of this series *.

PROP. II.

If three points D, E, F, be taken in the sides BC, CA, AB, of a triangle ABC, so that either only one, *or* all three, *be in the extensions of the sides, and that the segments of the sides made by these points be such that*

$$AE.BF.CD = EC.FA.DB;$$

then D, E, F, will be situated in one straight line.

[*See the figures to preceding proposition.*]

FOR if they be not in one line, let FD cut AC in some other point E′, and join E′D: then,

$$AE' : E'C :: AF.BD : FB.DC, \ (\textit{pr. } 1, \textit{cor. } 1); \text{ and}$$
$$AE : EC :: AF.BD : FB.DC, \ (\textit{hypoth.}); \text{ whence}$$
$$AE' : E'C :: AE : EC; \text{ or } (\textit{composition} \text{ or } \textit{division}),$$
$$AE' \pm EC' : AE' :: AE \pm EC : AE.$$

But $AE' \pm E'C = AC = AE \pm EC$; and hence $AE' = AE$, which is impossible. The line FD, therefore, passes also through E.

Scholium.

This theorem, the converse of the preceding, often furnishes a very effective method of proving that three points are in one line.

It is essential to remark, that this only holds good when an *odd* number of the sides is cut in extension.

---

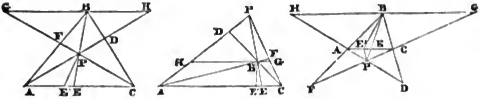
* This theorem, which was probably known to Hipparchus, and was certainly known to Menelaus and Ptolemy, has been made by Carnot the foundation of his system of transversals. His first demonstration was by a complex algebraical process (*Géom. de Position*, p. 276): but he afterwards (*Essai sur les Transversales*, p. 67) gave a demonstration similar to the preceding.

The manner, too, in which it is readily rendered subservient to the investigation of a great number of porisms, renders it probable that it was also known to Euclid. It was familiarly known amongst the Arabian mathematicians, from whom it was brought into Europe; and either notices of it, or the actual use, are frequent amongst the early geometers of the west. It is given in Ptolemy's *Almagest*, *l. i. pr.* 16; and in Menelaus *Sphær. lib. iii. pr.* 1, 2. We can scarcely doubt that Carnot rediscovered it, as originally given in the *Géom. de Pos.* above referred to: and it is very likely, that from having discovered the simple demonstration quoted in the text, he was led to reconsider the methods he had employed, and to give the greatly improved system of investigation contained in his *Essai sur la Théorerie des Transversals*, three or four years afterwards, as above referred to.

PROP. III.

If three straight lines AD, BE, CF, be drawn from the angles A, B, C, of a triangle through any point P in the plane of the triangle, to cut the opposite sides in D, E, F: then will

$$AE \,.\, BF \,.\, CD = EC \,.\, FA \,.\, BD.$$



For through B draw BH parallel to AC, meeting AP, CP, in H and G: then

$$AE \,.\, BG = EC \,.\, BH \text{ (parallels AC, HG, cut by HP, BP, GP)},$$
$$AC \,.\, BF = BG \,.\, FA \text{ (sim. trians. FAC, FBG)},$$
$$BH \,.\, CD = AC \,.\, BD \text{ (sim. trians. DCA, DBH)};$$

and compounding these, we have at once

$$AE \,.\, BF \,.\, CD = EC \,.\, FA \,.\, BD.$$

Corollary.

This may also be put under the forms in *pr.* 1, *cor.* 1; viz.

$$AE : EC :: AF \,.\, BD : FB \,.\, DC;$$
$$BF : FA :: BD \,.\, EC : AE \,.\, DC;$$
$$CD : DB :: EC \,.\, AF : AE \,.\, FB.$$

Scholium.

The earliest author whose known writings contain this elegant proposition is the Marquis Ceva (1678): but it has been usually attributed to John Bernouilli. There is no doubt, however, that it was known earlier than Ceva's time, for it is *implied* in the researches of Desargues and Pascal. See Chasles, *Aperçu Historique des Methodes en Géométrie, p.* 83.

Carnot's method of proof is very simple and elegant, and is therefore annexed.

The triangles ABD, ADC, cut by the transversals FC, BE, give respectively,

$$BC \,.\, DP \,.\, AF = CD \,.\, AP \,.\, FB, \text{ and } DB \,.\, CE \,.\, AP = BC \,.\, EA \,.\, PD;$$

which, upon compounding, yield the same result.

PROP. IV.

If the sides BC, CA, AB, of a triangle ABC be divided in D, E, F, so that either two *or* none *of the points be in the extensions of the sides, and so as to fulfil the condition,*

$$AE \,.\, CD \,.\, BF = EC \,.\, DB \,.\, FA;$$

then the lines AD, BE, CF, will pass through the same point, P.

[*See the figures to the preceding proposition.*]

For if not, let two of them AD, CF, meet in P, and draw BP to meet AC in E′. Then

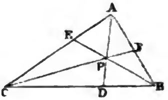
$$AE' : E'C :: AF \,.\, BD : FB \,.\, DC \text{ (prop. iii.), and}$$
$$AE : EC :: AF \,.\, BD : FB \,.\, DC \text{ (hypoth): whence}$$
$$AE' : E'C :: AE : EC; \text{ and } AE' = AE, \text{ as in the reasoning of}$$

prop. ii., which is impossible. The line BP does not, therefore, differ from BE; or all the three lines pass through one point.

This theorem, independently of the systematic use to be made of it hereafter, furnishes proofs remarkable for their simplicity and brevity, of several elementary ones. This is illustrated in the deductions or corollaries which follow.

Corollary 1.

Lines AD, BE, CF, drawn to bisect the angles A, B, C, of a triangle intersect in one point.



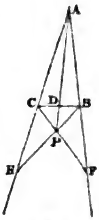
For we have (*Euc. vi.* 3),

CD : DB :: CA : AB, or CD . AB = DB . CA,
BF : FA :: BC : CA, or BF . CA = FA . BC,
AE : EC :: AB : BC, or AE . BC = EC . AB.

Whence, by compounding, CD . BF . AE = DB . FA . EC : which by the proposition is the criterion of the lines AD, BE, CF, passing through one point P.

Corollary 2.

Lines AD, BE, CF, of which one, AD, bisects the interior angle BAC, and the two others the opposite exterior angles CBF, BCE, of the triangle ABC, will pass through the same point, P.

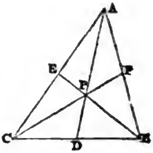


For, CD : DB :: CA : AB, or CD . AB = DB . CA (*Euc. vi.* 3)
BF : FA :: BC : CA, or BF . CA = FA . BC (*Euc. vi. A.*)
AE : EC :: AB : BC, or AE . CB = EC . AB (*ibid.*);

whence, compounding, the criterion is fulfilled.

Corollary 3.

Lines AD, BE, CF, drawn from the angles to bisect the opposite sides of the triangle ABC in D, E, F, meet in one point.



For CD = DB, BF = FA, and AE = EC; whence the criterion is fulfilled by the equality of the corresponding lines.

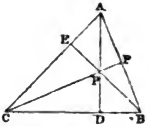
Corollary 4.

The lines AP, BP, CP, are respectively the doubles of PD, PE, PF.

For the triangle ADB cut by CF gives BC : CD :: BF AP : FA.PD :: AP : PD, since BF = FA. Whence, since BC = 2CD, we have also AP = 2PD; and similarly for the other two cases.

Corollary 5.

Perpendiculars AD, BE, CF, drawn from the angles of a triangle to the opposite sides, pass through the same point. For,



CE : CD :: CB : CA, or EC . CA = BC . CD (*sim. trians.* CBE, CAD)
DB : BF :: BA : BC, or DB . BC = FB . BA (.......... DBA, CFB)
FA : AE :: AC : AB, or FA . AB = EA . AC (.......... FAC, AEB).

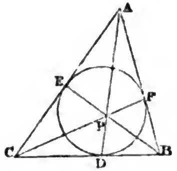
By compounding these, the criterion is again fulfilled.

Corollary 6.

If two sides of a triangle be cut proportionally, the lines drawn from the points of section to the opposite angles, intersect in the line drawn from the remaining angle to the middle of the base.

Also, the converse of this is true.

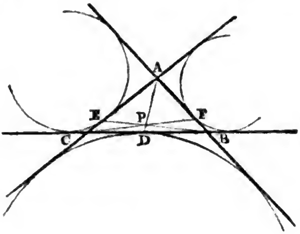
Corollary 7.



Lines AD, BE, CF, drawn from the angles of a triangle to the points of contact D, E, F, of the inscribed circle with the opposite sides BC, CA, AB, all pass through one point.

For CD = CE, BD = BF, and EA = AF; whence the criterion is fulfilled.

Corollary 8.



Lines AD, BE, CF, drawn from the angles of a triangle to the three points of contact of the escribed circles, all pass through one point.

For CE = BF, EA = BD, and FA = CD; whence the criterion is fulfilled, as before.

PROP. V.

If lines AD, BE, CF, be drawn from the angles A, B, C, of a triangle, to cut the opposite sides in D, E, F, and each other in P, then will

$$AP \,.\, BP \,.\, CP : DP \,.\, EP \,.\, FP :: AB \,.\, BC \,.\, CA : AF \,.\, BD \,.\, CE.$$

[*See figures to prop. iii.*]

For,

PA . EC . BD = DP . AE . CB (*trian.* ADC *cut by* EB),
PB . FA . CE = EP . BF . AC (*trian.* AEB *cut by* CF),
PC . DB . AF = FP . CD . BA (*trian.* BCF *cut by* AD);
FB . DC . EA = AF . BD . CE (*prop. iii.*):

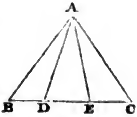
whence, compounding and forming the proportion, we have the stated result.

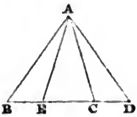
PROP. VI.

If a transversal be drawn to cut any three lines AB, AC, AD, which meet in a point A, in B, C, D, there will be the following relation amongst the segments of the transversal and the segments it cuts from the given line:

$$AC^2 . BD \pm AB^2 . CD = AD^2 . BC \pm BD . DC . CB;$$

the upper sign applying to the case where AD is between AB and AC, and the lower where AD is without them.





For draw AE perpendicular to the transversal BC: then (*Euc. ii.* 12, 13) we have

$AC^2 = AD^2 + DC^2 - 2DC.DE$; or $AC^2.BD = AD^2.BD + DC^2.BD - 2BD.DE.DC$,
$AB^2 = AD^2 + DB^2 \pm 2BD.DE$; or $AB^2.CD = AD^2.CD + BD^2.CD \pm 2BD.DE.DC$;
whence, adding or subtracting to exclude the last terms, we have

$$AC^2.BD \pm AB^2.CD = AD^2\{BD \pm DC\} + DC.DB\{CD \pm DB\}$$
$$= AD^2.BC \pm BD.DC.CB.$$

The earliest notice of this theorem is by Dr. Matthew Stewart, it being one amongst his many general theorems, published when a candidate to succeed Maclaurin as Professor of Mathematics in the University of Edinburgh, in 1746. It is used by Dr. Simson, in his *Restoration of the Plane Loci*, 1748; and by Professor Simpson, in his *Select Exercises*, 1752. Since that time it has been well known in this country, though it does not appear to be so amongst continental writers, till the use made of it by Carnot in 1803 (*Géom. de Pos. p.* 263). A trigonometrical investigation is given in *vol. i. p.* 458.

Chasles remarks, that this proposition includes eight of the lemmas of Pappus for plane loci. It also contains several other and well known properties of triangles, as particular cases, or corollaries, as shown below.

Corollary 1.

Consider ABC as a triangle, the base of which is bisected in D: then $BD = DC$, and we have

$$AB^2 + AC^2 = 2AD^2 + 2DB^2. \text{ (vol. i. p. 312.)}$$

Corollary 2.

Let $BD : DC :: BA : AC$, then when D is between C and B (*fig.* 1),

$$AB.CD = AC.BD, \text{ and } CD\{BA + AC\} = BC.CA;$$

whence $BA.AC - BD.DC = AD^2$. (*Sims. Euc. vi. B.*)

Corollary 3.

Let $BD : DC :: BA : AC$, and D be taken in extension of AB (*fig.* 2); then

$$BD.AC = BA.DC, \text{ and } BD\{BA - AC\} = AB.BC; \text{ whence}$$
$$BD.DC - BA.AC = AD^2:$$

a theorem analogous to the preceding corollary.

Corollary 4.

Let BAC be isosceles, and D be within the base: then $AC = AB$, and the theorem becomes

$$AB^2\{CD + DB\} = BC\{AD^2 + CD.DB\}, \text{ or}$$
$$AB^2 - AD^2 = CD.DB.$$

Corollary 5.

Let BAC be isosceles, and D in the extension of the base: then

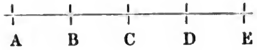
$$AD^2 - AC^2 = BD.DC.$$

Corollary 6.

Denoting D in *cor.* 2 and *cor.* 3 by D and D′ respectively, we get

$$BA.AC - BD.DC = AD^2, \text{ and } BD'.D'C - BA.AC = AD'^2: \text{ whence}$$
$$BD'.D'C - BD.DC = AD^2 + AD^2.$$

Corollary 7.



The same holds true when the point A is in the transversal. In this case we have, as regards the four points A, B, C, E, the equation, (taking, for instance, A to the *left* of B in the second figure),

$$AE^2 . BC + CE^2 . AB = BE^2 . AC + AB . BC . CA.$$

Corollary 8.

If we take another point D in the same line, we shall have, as in the last corollary,

$$BE^2 . CD + DE^2 . BC = CE^2 . BD + BC . CD . DB.$$

Multiply this result by AB.AC, and that of the preceding corollary by BD.DC, and arrange the results, we get

(1). $DE^2.AB.BC.CA - CE^2.AB.AC.BD = -BE^2.AC.CD.AB + AB.BC.AC.BD.CD.$

(2). $AE^2.BC.CD.DB + CE^2.AB.BD.DC = BE^2.AC.CD.DB + AB.BC.AC.BD.CD;$

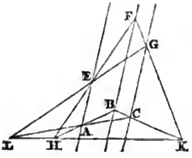
and subtracting the former from the latter, we get, after transposition,

$$AE^2.BC.CD.DB + CE^2.AD.DB.BA = BE^2.AC.CD.DA + DE^2.AB.BC.CA;$$

a theorem of great beauty, due to M. Chasles. See *Ladies' Diary*, 1843.

PROP. VII.

Two co-polar triangles, ABC, EFG, are co-axial (*def.* 7, *p.* 215); *or the points L, H, K, of intersection of the corresponding sides AB and EF, BC and FG, and CA and GE, are in one line.*



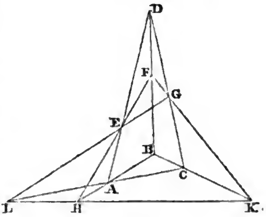
First. Let the lines AE, BF, CG, upon which the angles are situated, be parallel, and either all in the same plane or not. Then the following pairs of triangles will be similar, viz.

HEA, HBF; KCG, KBF; and LAE, LCG.

Whence,

$$FB . HA = BH . AE,$$
$$GC . KB = CK . BF,$$
$$EA . LC = AL . CG;$$

and compounding, LA . HB . KC = AH . BK . CL; which is the criterion (*prop. ii.*) of the points L, H, K, in the sides of the triangle ABC, being in one line.



Secondly. Let the lines AE, BF, CG, meet in some point D, and be either in one plane or not.

Then, the triangles ADB, BDC, CDA, cut by the transversals EFH, FGK, GEL, respectively, give (*prop. i.*) the following:

$$FD . EA . HB = DE . AH . BF,$$
$$DG . CK . BF = GC . KB . FD,$$
$$ED . GC . LA = DG . CL . AE;$$

whence, *comp.*,

LA . HB . KC = AH . BK . CL; the criterion as before *.

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* The cases where the three lines AE, BF, CG, are not in one plane, follows from very simple considerations, and without the aid of any ratios; thus furnishing one example, amongst

Corollary.

Let AG and EC intersect in M, AF and EB in N, and CF and BG in P: then M, N, P, L, H, K, will be in one plane.

For the line NP tends to L, MP tends to H, and MN to K; whence they are all in the plane through LH.

Scholium.

This remarkable theorem is due to Desargues, an author of such transcendent merit as to be justly termed by Poncelet, "the Monge of his age"—an author whose misfortune it was to live before the time in which his views could be appreciated, and even of whose printed works not a vestige is known now to exist, though he lived only two centuries ago. All we know of his works is from the writings of others, and these either his detractors, or those who could not comprehend the spirit and character of his writings. From such sources, however, enough can be gleaned to create the deepest regret for the loss of his original and profound researches.

PROP. VIII.

Two co-axial triangles, ABC, EFG, are co-polar (*def.* 6, *p.* 215); *that is, if the sections H, K, L, of the homologous sides be in one line, the homologous angular points will be situated upon three straight lines, which are either parallel, or pass through one point D.*

[*See fig.* 2, *prop. vii.*]

First. Let the two triangles ABC, EFG, be in one plane. Draw BF, CG, to intersect in D′; and if AE do not also pass through ED′, let it cut BF in D. Draw D′E to meet AB in A′; join A′C, and let it cut EG in L′, and join L′H.

Then A′BC and EFG are co-polar triangles, having D′ for their pole: whence (*prop. vii.*), K, H, L′, are in one straight line. But (*hypoth.*) K, H, L, are in one straight line. Wherefore, the straight lines LH, L′H, have a common segment HK; which is impossible. Hence the three lines AE, BF, CG, cannot pass otherwise than through one point D′; that is, the triangles are co-polar.

Secondly. Let the two triangles not be in one plane. [*See fig.* 2, *prop. vii.*] Since AC, EG, meet in L (*hypoth.*), they are in one plane; and hence AE, CG, are also in that plane, and therefore AE, CG, are either parallel or intersect in the plane AECG.

In like manner, AE, FB, are in one plane, and are either parallel, or intersect in that plane; and FB, CG, are in one plane, and are either parallel or intersect in that plane.

The three lines, AE, BF, CG, will therefore be in the intersections of the three planes, and hence will either be parallel, or meet in one point.

The two triangles are therefore co-polar.

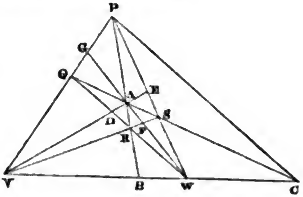
---

a great number that may be adduced, of the cases of a theorem which are most removed from our first principles, being easier of demonstration than those which lie nearer the threshold of the science. This circumstance suggests some important reflections upon geometrical methods; for the statement of which, however, this is not the proper place. See *Ladies' Diary*, 1842.

The planes ABFE, ABC, and EFG, intersect each the other two; and hence the lines AB, FE, in which ABFE cuts EFG and ABC, meet each other in the line of section EFG, ABC. In the same way EG, AC, meet in the intersection of EFG, ABC; and FG, BC, also meet in the intersection of EFG, ABC. That is, the three points are in the same straight line—that of intersection of the planes of the co-polar triangles. This method of treating such theorems has been much cultivated in the school of Monge.

PROP. IX.

The three diagonals, QS, PR, VW, of a complete quadrilateral being drawn, each is divided harmonically by the other two; that is,



(1) VB : BW :: VC : CW,
(2) QA : AS :: QC : CS,
(3) PA : AR :: PB : BR.

For, viewing VPW as a triangle, we have (*props. i.* and *iii.*),

VQ . PS : QP . SW :: VC : CW,
VQ . PS : QP . SW :: VB : BW;

whence,

VB : BW :: VC : CW.

Again, viewing PQS as a triangle, we have

QV . PW : VP . WS :: QC : CS,
QV . PW : VP . WS :: QA : AS; whence,
QA : AS :: QC : CS.

Lastly, viewing PQR as a triangle, we have

PV . QW : VQ . WR :: PB : BR,
PV . QW : VQ . WR :: PA : AR; whence,
PA : AR :: PB : BR.

Corollary 1.

If VA, WA, be drawn to cut the sides QR, PS, and RS, QP, in D, E, and F, G; the segments FG, AW, DE, VA, are all harmonically divided, each at the other two points in them.

Corollary 2.

The lines GE, DF (not in the figure) will pass through C; and the lines GD, EF, will pass through B.

Corollary 3.

If VW be any line, harmonically divided in B and C, and lines WP, BP, VP, be drawn to any point P; and if WQ, VS, intersecting in any point R in BP be drawn to meet PV, PW, respectively in Q and S: then SQ will always tend to the fourth point C.

The converse is also true: that if through C lines be drawn to cut PV, PW, in Q and S; the lines WQ, VS, will always cut each other in a point situated in PB.

Corollary 4.

If we conceive a transversal to be cut by two diagonals, PR, SQ, in *a* and *b*, and by lines drawn from the extremities W, V, of the other diagonal, through the intersection A of the former two, in *c* and *d*: then *ab* will be harmonically divided in *c* and *d*.

Scholium.

This property of the figure, which, with a slight difference in the manner of its enunciation, was known to the ancients, is the foundation of nearly all the problems given in our first volume on *lineation;* and by means of it, the student will be able to form the proofs of those which are not formally detailed in that place.

PROP. X.

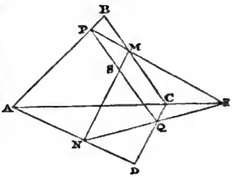
The opposite pairs of sides, AB, CD, and AD, BC, of a twisted quadrilateral, ABCD, are divided in the same ratio; viz.

$$BM : MC :: AN : ND,$$

$$\text{and } AP : PB :: DQ : QC;$$

then the lines PQ, NM, being drawn, will lie in the same plane, and intersect in a point S, such that

$$PS : SQ :: MS : SN.$$



For, let the planes of the two component triangles meet in AC, and draw PM, NQ, in those planes to meet AC in R and R′, supposed for the moment different points.

Then, (by *prop. i.* and the *hypothesis*,) we have

$AP.BM.CR = PB.MC.RA$ (ABC, and PM)

$ND.QC.R'A = AN.DQ.R'C$ (ADC, and NQ)

$MC.AN = BM.ND$ (*first hyp.*)

$PB\,DQ = AP.QC$ (*second hyp.*)

Whence, compounding, $AR.R'C = AR'.RC$; or R and R′ coincide. The four points, P, M, Q, N, are therefore in one plane, and hence PQ, MN, are in one plane, and intersect in a point S.

Again, in the same manner,

$PS.QN.RM = SQ.NR.MP$ (PQR and MN)

$MP.RQ.NS = PR.QN.SM$ (MRN and PQ)

$NR.QC.AD = RQ.CD.AN$ (NDQ and AC)

$PR.MC.BA = RM.CB.AP$ (PBM and AC)

$BC.AN = AD.CM$ (*first hypoth.*)

$CD.AP = AB.CQ$ (*second hypoth.*)

Whence, comp., $PS.SN = QS.SM$; or as a proportion,

$$PS : SQ :: MS : SN.$$

Corollary.

NP and QM will meet in the other diagonal DB, as may be similarly shown.

Scholium.

A different and very elegant solution on other principles may be seen in Legendre's Geometry (*translation, p* 129); but the method of transversals has been employed here for the sake of uniformity and illustration.

The property may be stated somewhat more generally; viz. when

$$\frac{AP}{PB} = m \cdot \frac{DQ}{QC} \text{ and } \frac{AN}{ND} = m \cdot \frac{BM}{MC},$$

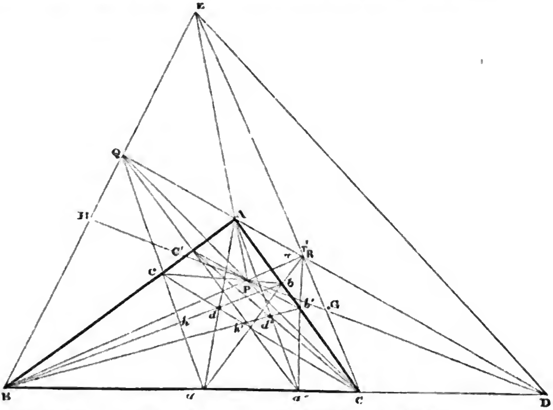
m being of any value whatever, the lines PQ, NM, are still in one plane, *etc.*

The chief value of the proposition is its subserviency to the investigation of the *hyperboloïd of one sheet*, which is generated in two different ways by the motion of one line upon three fixed lines. We must refer the reader to the writings especially of Hachette and Chasles on this subject.

The particular case of the hyperboloïd, being one of revolution, has been slightly noticed at *p.* 197 of this volume. It is easy to infer from that, that the surface may be generated by the revolution of two different lines.

PROP. XI.

If two triangles, abc, a′b′c′, be inscribed co-polars to a third ABC; and if R, P, Q, be the homologous sections of ab and a′b′, of bc and b′c′, and of ca and c′a′ respectively: then the triangle PQR is an escribed co-polar to each of the triangles abc, a′b′c′; and a circumscribed co-polar with the original triangle ABC.



For we have (*prop.* i. and *prop.* vii.) the following equalities:—

$Pb \,.\, Ab' \,.\, cc' = Pc \,.\, Ac' \,.\, bb'$, from triangle A*bc* and transv. P*b′c′*;
$Qc \,.\, Bc' \,.\, aa' = Qa \,.\, Ba' \,.\, cc'$ .... B*ca*.......... Q*c′a′*;
$Ra \,.\, Ca' \,.\, bb' = Rb \,.\, Cb' \,.\, aa'$ .... C*ab*......... R*a′b′*;
$Ac' \,.\, Ba' \,.\, Cb' = Ab' \,.\, Ca' \,.\, Bc'$, from co-polar triangles ABC, *a′b′c′*.

From which, by compounding, we obtain

$$Pb \,.\, Qc \,.\, Ra = Pc \,.\, Qa \,.\, Rb;$$

which is the criterion of PQR, being co-polar with *abc*; and it is escribed in *abc* by the construction.

In the same way we obtain the criterion of PQR being co-polar with *a′b′c′*; viz. the equation,

$$Pb' \,.\, Qc' \,.\, Ra' = Pc' \,.\, Qa' \,.\, Rb'.$$

Again, draw AQ to meet BC in D; and if it do not cut *ab* and *a′b′* at their point of intersection R, let it cut the former in *r* and the latter in *r′*.

Now since B is the intersection of the sides A*c*, C*a*, and *d* that of the diagonals C*c*, A*a*, of the quadrilateral AC*ac*, we have (*prop. ix. corollaries*), $Bh : hd :: Bb : bd$, and that the lines BC, *h*Q, A*a*, and *ab*, are drawn from *a* to B, *h*, *d*, *b*, these lines form a system of harmonical sectors; and therefore the line AQ crossing them is so cut that

$$QA : QD :: rA : rD.$$

Similarly, $QA : QD :: r'A : r'D$.

Wherefore, $Ar : Ar + rD :: Ar' : Ar' + r'D$, or $Ar = r'A$.

The points *r* and *r′*, therefore, being in the line AQ equidistant from A, coalesce; and, being also in the lines *ab*, *a′b′*, which intersect, are at their point of intersection R. The side QR of the triangle PQR consequently passes through the angle A of the triangle ABC. In the same manner, PR passes through B, and PQ through C.

Lastly, BQ, PA, CR, will pass through the same point.

For, draw PD cutting RC, BQ, in G and H; also let BQ, RC, meet in E, and draw DE, AE, and PE; and if AE do not pass through P, let it cut HP in some other point P′. Then, the sides BC, QR, of the quadrilateral BCRQ intersecting in D, and the diagonals BR, CQ, in P, the line HG is harmonically divided in P and D; or,

HP : PG :: HD : DG.

Also, since AD is harmonically divided in R and Q, as before shown, and lines are drawn to a point E from the points of section, the line HDGP′ drawn to cut them is also divided harmonically, or so that

HP′ : P′G :: HD : DG.

It follows, therefore, that HP′ : P′G :: HP : PG, or reasoning as in the preceding case, HP′ = HP. Whence P and P′ coincide; or A, P, E, are in one line, and the three, AP, BQ, CR, tend to one point, and the triangles ABC, PQR, are therefore co-polar.

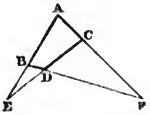
Corollary.

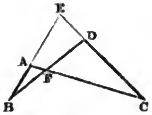
The triangles ABC, PQR, are co-axial; that is, if the sides of the one be respectively produced to cut those of the other, the three homologous sections will be in one line (*prop. vii.*).

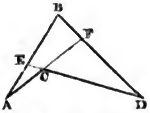
PROP. XII.

If the simple quadrilateral ABDC be completed by continuing the opposite sides till they meet, viz. AB, CD, to meet in E, and AC, BD, to meet in F; then,

BA . AC : BD . DC :: EA . AF : ED . DF,
BE . EC : AE . ED :: CF . FB : DF . FA,
AB . BD : AC . CD :: EB . BF : EC . CF.







First. The triangles AEC, ABF, cut by the transversals BF, EC, give

AB . ED . CF = BE . DC . FA,
EB . DF . CA = AE . BD . FC;

whence, compounding, and then forming the proportion, we have

BA . AC . ED . DF = BD . DC . EA . AF, or
BA . AC : BD . DC :: EA . AF : ED . DF.

Secondly. The triangles EBD, EAC, and transversals AF, BF, give

BA . EC . DF = AE . CD . BF,
BE . DC . FA = AB . ED . CF;

whence, compounding, *etc.*, we get the second proportion.

Thirdly. The triangles ABF, EBD, and transversals EC, AF, give

CF . BD . EA = AC . FD . BE,
AB . EC . DF = AE . CD . FB;

whence, compounding, *etc.*, we get the third proportion.

Scholium.

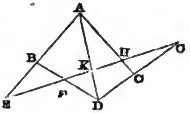
It is evident from a slight survey, that the three properties here deduced are in fact but the repetitions of a single one as related to the three different species of simple fundamental quadrilaterals. The proof of the second proposition, in reference to the first figure, is the same as that of the first in reference to the second figure; and a like interchange may be made between the first and third, or the second and third. This occurs perpetually in reference to the quadrilateral, and in actual investigations shortens the work considerably.

It may also be remarked, that these properties might have been deduced from those in *prop.* i. *cor.* 2.

PROP. XIII.

If the successive sides AB, BD, DC, CA, of a quadrilateral ABDC be cut by a transversal in E, F, G, H: then the sides will be so divided that

$$BE.CG : EA.GD :: CH.BF : HA.FD.$$



For draw the diagonal AD meeting the transversal in K; then (*prop.* i.) the triangles ABD, ACD, cut by the transversal, give

$$BE : EA :: KD.FB : DF.KA,$$
$$CG : GD :: KA.HC : KD.AH;$$

whence, compounding, we obtain the affirmed result.

Corollary.

In the form of an equality, the result just obtained becomes

$$AE.BF.DG.CH = EB.FD.GC.HA;$$

in which each side is composed of the alternate segments of the sides of the quadrilateral; or such as have not in any case a common point of junction.

Scholium.

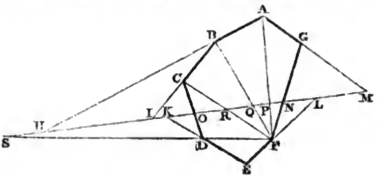
The mode of proof employed above applies to every species of simple quadrilateral, and to any position of the transversal. It will conduce to the student's improvement to sketch a set of figures adapted to all the cases that can arise.

It may be remarked, that in analogy to the property established in *prop. xii.* there are two other proportions deducible from the joint consideration of the three figures; or else by separate investigations, as I gave them in the *Phil. Mag.* vol. lxviii. (1826), pp. 117, 118.

It will be well worth the student's trouble to examine the cases of this proposition in detail, especially when the transversal passes through each of the points of section of the sides and diagonals of each of the simple quadrilaterals, and of the complete quadrilateral, either one intersection or two at a time. Many of the results are interesting, and a few are given amongst the exercises at the end. Amongst other results readily obtained from it is the *prop.* 139, *lib. vii.* of the *Mathematical Collections of Pappus*, of which another proof is given a little further on (*prop. xviii.*); also by means of transversals. *Phil. Mag.* vol. lxviii. p. 119.

PROP. XIV.

If a transversal cut the sides of any plane polygon, it will divide them so that the compound of one set of alternate segments (as in the preceding proposition) will be equal to the compound of all the other segments.



The method of proof being the same in all cases, it will be sufficient to give it in detail for the heptagon ABCDEFG cut by the transversal SM.

Draw lines from one of the angular points, as F, to all the others A, B, . . . (except the two adjacent ones, which are already drawn in the structure of the figure) meeting the transversal in P, Q, R, S; and let H, I, K, L, M, N, O, be the points in which the transversal cuts the sides of the polygon. Then (*prop.* i.),

$$\begin{aligned}
AM\,.\,GN\,.\,FP &= MG\,.\,NF\,.\,PA \text{ by the triangle } GFA;\\
AP\,.\,FQ\,.\,BH &= PF\,.\,QB\,.\,HA \ldots\ldots\ldots\ldots\ AFB;\\
BQ\,.\,FR\,.\,CI &= QF\,.\,RC\,.\,IB \ldots\ldots\ldots\ldots\ BFC;\\
CR\,.\,FS\,.\,DO &= RF\,.\,SD\,.\,OC \ldots\ldots\ldots\ldots\ CFD;\\
DS\,.\,FL\,.\,EK &= SF\,.\,LE\,.\,KD \ldots\ldots\ldots\ldots\ DFE;
\end{aligned}$$

all the triangles being cut by the same transversal SM. Hence, compounding and cancelling common terms,

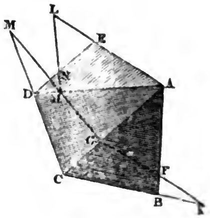
$$AM\,.\,BH\,.\,CI\,.\,DO\,.\,EK\,.\,FL\,.\,GN = MG\,.\,HA\,.\,IB\,.\,OC\,.\,KD\,.\,LE\,.\,NF.$$

Scholium.

Analogous to the general plan pursued by the ancients (*vide Euc.* vi. 18, *etc.*), we might have shown this for the quadrilateral as composed of two triangles; the pentagon as a quadrilateral and a triangle; the hexagon as a pentagon and triangle, and so on. The methods virtually agree, and the next proposition is demonstrated in this form for illustration of the difference of details.

PROP. XV.

If a polyhedral polygon be cut by a transversal plane, the sides will be divided into segments, such that the compound of one alternate set will be equal to the compound of the other set.



First. Let ABCD be a quadrilateral, no three of whose sides are in the same plane; and let the planes ABC, ADC, which respectively contain the sides AB, BC, and AD, DC, be cut by the transversal plane in KF and HM; then (*prop.* i.),

$$\begin{aligned}
AF\,.\,BK\,.\,CG &= FB\,.\,KC\,.\,GA,\\
AG\,.\,CM\,.\,DH &= GC\,.\,MD\,.\,HA;
\end{aligned}$$

whence,

$$AF\,.\,BK\,.\,CM\,.\,DH = FB\,.\,KC\,.\,MD\,.\,HA.$$

The property is therefore proved for the *polyhedral quadrilateral.*

Second. Let ABCDE be a polyhedral pentagon. It may be considered as formed of the quadrilateral ABCD and the triangle ADE. Let the transversal plane cut the planes ABC, ACD, and ADE, in KFG, GHM, and HNL, and these lines cut the sides of the triangles, as in the figure.

Then (1*st case*), $AF\,.\,BK\,.\,CM\,.\,DH = FB\,.\,KC\,.\,MD\,.\,HA$,

and (*prop.* i.) $AH\,.\,DN\,.\,EL = HD\,.\,NE\,.\,LA$; whence,

$$AF\,.\,BK\,.\,CM\,.\,DN\,.\,EL = FB\,.\,KC\,.\,MD\,.\,NE\,.\,LA.$$

In the same way, the hexagon, considered as a pentagon and a triangle, gives the corresponding property of the hexagon; and so on to any extent, as the diagonal which connects the preceding polygon with the new triangle has its segments always eliminated by the new combination.

Scholium.

Carnot proves this theorem by projecting the system on a plane (*Géom. de Pos.* p. 296, and *Essai*, p. 71); but that process is both inelegant and unnecessary.

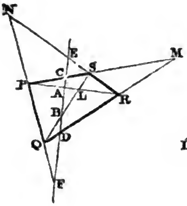
PROP. XVI.

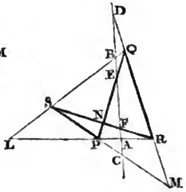
If the four sides PQ, QR, RS, SP, and the two diagonals PR, QS, of a simple quadrilateral PQRS be cut by a transversal in F, D, E, C, A, B, respectively; then the seven following relations will exist cotemporaneously amongst the segments of the transversal:—

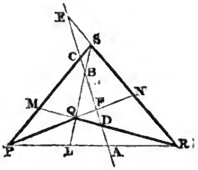
$$FA \cdot AE \cdot DB \cdot BC = EB \cdot BF \cdot DA \cdot AC;$$
$$AC \cdot CB \cdot ED \cdot DF = AD \cdot DB \cdot EC \cdot CF;$$
$$AE \cdot EB \cdot CF \cdot FD = AF \cdot FB \cdot CE \cdot ED;$$

and,

$$FA \cdot DE \cdot BC = AD \cdot EB \cdot CF;$$
$$AE \cdot DB \cdot FC = ED \cdot BF \cdot CA;$$
$$EC \cdot AF \cdot DB = CA \cdot FD \cdot BE;$$
$$CB \cdot FD \cdot AE = BF \cdot DA \cdot EC.$$







First. Let PQ, RS, intersect in L; then the triangle ALB cut by the four sides of the quadrilateral successively, gives

$$AE \cdot BS \cdot LR = EB \cdot SL \cdot RA \text{ (transversal SR)},$$
$$BD \cdot AR \cdot LQ = DA \cdot RL \cdot QB \text{ (.......... RQ)},$$
$$FA \cdot PL \cdot QB = BF \cdot AP \cdot LQ \text{ (.......... QP)}.$$
$$BC \cdot AP \cdot LS = CA \cdot PL \cdot SB \text{ (.......... PS)};$$

whence, compounding and cancelling, we obtain

$$FA \cdot AE \cdot DB \cdot BC = EB \cdot BF \cdot DA \cdot AC,$$

which is the first of the seven equalities stated in the proposition.

Secondly. Let PS, QR, intersect in M; then the triangle CDM cut by the four sides of the quadrilateral, successively, gives

$$DE \cdot CS \cdot MR = EC \cdot SM \cdot RD \text{ (transversal RS)},$$
$$CB \cdot DQ \cdot MS = BD \cdot QM \cdot SC \text{ (......... SQ)},$$
$$DF \cdot CP \cdot MQ = FC \cdot PM \cdot QD \text{ (......... QP)},$$
$$CA \cdot DR \cdot MP = AD \cdot RM \cdot PC \text{ (......... PR)};$$

and compounding, *etc.*, as in the former case, we have

$$AC \cdot CB \cdot ED \cdot DF = AD \cdot DB \cdot EC \cdot CF,$$

and this constitutes the second equality.

Thirdly. Let PQ, RS, meet in N; then the triangle ENF, cut by the four sides of the quadrilateral successively, gives

$$\begin{aligned} CF \,.\, PN \,.\, SE &= EC \,.\, FP \,.\, NS \text{ (transversal PS)},\\ BE \,.\, SN \,.\, QF &= FB \,.\, ES \,.\, NQ \text{ (......... SQ)},\\ DF \,.\, QN \,.\, RE &= ED \,.\, FQ \,.\, NR \text{ (........ QR)},\\ AE \,.\, RN \,.\, PF &= FA \,.\, ER \,.\, NP \text{ (......... RP)}; \end{aligned}$$

and compounding, as before,

$$AE \,.\, EB \,.\, CF \,.\, FD = AF \,.\, FB \,.\, CE \,.\, ED,$$

which is the third of the given properties.

Any of these equalities taken in reference to the three species of simple quadrilaterals, becomes successively each of the other two in reference to the remaining two species of quadrilaterals; so that in fact these three equalities do express only one single property taken in each of the figures. Independently of this consideration, it would be easy, from having any one of the three, to deduce the two others; by forming differently the values of the lines themselves as the points are situated with respect to each other. These transformations, however, we have not space to give here.

The second series of equalities are also essentially contained in those already established. To secure order in our investigations, denote the first and second sides of the first equation by a_1 and a_2, those of the second by b_1 and b_2, and those of the third by c_1 and c_2; then all the combinations of these in *threes*, so as to be essentially different in their composition, will be

$$\begin{array}{c|c} a_1\, b_1\, c_2 = a_2\, b_2\, c_1 & a_1\, b_2\, c_2 = a_2\, b_1\, c_1 \\ a_1\, b_2\, c_1 = a_2\, b_1\, c_2 & a_1\, b_1\, c_1 = a_2\, b_2\, c_2. \end{array}$$

The combinations being made according to this model, will give the results stated in the proposition. The process for one only is given here, the others being exactly similar.

$$\begin{aligned} FA \,.\, AE \,.\, DB \,.\, BC &= EB \,.\, BF \,.\, DA \,.\, AC,\\ AC \,.\, CB \,.\, ED \,.\, DF &= AD \,.\, DB \,.\, EC \,.\, CF,\\ AF \,.\, FB \,.\, CE \,.\, ED &= AE \,.\, EB \,.\, CF \,.\, FD; \end{aligned}$$

in which, effacing all the terms which are common, there is left the equation

$$FA^2 \,.\, CB^2 \,.\, DE^2 = EB^2 \,.\, DA^2 \,.\, CF^2; \text{ or } FA \,.\, CB \,.\, DE = EB \,.\, DA \,.\, CF.$$

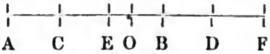
Scholium.

The first three of these equalities are due to Desargues, who called a line so divided an *involution of six points*. The four last were known to the ancients, and form prop. 130, book vii. of the *Mathematical Collections* of Pappus. The points A and B are called *conjugate points;* and, in like manner, C and D, and likewise E and F, are pairs of conjugate points.

A more general statement of this property will be given in *prop. xx. p.* 233; viz. where instead of the diagonals of the quadrilateral, the curve of a conic section circumscribing the quadrilateral PQRS is substituted. I shall add here, however, one remarkable and extremely useful property of this division of a line.

There is a point O in the line of involution such that

$$OA \,.\, OB = OC \,.\, OD = OE \,.\, OF.$$



For, such a point can always be found as shall fulfil one of these equations, which we will suppose to be

$$OA \,.\, OB = OC \;\; OD. \quad \text{Wherefore,}$$

$$\begin{array}{c|c} OA : OC :: OD : OB; \text{ whence} & OA : OD :: OC : OB; \text{ whence} \\ OA - OC : OD - OB :: OA : OC, \text{ or} & OA + OD : OC + OB :: OC : OB, \text{ or} \\ AC : BD :: OA : OC. & AD : BC :: OC : OB. \end{array}$$

Compounding these two equations, we get

$$CA \cdot AD : CB \cdot BD :: OA \cdot OC : OB \cdot OC :: OA : OB.$$

Now if the same point O do not fulfil the condition for the four points A, B, and E, F, let O′ be the point which fulfils it. Then we get in the same manner,

$$FA \cdot AE : FB \cdot BE :: O'A : O'B.$$

But by the proposition itself, written as a proposition,

$$CA \cdot AD : CB \cdot BD :: FA \cdot AE : FB \cdot BE; \text{ whence,}$$

$$OA : OB :: O'A : O'B, \text{ or } OA + OB : OA :: O'A + O'B : O'A.$$

Consequently, $O'A = OA$; and O′ must coincide with O.

Conversely. If $OA \cdot OB = OC \cdot OD = OE \cdot OF$, then these six points, A, B, C, D, E, F, will be in involution.

For,

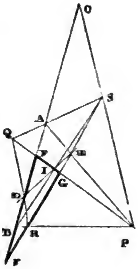
$$CA \cdot AD : CB \cdot BD :: AO : OB :: FA \cdot AE : FB \cdot BE;$$

which is the first or fundamental property of this division of a line, and that from which all the rest may be derived.

For a development of numerous other curious and useful relations, the student may consult Chasles's *Aperçu Historique des Methodes en Géométrie*, notes *x.*, *xv.*, and *xvi.*

PROP. XVII.

Upon QS, BP, the opposite sides of a simple quadrilateral SQBP, as bases, describe triangles BAP, QRS, the vertex of each being in the base of the other; and let AB, QR, intersect in D, and AP, SR, in H: then the line DH will pass through I the intersection of the diagonals PQ, BS.



For, let AB meet PS in C, PQ in F, and SR in E; and let PQ meet RS in G. Then the triangle EFG is cut by the four transversals PC, QR, AP, BS: from which we have

$$GP \cdot FC \cdot ES = PF \cdot CE \cdot SG,$$
$$GQ \cdot FD \cdot ER = QF \cdot DE \cdot RG,$$
$$GH \cdot EA \cdot FP = HE \cdot AF \cdot PG,$$
$$GS \cdot EB \cdot FI = SE \cdot BF \cdot IG.$$

Also, the figure PQRS is a quadrilateral (of the third species) and QS, PR, are the diagonals. These are cut by the transversal CE in A, B, and the opposite pairs of sides are cut in C, D, and E, F. The line CE is therefore divided in involution. Compounding, therefore, either three of the above equations with a corresponding one of the first three of the preceding proposition: for instance, the last of those equations with the first, third, and fourth of these, we shall have

$$AF \cdot FB \cdot CE \cdot ED = AE \cdot EB \cdot CF \cdot FD,$$
$$GP \cdot FC \cdot ES = PF \cdot CE \cdot SG,$$
$$GH \cdot EA \cdot FP = HE \cdot AF \cdot PG,$$
$$GS \cdot EB \cdot FI = SE \cdot BF \cdot IG;$$

and hence, by composition,

$$ED \cdot FI \cdot GH = DF \cdot IG \cdot HE;$$

which (*prop. ii.*) is the criterion of the sides of the triangle EFG having the three points D, I, H, situated in its sides, in one straight line.

Scholium.

This theorem forms *prop.* 139, *lib. vii.* of the *Mathematical Collections* of Pappus. The demonstration in Pappus is extremely elegant; but the method above employed is selected on account of its forming an illustration of the method of transversals. It is a particular case of *prop. xxi.* of this series; and another proof, also by means of transversals, may be seen in my paper on the *trapezium*, in the *Phil. Mag. vol. lxviii.* (1826).

PROP. XVIII.

Let any transversal KH be drawn conjugate to the angular point A of the triangle AMP inscribed in the conic section, to cut the curve in G, H, the sides of the triangle in K, L, and the base MP in N: then

$$KN \cdot NL = GN \cdot NH.$$

For produce MP to meet the tangent at A in D; then AD is parallel to KH; and we have, *by parallels,*

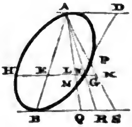
$$DM : MN :: DA : NL, \text{ and } DP : PN :: DA : NK;$$

whence,

$$MD \cdot DP : MN \cdot NP :: DA^2 : KN \cdot NL.$$

But, $MD \cdot DP : MN \cdot NP :: DA^2 : GN \cdot NH$

Whence (*ex eq.*), $KN \cdot NL = GN \cdot NH$.



Corollary 1.

Produce AM, AN, AP, to meet the opposite and parallel tangent at B, the other extremity of the diameter AB, in Q, R, S; then

$$QB \cdot BS = RB^2.$$

For this is the limiting state of the lines in the proposition.

Corollary 2.

Let the transversal cut the diameter AB in E, then the system of segments of the transversal is similar to that of the tangent SB, and we have by similar figures,

$$LE \cdot EK = EN^2, \text{ and } GN \cdot NH = EN^2.$$

PROP. XIX.

If a transversal, KL, be drawn parallel to AB, one of the sides of a quadrilateral, ABDC, inscribed in a conic section, cutting the curve in KL, the sides AC, BD, adjacent to the side AB in F and G, and the side opposite to AB in E: then will the transversal be divided by the curve and the sides, so that

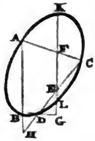
$$FE \cdot EG = KE \cdot EL.$$

For, produce AB, CD, to meet in H; then by similar triangles,

$$AH : FE :: CH : CE, \text{ and } HB : EG :: HD : ED; \text{ whence,}$$

$$AH \cdot HB : FE \cdot EG :: CH \cdot HD : CE \cdot ED,$$
$$:: AH \cdot HB : KE \cdot EL.$$

Whence, the first and third terms being identical, the second and fourth are equal; or $FE \cdot EG = KE \cdot EL$.

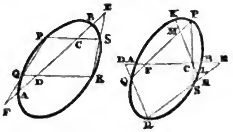


Corollary.

When A and B coalesce, AB will become a tangent, and the theorem is reduced to that of the preceding proposition.

PROP. XX.

Let a transversal cut a conic section in A and B, and the sides PQ, QR, RS, SP, of an inscribed quadrilateral, in F, D, E, C: then the transversal will be divided in involution; or such that the seven equations following are fulfilled amongst its segments.



$$\begin{aligned}
FA.AE.DB.BC &= EB.BF.DA.AC;\\
AC.CB.ED.DF &= AD.DB.EC.CF;\\
AE.EB.CF.FD &= AF.FB.CE.ED;
\end{aligned}$$

and,

$$\begin{aligned}
FA.DE.BC &= AD.EB.CF;\\
AE.DB.FC &= ED.BF.CA;\\
EC.AF.DB &= CA.FD.BE;\\
CB.FD.AE &= BF.DA.EC.
\end{aligned}$$

As the first three equations are deduced exactly in the same manner, it will be sufficient to establish any one of them, as a pattern for the investigation of the other two. Moreover, the last four are deduced from the three preceding ones, exactly as is done in the corresponding case of *prop. xv.*, and need not, therefore, be repeated here.

First. Let PS be parallel to QR; then, by similar triangles,

$$FD : CF :: DQ : CP, \text{ and } ED : CE :: DR : CS; \text{ whence,}$$

$$\begin{aligned}
ED.DF : EC.CF &:: QD.DR : PC.CS,\\
&:: AD.DB : AC.CB.
\end{aligned}$$

Second. Let PS be not parallel to QR; then through C draw KL parallel to QR, meeting the curve in K, L, and the sides QP, RN, in M, N. Then, by similar triangles,

$$FD : FC :: DQ : CM, \text{ and } DE : EC :: DR : CN; \text{ whence,}$$

$$\begin{aligned}
ED.DF : EC.CF &:: RD.DQ : MC.CN = KC.CL \ (\textit{prop. xix.})\\
&:: AD.DB : AC.CB.
\end{aligned}$$

Both of which results are identical with the second equation of the proposition.

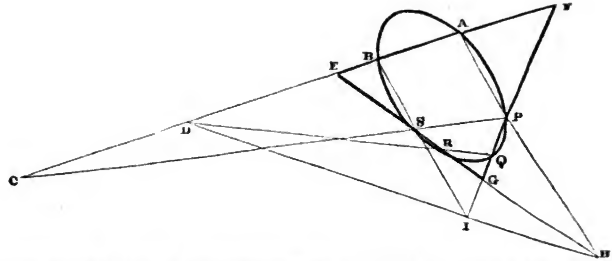
Scholium.

This is the celebrated proposition of Desargues regarding the *involution of six points*, spoken of in the scholium on *prop. xvi.*

PROP. XXI.

If the three pairs of opposite sides, AB and QR, BS and PQ, SR and AP, of a hexagon inscribed in a conic section, ABSRQP, be produced to meet, the three points of intersection D, I, H, will be in one line.

For, produce the three alternate sides, AB, PQ, RS, to meet and form the triangle EFG; and draw PS to meet AB in C.



Then the inscribed quadrilateral PQRS is cut by the transversal AB, which cuts the curve in A, B; and the triangle EFG is cut by the four transversals PS, QR, AP, BS. From the latter we get

$GP \cdot FC \cdot ES = PF \cdot CE \cdot SG$ from the transversal PS;
$GQ \cdot FD \cdot ER = QF \cdot DE \cdot RG$ .... QR;
$GH \cdot EA \cdot FP = HE \cdot AF \cdot PG$ .... AP;
$GS \cdot EB \cdot FI = SE \cdot BF \cdot IG$ .... BS.

Combining any three of these with a corresponding one of the first three equations in *theorem xxi.*, we shall get the criterion of the asserted conclusion.

Take, for instance, the third of the equations in *prop. xxi.*, and the first, third, and fourth of the preceding ones in this; and combining them we have,

$$AF \cdot FB \cdot CE \cdot ED = AE \cdot EB \cdot CF \cdot FD,$$
$$GP \cdot FC \cdot ES = PF \cdot CE \cdot SG,$$
$$GH \cdot EA \cdot FP = HE \cdot AF \cdot PG,$$
$$GS \cdot EB \cdot FI = SE \cdot BF \cdot IG;$$

and hence,

$$ED \cdot FI \cdot GH = DF \cdot IG \cdot HE;$$

which (*prop. ii.*) is the criterion of the sides of the triangle EFG having its three points of section, D, I, H, in one line.

PROP. XXII.

If a transversal, as FE, be drawn parallel to a given line CV, and cut the conic section in P, the chord of contact AB in F, and the tangents AC, BC, in E and D; then, wherever the transversal be drawn, $DP \cdot PE : PF^2$ will be a given ratio [*].

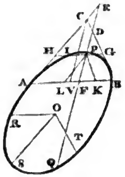
For, draw HG through P parallel to AB, meeting the tangents in H, G, and

---

[*] This theorem is one of considerable importance in the theory of transversals, and I am not aware that it has been *geometrically* established till now. Carnot has given an algebraic investigation of the problem, "to find the locus of P, so that $FP^2 : DP \cdot PE$ shall be a given ratio" (*Geom. de Pos. p.* 443); but it is neither elegant, nor consistent with his general plan; though it appears to have occupied much of his attention to accomplish it otherwise. His remark is:—"*Ce problême n'est qu'un cas particulier de celui que l'on connoit sous le nom de* problême de Pappus, *résolu par Descartes (voyez sa Géométrie, l. ii. p. ii.). Ce problême, fameux autrefois par sa difficulté, n'en a plus aujourd'hui: je ne le donne que comme exemple du changement des co-ordonnées, et parceque j'ai à en tirer diverses conséquences curieuses que je crois n'être pas connues.*"

The geometrical difficulty attached to this problem in its present shape, it is clear, from the foregoing demonstration, is greatly overrated by Carnot. The co-ordinate method employed by him is very inelegant; and as the general character of his work required a demonstration strictly geometrical, we cannot help regretting that so able a geometer had not considered it worthy of further investigation by the principles of pure geometry.

The "curious propositions" of Carnot, with many others, are given in this work.



the curve again in I; also draw PL, PK, parallel to AC, BC; and the semi-diameters OR, OS, OT, parallel to AB, AC, BC. Then, since GH is parallel to the chord of contact, HI = PG, and HP = IG.

Again, by similar triangles,

$$HP : PE :: AV : VC, \text{ and } PG : PD :: BV : VC;$$

whence,

$$HP . PG : DP . PE : AV . VB : VC^2.$$

$$\text{Also, } PH . HI : HA^2 :: OR^2 : OS^2,$$

$$IG . GP : GB^2 :: OR^2 : OT^2;$$

therefore, since $PH . HI = IG . GP = HP . PG$, $HA^2 = PL^2$, and $GB^2 = PK^2$, we have, by compounding,

$$HP . PG : KP . PL :: OR^2 : OS . OT.$$

$$\text{Also, } KP . PL : PF^2 :: AC . CB : CV^2 \text{ by sim. trians.};$$

$$\text{and } DP . PE : HP . PG :: VC^2 : AV . VB, \text{ as before found};$$

wherefore, compounding,

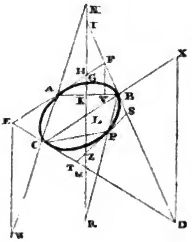
$$DP . PE : PF^2 :: OR^2 . AC . CB : OS . OT . AV . VB,$$

$$:: \frac{OR^2}{OS . OT} : \frac{AV . VB}{AC . CB};$$

which assigns the constant and given ratio of DP . PE : PF².

PROP. XXIII.

Let any triangle ABC be inscribed in a conic section, and to the points A, B, C, tangents to the curve be drawn, forming the triangle DEF; and let any transversal IN, having a given inclination, be drawn to cut the sides of the inscribed triangle AB in K, BC in L, and CA in N; and the sides of the circumscribed triangle, EF in H, FB in I, and DE in M: then, if G be one of the points of section of the transversal with the curve, GK . GL . GN : GH . GI . GM, will be a given ratio.



For, denote the ratio found in the last proposition by k_1^2, and the corresponding ones for the chords BC, and CA by k_2^2, and k_3^2; also, express for brevity the three distances estimated from G upon the transversal to the sides of the inscribed triangle (the chords), by Y_1, Y_2, Y_3; and the distances from the sides of the circumscribed triangle (the tangents), by Z_1, Z_2, Z_3.

Then, by the last proposition, we have

$$Y_1^2 = k_1^2 Z_2 Z_3; \quad Y_2^2 = k_2^2 Z_3 Z_1;$$

$$\text{and } Y_3^2 = k_3^2 Z_1 Z_2.$$

Compounding these, we have

$$Y_1^2 Y_2^2 Y_3^2 = k_1^2 k_2^2 k_3^2 Z_1^2 Z_2^2 Z_3^2;$$

$$\text{or } Y_1 Y_2 Y_3 = k_1 k_2 k_3 Z_1 Z_2 Z_2,$$

which establishes the proposition.

Corollary 1.

The same proposition holds with respect to the quadrilateral, the pentagon, and generally for all polygons.

For let ABPC be a quadrilateral inscribed, and FSTE the corresponding circumscribed one. Then, denoting by k_1^2, k_2^2, k_3^2, k_4^2, the ratios in reference to each of the systems of one chord and its two tangents, we shall get, in the same way,

$$Y_1\,Y_2\,Y_3\,Y_4 = k_1\,k_2\,k_3\,k_4\,Z_1\,Z_2\,Z_3\,Z_4.$$

And similarly the theorem is extensible to any polygon whatever.

Corollary 2.

If we seek the actual expression of the ratio $k_1\,k_2\,k_3$, in the case of the triangle, draw FV, FW, DX, parallel to the transversal; then the ratio will obviously be that of unity to $\frac{AV.VB}{AC.CB}\cdot\frac{AW.WC}{AC.EC}\cdot\frac{BX.XC}{BD.DC}$, or that of

$$AV.VB.AW.WC.BX.XC : AC.CB.AE.EC.BD.DC.$$

For, in the preceding figure, the antecedent of the ratio is

$$\frac{OR^2}{OS.OT}\cdot\frac{OS^2}{OT.OR}\cdot\frac{OT^2}{OR.OS}, \text{ or unity};$$

and the same is extensible to all other polygons.

Corollary 3.

From this system of relations we may also deduce some others:—

$$Y_1^2 = k_1^2\,Z_1\,Z_3,\quad Y_2^2 = k_2^2\,Z_2\,Z_4,$$
$$Y_3^2 = k_3^2\,Z_3\,Z_1,\quad Y_4^2 = k_4^2\,Z_4\,Z_2.$$

These give

$$Y_1\,Y_2 = k_1\,k_2\,\sqrt{Z_1\,Z_2\,Z_3\,Z_4};$$
$$Y_3\,Y_4 = k_3\,k_4\,\sqrt{Z_1\,Z_2\,Z_3\,Z_4}.$$

Wherefore,

$$Y_1\,Y_2 : Y_3\,Y_4 :: k_1\,k_2 : k_3\,k_4.$$

That is to say: if a transversal with a given inclination be cut by a conic section and its inscribed quadrilateral, then the rectangle under the segments of the transversal, estimated between either of its intersections with the curve and one pair of opposite sides, has a given ratio to the rectangle under the segments between the same point in the curve and the other pair of opposite sides.

This is a particular case of the proposition at *p.* 183: viz. when the two transversals of that proposition coalesce.

PROP. XXIV.

If a hexagon, ABCDEF, inscribed in a conic section, be cut by a transversal KL, which also cuts the curve in P: then the solid under the distances of P from H, K, M, the intersections of the transversal with the second, fourth, and sixth sides, will have a given ratio to the solid under the distances of P from G, I, L, the intersections of the first, third, and fifth sides of the hexagon.

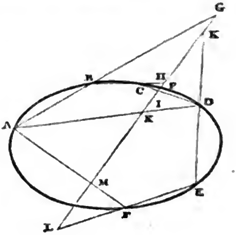
For join AD, cutting the transversal in N: then, if m_1, m_2, be the ratios denoted by $k_1\,k_2$ and $k_3\,k_4$ of the preceding proposition, as applied to the two quadrilaterals ABCD, ADEF, cut by the transversal, we shall have

$$PH.PN = m_1\,PI.PG,$$
$$PM.PK = m_2\,PN.PL;$$

whence, compounding, the segment PN disappears, and we have

$$PH.PK.PM = m_1\,m_2\,PG.PI.PL,$$

which is the property affirmed.



Corollary 1.

An inscribed octagon may be divided into a hexagon and quadrilateral, a decagon into an octagon and a quadrilateral, and so on. Whence the property extends to all polygons of an *even number of sides.*

Corollary 2.

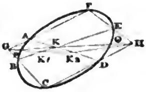
If the number of sides of the inscribed polygon be *odd*, a tangent at any one of the angular points may be substituted for the deficient one; and taking this either to belong to the odd or even series, the preceding theorem will apply.

Corollary 3.

If the angles of any inscribed polygon be drawn to form a circumscribed one, the sides of the inscribed may be considered as the even sides, and those of the circumscribed one as the odd sides of the inscribed one; then the theorem becomes identical with *prop. xxiii.*

PROP. XXV.

Let the opposite pairs of adjacent sides of a hexagon, ABCDEF, inscribed in a conic section, be produced to meet in G and H; and draw the diagonals AD, BE, to intersect in K; then G, H, K, will be in one line.



For if not, let GH cut AD in K_2, and BE in K_1. Then, considering the six lines, AD, DC, CB, BE, EF, FA, in succession to constitute a new hexagon, cut by the transversal GH, which cuts the curve in P, Q, and the *opposite lines* AD, BE, in K_2, K_1, we have (*prop. xxiv.*),

$$PK_1 . PG . PH = m . PH . PK_2 . PG, \text{ and}$$
$$QK_1 . QG . QH = m . QH . QK_2 . QG.$$

Whence, $QK_1 . PK_2 = QK_2 . PK_1$, or adding $QK_1 . QK_2$,

$$QK_1 (QK_2 + K_2P) = QK_2 (QK_1 + K_1P).$$

Now $QK_2 + K_2P = QP = QK_1 + K_1P$; whence,

$$QK_1 = QK_2,$$

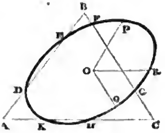
which is impossible. The line GH, therefore, passes through K.

Scholium.

This is one of the many elegant cases of Pascal's hexagram; but this demonstration cannot be made to apply (as Carnot, by some oversight, inferred) to the cases where the line GH does not cut or touch the curve, the essence of the proof requiring P and Q to be in the curve.

PROP. XXVI.

A triangle ABC is cut by a conic section in D, E, F, G, H, K, as in the figure; then AD . AE . BF . BG . CH . CK = AK . AH . BE . BD . CF . CG.



For draw the semi-diameters OP, OQ, OR, parallel to the sides of the triangle: then,

$$AD . AE : AK . AH :: OP^2 : OR^2,$$
$$BF . BG : BE . BD :: OQ^2 : OP^2,$$
$$CH . CK : CG . CF :: OR^2 : OQ^2;$$

whence,

$$AD.AE.BF.BG.CH.CK = AK.AH.BE.BD.CF.CG.$$

Corollary 1.

The same process applies to any polygon whatever, and the same conclusion is deducible.

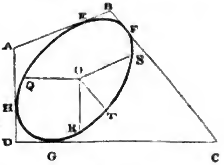
Corollary 2.

When the conic touches the several sides of the figure, the general proposition becomes identical with the following one.

PROP. XXVII.

If a polygon, as ABCD, circumscribe a conic section, touching it in EFGH; then the continued product of the one set of alternate segments is equal to that of the other set: viz.

$$AE.BF.CG.DH = EB.FC.GD.HA.$$



For, draw the semi-diameters OQ, OR, OS, OT, parallel to the sides: then,

EB : BF :: OS : OT
FC : CG :: OT : OQ
GD : DH :: OQ : OR
HA : AE :: OR : OS;

whence, compounding, we obtain the stated result.

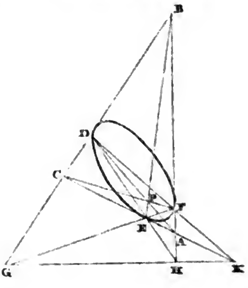
The same proof evidently applies, however many the sides of the polygon may be.

Scholium.

The *contact of the general conic section*, with any polygon whatever, therefore, gives a result analogous to the *intersection of a right line in prop. xix.*

PROP. XXVIII.

Let ABC be a triangle, touching a conic section in D, E, F, and DEF the triangle formed by the chords of contact; let also the opposite sides of the two triangles be produced to meet in G, H, K; and, finally, let AD, BE, CF, be drawn: then AD, BE, CF, will meet in one point, and G, H, K, will be in one line.



For, AF . BD . CE = FB . DC . EA (*prop. xxvii.*); whence, AD, BE, CF, pass through one point, P.

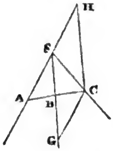
Again (*prop. xxvii.*), the corresponding angles of the two triangles being upon lines AD, BE, CF, which pass through one point, P, their corresponding intersections G, H, K, are in one line.

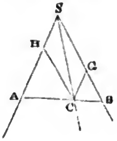
Scholium.

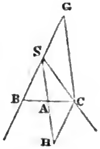
In the language of our definitions 6 and 7. the two triangles so constituted are *copolar* and *co-axial*. The theorem itself is very efficient in the investigation of properties of the conic sections.

PROP. XXIX.

If three straight lines be drawn from any point S, viz. SA, SB, SC, and be cut by any transversal in A, B, C, respectively: then the ratio CA . BS :: BC . SA will be constant for all positions of the transversal.







For, through C draw CG, CH, parallel to SA, SB: then,

$$AC : CH :: AB : BS, \text{ or } AC . BS = AB . CH, \text{ and}$$
$$AB : BC :: AS : CG, \text{ or } BC . AS = AB . CG; \text{ whence,}$$
$$AC . BS : BC . AS :: AB . CH : AB . CG :: CH : CG :: SG : GC;$$

and this is a constant ratio, since the triangle SCG is given in species.

Corollary 1.

In the continental method of writing this class of results, we may express it by

$$\frac{CA}{SA} : \frac{CB}{SB} = \frac{CG}{SG} : \frac{CH}{SH} = \text{constant}.$$

Corollary 2.

The following form is also an exceedingly convenient one:

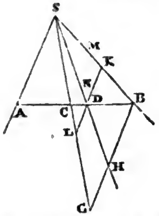
$$CA . BS : BC . SA :: \sin ASC : \sin CSB.$$

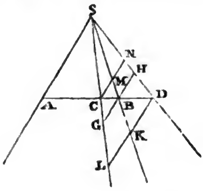
For, $\frac{CA}{SA} = \frac{\sin CSA}{\sin SCA}$, $\frac{CB}{SB} = \frac{\sin CSB}{\sin SCB}$, and $\sin SCA = \sin SCB$; whence, $\frac{CA}{SA} : \frac{CB}{SB} :: \frac{\sin ASC}{\sin SCA} : \frac{\sin CSB}{\sin SCB} :: \sin ASC : \sin CSB$, and the conclusion follows.

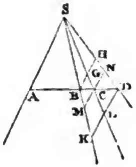
PROP. XXX.

If four lines, SA, SB, SC, SD, which meet in one point S be cut by any transversal in A, B, C, D, respectively, then the three following anharmonic ratios will be constant:—

AC . BD : BC . AD, where A, B, are conjugates,
BC . AD : AB . CD...... A, C, ...
AB . CD : AC . BD...... A, D, ...







For, draw through B, C, D, lines parallel to SA: and let them meet SC, SD, in G, H; SD, SB, in N and M; and SC, SB, in L and K. Then,

AC : BC :: AS : BG, and BD : DA :: HB : AS; whence,
AC . BD : BC . AD :: HB : BG, a constant ratio.

Again,

BC : BA :: CM : AS, and DA : DC :: AS : CN; whence,
BC . AD : AB . CD :: CM : CN, a constant ratio.

Lastly,

CD : CA :: DL : AS, and BA : BD :: AS : DK; whence,
AB . CD : AC . BD :: DL : DK, a constant ratio.

Another demonstration.

By *prop. xxix. cor.* 2, we have,
CA . BS : CB . AS :: sin ASC : sin BSC, and
DB . AS : DA . BS :: sin BSD : sin ASD; whence,
CA . BD : CB . AD :: sin ASC sin BSD : sin ASD sin BSC.
Similarly, AC . BD : AB . CD :: sin ASC sin BSD : sin ASB sin CSD,
and AB . CD : AD . BC :: sin ASB sin CSD : sin ASD sin BSC.

Corollary 1.

Since the lines CM, LK, GB, are similarly divided, the constant ratios may be expressed in terms of the segments of any one of them: for instance, that through D in each of the figures.

AC . BD : BC . AD :: DK : KL,
BC . AD : AB . CD :: KL : LD,
AB . CD : AC . BD :: LD : DK.

Corollary 2.

Let another transversal cut the four lines in A′, B′, C′, D′: then,

$$AB\,.\,CD + AC\,.\,BD = BC\,.\,AD.$$

$$\begin{array}{l|l}
\dfrac{AB\,.\,CD}{BC\,.\,AD} + \dfrac{A'C'\,.\,B'D'}{BC'\,.\,A'D'} = 1 & \dfrac{A'B'\,.\,C'D'}{B'C'\,.\,A'D'} + \dfrac{AC\,.\,BD}{BC\,.\,AD} = 1 \\
\dfrac{B'C'\,.\,A'D'}{A'B'\,.\,C'D'} - \dfrac{BD\,.\,AC}{AB\,.\,CD} = 1 & \dfrac{BC\,.\,AD}{AB\,.\,CD} - \dfrac{B'D'\,.\,A'C'}{A'B'\,.\,C'D'} = 1 \\
\dfrac{B'C'\,.\,A'D'}{A'C'\,.\,B'D'} - \dfrac{AB\,.\,CD}{AC\,.\,BD} = 1 & \dfrac{BC\,.\,AD}{AC\,.\,BD} - \dfrac{A'B'\,.\,C'D'}{A'C'\,.\,B'D'} = 1.
\end{array}$$

Scholium.

The three cases of this theorem, like so many others in transversals, are but one single theorem adapted to the three figures. The property itself forms *vii. prop.* 129, of the *Mathematical Collections* of Pappus. The general proposition and its corollaries are made, with promising effect, the foundation of a general theory of the conic sections and their corresponding surfaces, by M. Chasles, in the notes to his *Aperçu Historique des Méthodes en Géometrie.* Whether such form as he has given to the inquiry be really the best that it admits of, is a question it would at present be premature to discuss, and much more premature to affirm or deny; but, there is considerable reason to believe that it was upon this principle that the discussions of the properties of the conic sections were conducted in the later period of the Alexandrian school of geometry. See also the note at *p.* 242.

When the ratios in the proposition are those of *equality*, the division is the *harmonical*, of which considerable use (as is evident from Pappus) was made by the later Greek Geometers.

PROP. XXXI.

(See figure to preceding proposition.)

If from any four fixed points, A, B, C, D, situated in one line, straight lines be drawn to any point S; then the following anharmonic ratios will be constant:

$$\sin ASC \sin BSD : \sin ASD \sin BSC,$$
$$\sin ASC \sin BSD : \sin ASB \sin CSD,$$
$$\sin ASB \sin CSD : \sin ASD \sin BSC.$$

For, by the second proof of *prop. xxxi.* we have,

$$\sin ASC \sin BSD : \sin ASD \sin BSC :: AC . BD : AD . BC;$$

and since A, B, C, D, are given, the latter ratio is given, and hence the former one is constant. In the same manner the other two cases follow.

Corollary 1.

When AC . BD = AD . BC, the division becomes the *harmonical*, and we thus obtain a property of the angles formed by the system of harmonical sectors that has not been much noticed; viz. that the ratios expressing the proposition are those of equality.

Corollary 2.

If there be given four points in a line, or four lines meeting in a point, all the anharmonic ratios that result from drawing lines from the given points to meet in a new point, cutting these by a new transversal, drawing lines from the new points of section to another point, and so on: these will all be constant, whether the systems be in one or in different planes.

Corollary 3.

If two lines are divided in the same anharmonic ratio in A, B, C, D, and A′, B′, C′, D′, and if they be so situated that AA′, BB′, CC′, tend to one point S, then also DD′ will tend to that point.

Corollary 4.

When two lines, as in the preceding corollary, are divided similarly, they may be in innumerable ways placed upon the same system of anharmonic sectors.

Corollary 5.

If four planes meet in one line, the anharmonic ratios of the transversals which cut them will be constant.

Corollary 6.

If through any four points in a straight line planes be drawn to meet in one line, the anharmonic ratios of their dihedral angles will be constant.

Scholium.

It may be well to apprise the student, that though M. Chasles has established this corollary upon *theorem xxxi.*, this process is not at all indispensable, and that the theorem of Pappus is in itself adequate to establish the equality of the conjugate ratios of the segments of AB, A′B′, A″B″, *etc.*

In fact, the theorem of Pappus gives directly the equality of the anharmonic ratios of the segments of AB and AB, by the system of lines drawn to S, those of A′B′ and A″B″ by those drawn to S″, and so on, without the intervention of any consideration of the angles formed at S, S′, S″, *etc.* It does not appear, therefore, at all unlikely, that many of the *properties* and *uses* of the anharmonic system were known to the ancients; though I am not able to adduce any direct proof from the mathematical collections of Pappus (the only sketch of the ancient systems of geometrical inquiry that have survived the general wreck of science and literature[*]) for what purpose it was used, and to what extent its applications had been carried.

PROP. XXXII.

If four lines, A, B, C, D, cut three other lines, L, L′, L″, of which no two are in the same plane, in abcd, a′b′c′d′, and a″b″c″d″,: then ad, a′d′, a″d″, are divided in the same anharmonic ratio.

For, through L″ draw planes to rest upon A, B, C, D: then these

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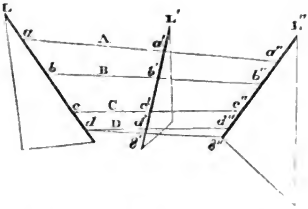
[*] The mathematical collections of Pappus have always appeared to me to partake much more of the character of a rough note-book, than of a finished digest of the mathematical sciences as they existed in his time. The slightest glance at the contents of his several books will convince us, that even the most complete of his collections is in an unfinished state; and that they form only detached parts of a larger and complete work.

The distribution and arrangement into books is certainly the work of an editor, though probably done immediately after the death of Pappus. There is no natural order in the distribution of these books as to subject; and frequently we may detect an unnatural order in the propositions composing those books which are the most complete. The absence, too, of the very important subject of the Conic Sections, (and how important a subject of geometrical speculation this was amongst the ancients, is evinced by the extended treatise of Apollonius,) proves not only incompleteness of the final arrangement, but the essential incompleteness of the whole work.

The seventh book, especially, whilst it has proved the great extent of the researches of the Greeks, and informs us, with a greater or less degree of precision, of the methods which they employed, and the general problems which had been treated by the successors of Plato, is the most incomplete of all. It consists, in fact, of *a number of mere fragments*, put down without regard to order, and without any attempt at systematic arrangement. It seems to have been composed of all the fragments which did not naturally fall under the heads of the former books. After all, it is a prize even in its present state. At the same time we must be careful in our inferences from it, as to what the Greeks had *not* attained in mathematical science; and, on the other hand, may we not attribute to them all the natural and obvious consequences which could be derived from the propositions contained in this book (the seventh), and that in the most simple and obvious manner, consistent with *their* manner of investigating?

When we consider that Pappus lived at a period to be able to denominate Apollonius "an ancient," we can scarcely suppose that the subject of the sections of the cone was stationary for six hundred years. The treatise of Apollonius is come down to us, perhaps nearly as he left it; but none of the properties that are so explicitly stated by Pappus are employed in it. Are we not justified, then, in the inference, that the properties so much dwelt upon by Pappus, and which are shown by Chasles to lead to the entire properties of the conic sections with such extreme facility, were mainly destined to subserve his development of the properties of these curves in a form more simple than had been effected by Apollonius? The almost entire absence of extract from the *Conics* of Apollonius (except a few plane theorems which may have been used for other purposes) in the collections of Pappus, favours this view. The fragmentary character, indeed, of the seventh book of the Collections, justifies the belief of its being composed of several systems of inquiries *preparatory* to so many distinct treatises on the several subjects to which they relate.

Much may be added to enforce the idea under which this note was written: but as it is already unusually long for such a purpose, I shall only add, that I consider Chasles to be the true restorer of the conics of the modern-ancients; that is, of the conics between the periods in which Apollonius and Pappus flourished.



will cut L in *a*, *b*, *c*, *d*, and L′ in *a*′, *b*′, *c*′, *d*′. Consequently, the lines L and L′ cutting the four planes AL″, BL″, CL″, DL″, will be divided in the same anharmonic ratio (*prop. xxxi. cor.* 5).

In the same manner, planes through L and A, B, C, D, will be cut by L′ and L″ in the same anharmonic ratio, or *a*′*b*′*c*′*d*′, and *a*″*b*″*c*″*d*″, will be divided in the same anharmonic ratio.

Wherefore, the three lines L, L′, L″, are divided in the same anharmonic ratio by the four lines A, B, C, D, which rest upon them.

Corollary.

If two lines L′, L, be so divided in *a*′*b*′*c*′*d*′ and *abcd* as to have the same anharmonic ratio, and if the four lines A, B, C, D, or *a*′*a*, *b*′*b*, *c*′*c*. *d*′*d*, be drawn; then every straight line L″ which touches three of these lines A, B, C, will also meet the fourth, D.

EXERCISES UPON TRANSVERSALS.

1. If through any point, P, within the field of the triangle, ABC, lines be drawn from the angles A, B, C, to meet the opposite sides respectively in D, E, F: then,

$$\frac{AP}{AD}+\frac{BP}{BE}+\frac{CP}{CF}=1, \text{ and } \frac{DP}{DA}+\frac{EP}{EB}+\frac{FP}{FC}=2.$$

Also show what the relation becomes when P is without the field of the triangle, whether below the base or beyond the vertex.

2. If through any point P, in the plane of the triangle ABC, lines be drawn parallel to the sides of the triangle, meeting BC in D, D′, meeting CA in E and E′, and meeting AB in F, F′: then will

$$PD.PE.PF = PD'.PE'.PF' = DD'.EE'.FF' = AF.CD.BE = AE'.BD'.CF'.$$

3. If a transversal EFD cut the sides of a triangle ABC; viz. BC in D, CA in E, and AB in F; then will

$$AB\,.\,BC : FB\,.\,BD :: -AC^2 + CB^2 + BA^2 : -FD^2 + DB^2 + BF^2.$$

4. If in two lines AB, A′B′, any how situated in space, two points, O, O′, be taken; and any variable equal distances AO, OB, A′O′, O′B′, be taken, and AA′, BB′, be bisected in C, D, and AB′, A′B, be bisected in C′, D′: then C, D, will always be in one straight line, C′D will always be in another, and these two straight lines will intersect in a point, and be at right angles to each other.

5. Let there be given any polygon with an even number of sides, and likewise two other lines (all the lines concerned being capable of indefinite extension where required): then, if a variable transversal be drawn parallel to a given line, cut the sides of the polygon, and likewise the two given lines in two points, the continued product of the distances of all the points of section of the odd-numbered sides of the polygon from that of the first line, has to the continued product of the distances of all the even-numbered points from that of the second line, a given ratio.

6. From three given points in a straight line, draw lines to any arbitrary point; and from two of the given points draw lines to meet at an arbitrary point in the

third line, and to cut each line different from that in which it is situated: the line drawn through the two points of intersection will always pass through a fixed point in the line which contains the three given ones.

7. From each of three points, situated in one straight line, let two straight lines be drawn, so as by their mutual intersections to form three quadrilaterals: then the diagonals of these three quadrilaterals, being taken three and three (viz. one diagonal of each quadrilateral), will pass through the same point.

8. Let two pairs of points, A, C, and A′, C′, be given in a straight line, and NP any line cutting AB: let any two triangles, ABC, A′B′C′, be described having B, B′, in NP; from any other point P in NP draw PA, PC, PA′, PC′, cutting A′B′, B′C′, AB, BC, Q′, R′, Q′, R, respectively; then QR, and Q′R′, will meet in AB, or be both parallel to it.

9. Let ABC, A′B′C′, and PN be as in the preceding theorem, and draw PEE′, PFF′, cutting AB, A′B′, in E, E′, and BC, B′C′, in F, F′: then EF and E′F′ will meet each other in AB, or be both parallel to it.

10. Let there be a complete quadrilateral, the simple salient part being ABCD, and let AB, CD, meet in F, and AD, BC, in G: then, if circles be described about ABG, DCG, ADF, BCF, they will meet in one point P, and the centres of the four circles are in another circle passing also through P.

Also, if PA, PB, PC, PD, PF, PG, be drawn, we shall have,

$$PA \cdot BC \cdot CD = PB \cdot CD \cdot DA = PC \cdot DA \cdot AB = PD \cdot AB \cdot BC;$$
$$PD \cdot FB \cdot BG = PG \cdot BF \cdot FD = PB \cdot FD \cdot DG = PF \cdot DG \cdot GB;$$
$$PC \cdot GA \cdot AF = PG \cdot AF \cdot FC = PA \cdot FC \cdot CG = PF \cdot AG \cdot GC.$$

11. From P (*figure to last question*) draw perpendiculars upon the four sides of the quadrilateral, meeting them in *d*, *e*, *f*, *g*: these four points will be in one line.

12. Let the perpendiculars from the angles of the triangle upon the opposite sides of ADF meet in *h*, those similarly related to BCF, ABG, CDG, meet in *k*, *l*, *m*, respectively; then *h*, *k*, *l*, *m*, will be in one line parallel to *defg*.

13. If circles be described upon the three diagonals of a complete quadrilateral, they will all intersect in the same points when A is an acute angle, and tangents to them from a fixed line will be equal when A is obtuse.

14. Let the five points, A, B, C, D, E, in any order, be taken in one line; from E let any line BG be drawn, upon the bases AD, BC, let triangles AMD, BGC, be constituted, having their vertices M and G in EG; and AM, BG, cut in L, and DM, CG, cut in N; then the line LN will always tend to the same point, K.

15. Let each of three circles be cut by *any other* circle, then the three chords of intersection will intersect two and two upon three straight lines which pass through one point.

16. If *any two* circles cut the three circles of the preceding proposition, the chords which are in the first meet in *a*, those which are in the second meet in *b*, and those which are in the third meet in *c*; then the points *a*, *b*, *c*, are in one line.

17. If tangents be drawn to three unequal circles, two and two, the three points of section will be in one line. Also, the same when there are any three conic sections, similar and similarly situated.

Examine the cases analogous to *props*. 22, 23. *page* 82.

18. Let OV, *da*, two parallel lines, be cut by a transversal in V and *v*; and from a point O in one of them (OV) draw any four lines meeting the transversal in D, C, B, A, and the other parallel in *d*, *c*, *b*, *a*; then

$$CD \cdot ab : cd \cdot AB :: CV \cdot VD : AV \cdot VB.$$

19. If in the same quadrilateral conics be inscribed, their centres will all be upon the line which bisects the diagonals of the quadrilateral.

20. Two conic sections intersect in four points, and we construct:—

(1). The quadrilateral whose angles are the points of intersection.

(2). The two quadrilaterals formed by tangents drawn to the two curves at the two opposite points of intersection of the preceding quadrilateral.

(3). The two circumscribing quadrilaterals, whose contacts are the four points before mentioned.

(4). The quadrilateral formed by the four tangents which are common to the two curves.

(5). The two quadrilaterals inscribed in the two curves, and whose summits are the points of contact of two preceding ones.

Then the following relations will subsist:—

(1). The diagonals of these eight quadrilaterals will pass through the same point.

(2). The thirty-two sides of the eight quadrilaterals form eight sets of concurring lines, and the eight points of concourse are upon one straight line.

(3). The summits of the two circumscribing quadrilaterals which touch the curves at their common points, and those of the quadrilateral formed by the tangents common to the two curves, are distributed six and six upon two straight lines which are their common diagonals.

21. If any variable tangent to a conic section be drawn to cut the four sides of a given circumscribed quadrilateral, it will divide each opposite pair of sides in the same constant ratio.

22. When two parabolas are escribed to the same triangle, and in respect of the same sides, it is possible to describe an hyperbola through the six points of contact, and its asymptotes will be parallel to the axes of the parabolas.

23. Let ALSTK be any pentagon, having the alternate sides produced to meet; viz. AL and ST in D, LS and TK in P, ST and AK in E, TK and LA in C, and KA and LS in B; also draw DK and BL to meet in O, DK meeting LS in M, and EL meeting TK in N, and draw NB, MC, intersecting in R; then the points O, R, A, will be in one line.

24. Let BE, CD, be two straight lines intersecting in A, and B, E, two points in BE, and C, D, two points in CD, and let P be a variable point anywhere between these lines; from P draw PB, PC, cutting DC in L and BE in K, and draw DK, cutting BP, LE, in M and O, and let LE cut PC in N; and finally, draw NB, MC, intersecting in R; then OR always tends to a given point, however all the four named points be taken.

25. Through either angle C of a quadrilateral ABCD draw a line FCG limited by the sides AB and AD in F and G; join GB, FD, cutting CD and CB in P and O, and draw AO, AP, cutting GB, FD, in Q and R; then FQ, GR, will pass through E, the intersection of the diagonals AC, BD.

26. On the same base, FG, let any two triangles, PCG, PAG, be constructed; and from any point P in the base PG let lines be drawn, viz. PHK cutting GC, CF in H and K, and PIL cutting GA, AF, in I and L; then the three lines HI, KL, and CA being drawn, they will meet in one point, B.

27. Produce the alternate sides of a hexagon inscribed in a conic section, and draw lines to join the three opposite pairs of sections: these three lines will intersect in one point.

28. Let two triangles be inscribed in a conic section, and by their mutual intersections form a hexagon: the diagonals of this will meet in one point; and

a conic section described to touch five sides of the hexagon will also touch the sixth.

29. If two angles of constant magnitude turn about two fixed points, and if the intersections of two corresponding sides be always in the curve of a conic section passing through those points, then the intersection of the other pair of sides will also be in a conic section, likewise passing through those points.

30. If two tetrahedrons be co-polar (that is, have their summits upon four straight lines passing through the same point), the four lines in which the corresponding faces intersect will be situated in one plane.

31. When three conic sections have the same four points of intersection, any transversal is divided by them in involution.

32. When two conic sections are circumscribed to the same quadrilateral, every transversal which cuts the curves and either two opposite sides (or the diagonals) of the quadrilateral is divided in involution.

33. If a variable tangent to a conic section cut two fixed tangents, the rectangle under the distances of the point where it meets them from the points where these two tangents meet the diameter parallel to the chord of contact, is invariable.

34. If from the six points of involution of any line lines be drawn to any other point, the equation of involution will subsist amongst the sines of the angles subtended at that other point by the six involution-segments of the line.

35. If any other line be drawn to cut the six mentioned in the preceding theorem, it will be divided by them in involution.

36. Let E, F, be the respective intersections of the opposite pairs of sides AB, CD, and AD, BC; and draw any two lines EMP, FNL, to cut AD, BC, AB, CD, in P, M, L, N, respectively: then we shall have,

$$AL . BM . CN . DP = LB . MC . ND . PA.$$

In the preceding construction, form the quadrilateral PLMN: then the opposite pairs of sides will meet in AC and BD.

37. From the intersection of any two diagonals of the complete quadrilateral, draw a perpendicular to the third diagonal; and from the foot of this perpendicular draw lines to the extremities of the first two diagonals: then the angle contained by the lines drawn to the extremities of each diagonal (or its supplemental angle) will be bisected by the perpendicular.

38. If all the angles of a polygon, moveable in its plane, are subject to be situated constantly each upon one of a system of straight lines tending to one point; and that, moreover, all its sides, except one, pass constantly through fixed and given points: then the free side of the polygon, and likewise all the diagonals, will constantly pass each through its own fixed and determinable point.

39. When a conic section passes by the intersections E, F, of the opposite sides of the simple quadrilateral ABCD, and cuts the four sides in L, M, N, P; then there will subsist the relation

$$AL . BM . CN . DP = LB . MC . ND . PA.$$

40. The locus of the poles of a given straight line, in reference to all the conic sections which touch the four sides of a given quadrilateral, is another straight line.

41. The polars of a given point, in reference to a similar system of conic sections, is always tangent to another conic section.

THE GEOMETRY OF CO-ORDINATES OF TWO DIMENSIONS.

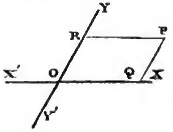
1. When only one equation is given between two undetermined quantities, $y = f(x)$, each of them, as x, admits of innumerable values; and for each value of x there are n values of y (n being the degree of the equation in terms of y), either real or imaginary.

2. If we suppose x and y to represent any two geometrical magnitudes determinable according to some specific law, and connected by the equation $y = f(x)$; then all the points determined by the extremity of y constitute a curve or straight line, which is called the *locus of the equation* $y = f(x)$.

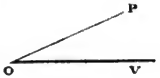
3. Conversely, $y = f(x)$, or $f(x, y = 0)$, is called the *equation of the curve*, the relation between the magnitudes x, y, of which is expressed by it.

4. The magnitudes x, y, however taken, are called the *co-ordinates of the curve*.

5. The most frequent modes of considering the equation of a curve are the method of *rectilinear co-ordinates, and that of polar co-ordinates*.



6. Let any two lines XX', YY', intersecting in O, and inclined to one another in a given angle XOY, be taken; and let OQ be one of the values of x; and QP or OR a corresponding value of y: then OQ, QP, are *the co-ordinates of the point P in* reference to the *rectilinear axes* XX', YY'; and the discussion of the properties of any geometrical figure by means of the equation between OQ and QP is called the *method of rectilinear co-ordinates*—or, when XOY is a right angle, that of *rectangular co-ordinates*.



7. Again, if O be a given point in a given line, and x represent any *variable angle* VOP; and let y, determined from the equation $y = f(x)$, be denoted by the line OP, correspondent to x: then the angle VOP and line OP are called the *polar co-ordinates* of the point P: O is called the *pole*, OV the *angular axis*, or *angular origin*, and OP the *radiant*, or *radius vector*. To designate to the eye, however, these separate systems of co-ordinates, the polar angle VOP is denoted by θ, or ω, and the radius-vector OP by r, v, or ρ. In what follows, they are designated as θ and r respectively.

8. Other methods of reference, or systems of co-ordinates, have been proposed, and have in certain cases the advantage over these: but for *elementary purposes*, they appear, in their present state, to be inferior to the rectilinear and polar systems.

9. The method of co-ordinates comprises two distinct and converse inquiries:—

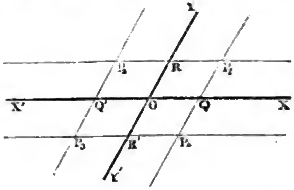
(1). Given the geometrical genesis and position of the co-ordinate axes of a curve, to find the equation of it.

(2). Given the equation of a curve, to determine its figure and position, in reference to the co-ordinate axes, as well as to assign its geometrical properties.

10. Lines are said to be of different orders, according to the highest degree of either of the co-ordinates (or product of them) that enters into the equation. Thus,

$$ay + bx + c = 0,$$
$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$
$$etc.,$$

are general types of equations, of the *first*, *second*, *etc.*, degrees, and the lines represented by them are said to be of the *first*, *second*, *etc*, orders, respectively.



11. In the rectilinear system, the designation of + and − values is to be in accordance with what is shown at *p.* 414, *vol. i.*; and the particular directions are precisely similar to those which occur in the sine and cosine of an angle of variable magnitude (*p.* 424, *vol. i*). When, therefore, the absolute magnitudes of x and y, the co-ordinates of a point P, are given, the particular angle in which the point P is situated, will be at once apparent from the *prefixed signs* of those values. Thus, if $\mathrm{OQ} = \mathrm{RP}_1 = q$, and $\mathrm{OR} = \mathrm{QP}_1 = r$, in respect of absolute values: then, if OQ, OR, be taken positive, the co-ordinates of the four points, P_1, P_2, P_3, P_4, situated in the first, second, third, and fourth angles respectively, will be

P_1	P_2	P_3	P_4
q, r	$-q, r.$	$-q, -r.$	$q, -r.$

12. The co-ordinates of a point in a locus are denoted by x, y; and when they are *given*, they are marked by subscribed accents, or subscribed numbers, and also often by accents above them, as $x_{,}, y_{,}, x_{,,}, y_{,,}$, *etc.*, or x_1, y_1, x_2, y_2, *etc.*, or x', y', x'', y'', *etc.* The superior accents are, however, in this work, reserved for the variable co-ordinates referred to a second, third, *etc*, pair of co-ordinate axes—a circumstance of frequent occurrence.

13. When it is necessary to change the given co-ordinate axes, for others which have given relations to the primitive ones, the transformation of the general equation to suit the new conditions is called the *transformation of co-ordinates*. Thus, if x, y, be the co-ordinates of *any point* P referred to one system of co-ordinates, and x', y', those of the same point referred to another system: the transformation consists in expressing x and y each in terms of x' and y', and given constants; and substituting these values of x, y, in the equation of the locus, constitutes the transformation. In like manner, if the polar equation in terms of r, θ, is to be transformed into another polar where the co-ordinates are r', θ', a similar process is employed: or again, if an equation in x, y, be required to be changed into one in r, θ, or the contrary, the determination of the co-ordinates of the given system in terms of those of the new one, constitutes the transformation of co-ordinates.

14. Any point designated by its co-ordinates, is written (x, y), or x, y, or simply xy, when no ambiguity would result from dropping the parenthesis or the comma. Also, if the points be *given*, they are written $(x_1\, y_1)$, $(x_2\, y_2)$, and called the points $x_1\, y_1$, $x_2\, y_2$, *etc.* Similarly, for the polar system, a point is designated as (r, θ), r, θ, or $r\theta$; or accentuated where the points are *given*.

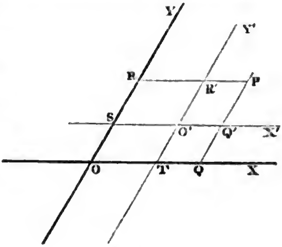
THE TRANSFORMATION OF CO-ORDINATES.

In a considerable number of researches, it is either convenient or essential to change the equations referred to one given system, so as to be adapted to another given system of co-ordinate axes. It consists in expressing the co-ordinates of a point referred to the first system, as functions of those of the same point, and of the lines and angles which define the relation of the new system of co-ordinates to the original one.

PROP. I.

When the new axes have a different origin but are parallel to the original ones.

Let the point P, referred to the axes OX, OY, have its co-ordinates OQ, QP, denoted by x, y; and let its co-ordinates, referred to the new system O'X', O'Y', be denoted by $x'y'$; these new axes O'X', O'Y', being parallel to the primitive ones OX, OY, respectively, and the co-ordinates of O', viz. OT, TO', denoted by a and b. Then,



$$\left.\begin{array}{l} x = \mathrm{OQ} = \mathrm{OT} + \mathrm{TQ} = \mathrm{OT} + \mathrm{O'Q'} = a + x' \\ y = \mathrm{QP} = \mathrm{QQ} + \mathrm{PQ'} = \mathrm{O'T} + \mathrm{PQ'} = b + y' \end{array}\right\} \ldots\ldots\ldots\ldots \quad (1).$$

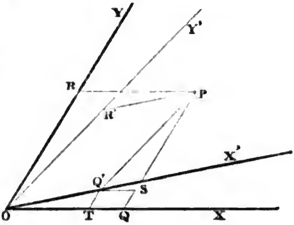
The conversion of the equation into the form suitable to the new conditions will only require the substitution of the values above found.

PROP. II.

When the new axes have the same origin as the primitive, but extend in different directions.

Let P, a point referred to the axes OX, OY, by the co-ordinates OQ $= x$, and QP $= y$, be referred to the new axes OX', OY', by the co-ordinates OQ' $= x'$, and Q'P $= y'$; the inclinations of the lines OX', OY, OY', to OX being β, α, γ; and, consequently, X'OY $= \alpha - \beta$, and Y'OY $= \alpha - \gamma$.

Draw Q'S parallel to OX, meeting PQ in S, and Q'T parallel to PQ, meeting OX in T. Then by trigonometry,



$$\mathrm{TQ}' = \mathrm{OQ}'.\frac{\sin \mathrm{Q'OT}}{\sin \mathrm{OTQ'}} = \frac{x' \sin \beta}{\sin \alpha}$$

$$\mathrm{SP} = \mathrm{PQ}'.\frac{\sin \mathrm{PQ'S}}{\sin \mathrm{PSQ'}} = \frac{y' \sin \gamma}{\sin \alpha}$$

$$\mathrm{OT} = \mathrm{OQ}'.\frac{\sin \mathrm{OQT}}{\sin \mathrm{OTQ'}} = \frac{x' \sin (\alpha - \beta)}{\sin \alpha}$$

$$\mathrm{Q'S} = \mathrm{PQ}'.\frac{\sin \mathrm{Q'PS}}{\sin \mathrm{PSQ'}} = \frac{y' \sin (\alpha - \gamma)}{\sin \alpha}$$

Wherefore, by addition and the geometrical structure of the figure,

$$\left.\begin{aligned} y = \mathrm{PQ} = \mathrm{TQ}' + \mathrm{SP} &= \frac{x' \sin \beta + y' \sin \gamma}{\sin \alpha} \\ x = \mathrm{OQ} = \mathrm{OT} + \mathrm{Q'S} &= \frac{x' \sin (\alpha - \beta) + y' \sin (\alpha - \gamma)}{\sin \alpha} \end{aligned}\right\} \ldots\ldots\ldots\ldots (1).$$

These two formulæ include all the cases of transformation of rectilinear co-ordinates; and in their present form may be considered to represent generally the case where both systems of co-ordinates are oblique. By rendering either or both of them rectangular, all the possible cases will be deduced.

Case I. *Let the primitive axes be rectangular and the new ones oblique:* then $\sin \alpha = 1$, $\sin (\alpha - \beta) = \cos \beta$, and $\sin (\alpha - \gamma) = \cos \gamma$: whence,

$$\left.\begin{aligned} y &= x' \sin \beta + y' \sin \gamma \\ x &= x' \cos \beta + y' \cos \gamma \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Case II. *Let the primitive be oblique and the new ones rectangular:* then, $\gamma = \beta + \frac{1}{2}\pi$, or $\sin \gamma = \cos \beta$; also, $\sin (\alpha - \gamma) = \cos \{\frac{1}{2}\pi - (\alpha - \gamma)\}$ $= \cos \{\frac{1}{2}\pi - \alpha + \beta + \frac{1}{2}\pi\} = \cos \{\pi - (\alpha - \beta)\} = - \cos (\alpha - \beta)$. Hence

$$\left.\begin{aligned} y &= \frac{x' \sin \beta + y_1 \cos \beta}{\sin \alpha} \\ x &= \frac{x' \sin (\alpha - \beta) - y' \cos (\alpha - \beta)}{\sin \alpha} \end{aligned}\right\} \ldots\ldots\ldots\ldots (3).$$

Case III. *Let both systems of axes be rectangular:* then we have, in the preceding equation, $\sin \alpha = 1$, $\sin (\alpha - \beta) = \cos \beta$, $\cos (\alpha - \beta) = \sin \beta$; and hence,

$$\left.\begin{aligned} y &= x' \sin \beta + y' \cos \beta \\ x &= x' \cos \beta - y' \sin \beta \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

In all these cases we have supposed that the new axes do neither of them coincide with the old: but should such be the case, it will only imply $\beta = 0$, when the axes of x, x', coincide, or $\gamma = \alpha$, when the axes of y, $y_{,}$, coincide. The formulæ above are therefore easily adapted to these cases.

PROP. III.

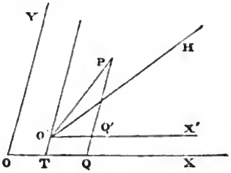
When the origin and direction of the axes are both changed.

This obviously implies the two preceding transformations, either successively or simultaneously performed. This gives,

$$\left.\begin{aligned} y &= b + \frac{x' \sin \beta + y' \sin \gamma}{\sin \alpha} \\ x &= a + \frac{x' \sin (\alpha - \beta) + y' \sin (\alpha - \gamma)}{\sin \alpha} \end{aligned}\right\} \ldots\ldots\ldots\ldots (5).$$

PROP. IV.

To transform the co-ordinates of a point given in reference to rectilinear co-ordinates into functions of polar co-ordinates.



Let OX, OY, be the given rectilinear axes to which the point P is referred; O' the pole, and O'H the axis of angular reference. Draw O'X' parallel to OX, cutting the ordinate PQ in Q'.

Let $OQ=x$, $QP=y$, $XOY=\alpha$, $X'O'H=\beta$, $PO'H=\theta$, and $O'P=r$. Also, let the rectilinear co-ordinates of O' be $OT=a$, and $TO'=b$.

Then $$PQ' = PO'.\frac{\sin PO'Q'}{\sin PQ'O'} = \frac{r\sin(\theta+\beta)}{\sin\alpha},$$

$$O'Q' = PO'\,\frac{\sin O'PQ'}{\sin PQ'O'} = \frac{r\sin\{\alpha-(\theta+\beta)\}}{\sin\alpha};$$

and hence,

$$\left.\begin{aligned} y = PQ = OT + PQ' &= b + \frac{r\sin(\theta+\beta)}{\sin\alpha} \\ x = OQ = OT + O'Q' &= a + \frac{r\sin\{\alpha-(\theta+\beta)\}}{\sin\alpha} \end{aligned}\right\}\ldots\ldots\ldots\ldots(6).$$

Corollary 1.

When the pole coincides with the rectilinear origin, then $a=0$, and $b=0$, in the preceding equation.

Corollary 2.

When the rectilinear axes are at right angles, the expressions become simpler; viz.

$$\left.\begin{aligned} y &= b + r\sin(\theta+\beta) \\ x &= a + r\cos(\theta+\beta) \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Corollary 3.

If also the pole is at the rectilinear origin, we have

$$y = r\sin(\theta+\beta),\ x = r\cos(\theta+\beta)\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Corollary 4.

If, lastly, the angular origin be coincident with the axis of x, we have

$$y = r\sin\theta,\ x = r\cos\theta\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

PROP. V.

To transform the polar co-ordinates to rectilinear ones.

Transpose a and b in equations (6), and divide the results: then,

$$\frac{x-a}{y-b} = \frac{\sin\{\alpha-(\theta+\beta)\}}{\sin(\theta+\beta)} = \sin\alpha\cos(\theta+\beta) - \cos\alpha, \text{ or}$$

$$\tan(\theta+\beta) = \frac{(y-b)\sin\alpha}{(x-a)+(y-b)\cos\alpha}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

Also, the line O′P from (a, b) to (x, y) is denoted by

$$r^2 = \mathrm{O'P}^2 = (x-a)^2 + 2(x-a)(y-b)\cos a + (y-b)^2 \ldots\ldots (11).$$

Corollary 1.

When the pole coincides with the rectilinear origin, these equations become

$$\left.\begin{array}{r} \tan(\theta+\beta) = \dfrac{y \sin a}{x + y\cos a} \\ r^2 = x^2 + 2xy\cos a + y^2 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

Corollary 2.

When the rectilinear angle is right, the equations become

$$\left.\begin{array}{r} \tan(\theta+\beta) = \dfrac{y-b}{x-a} \\ r^2 = (x-a)^2 + (y-b)^2 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

Corollary 3.

If also the pole is at the origin of rectilineal co-ordinates,

$$\tan(\theta+\beta) = \frac{y}{x}; \text{ and } r^2 = x^2 + y^2 \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

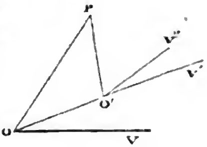
Corollary 4.

If, lastly, the angular origin coincide with the axis of x, we have

$$\tan\theta = \frac{y}{x}; \text{ and } r^2 = x^2 + y^2 \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

PROP. VI.

To transform from one polar system to another.



Let O be the origin of radiants and OV of angles in the given system; O′ and O′V″ those of the transformed. Draw OO′V′, and join O′P.

Then put VOP $= \theta$, VOV′ $= \beta$, V″O′P $= \theta_1$, V′O′V″ $= \beta_1$; also, OP $= r$, OO′ $= \rho$, and O′P $= r_1$.

Then, in the triangle POO′ we have at once,

$$\left.\begin{array}{r} r^2 = \rho^2 + 2r_1\rho\cos(\theta_1+\beta_1) + r_1^2 \\ \sin(\theta-\beta) = \dfrac{\pm\, r_1 \sin(\theta_1+\beta_1)}{\sqrt{\rho^2 + 2r_1\rho\cos(\theta_1+\beta_1) + r_1^2}} \end{array}\right\} \ldots\ldots\ldots\ldots (16).$$

which formulæ adapt themselves to the most general conditions of transformation.

Corollary 1.

When $\rho = 0$, or the origins coincide, then the equations become

$$r = r_1, \text{ and } \theta = \theta_1 + \beta + \beta_1.$$

Corollary 2.

When $\beta = 0$, the second of (16) is simplified on the left side: and when $\beta_1 = 0$ also, or separately, there is a slight simplification, too obvious to need further remark here.

PROBLEMS RELATING TO THE STRAIGHT LINE.

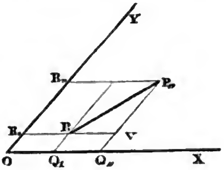
In the general investigations, the axes are supposed to have any inclination, a, and the formulæ are deduced in a general form: but where a difference of form results from the use of rectangular axes, the results are put down as cases of the general theorem.

PROP. I.

To assign the expression of the length of the line which connects two points whose co-ordinates are given.

Let (x_1, y_1) and (x_2, y_2) denote the co-ordinates: then $P_{,}VP_{,,} = \pi - a$; and we have,

$$P_{,}P_{,,} = \pm \sqrt{P_{,}V^2 - 2P_{,}V.VP_{,,} \cos P_{,}VP_{,,} + P_{,,}V^2}$$
$$= \pm \sqrt{(x_2 - x_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos a + (y_2 - y_1)^2} \ldots\ldots\ldots\ldots (1).$$



The double sign expressing that the actual length may be measured from either extremity $P_{,}$ or $P_{,,}$ *towards* the other $P_{,,}$ or $P_{,}$ respectively.

Rectangular system. Let $a = \frac{\pi}{2}$; then,

$$\cos a = 0, \text{ and}$$

$$P_{,}P_{,,} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ldots . (2).$$

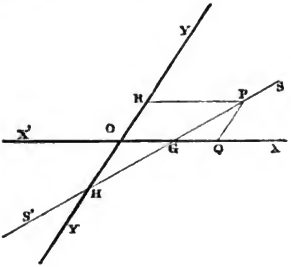
Ex. The co-ordinates of two points are (— 3, 5) and (2—3); and the axes are inclined in an angle of 60°; what is their distance?

Here $x_1 = -3$, $x_2 = 2$, $y_1 = 5$, $y_2 = -3$; and, therefore, $x_2 - x_1 = 2 + 3 = 5$, $y_2 - y_1 = -3 - 5 = -8$, and $\cos a = \cos 60° = \frac{1}{2}$. The value, therefore, is

$$P_{,}P_{,,} = \pm \sqrt{5^2 + 2 . 5 . (-8) . \tfrac{1}{2} + (-8)^2} = \pm 7.$$

PROP. II.

To find the general form of the equation between the co-ordinates of a straight line.



Let SS′ be the given line, and P a point in it, the co-ordinates of which, OQ, QP, are denoted by x and y respectively. Suppose the line to cut the axis of x in G under an angle XGP equal to β, and to cut off the segment OG from the centre equal to h. Then,

$$GPQ = PQX - PGQ = a - \beta,$$

and we have

$$\frac{y}{x-h}=\frac{PQ}{GQ}=\frac{\sin PGQ}{\sin GPQ}=\frac{\sin\beta}{\sin(\alpha-\beta)}.$$

Whence,

$$y=\frac{\sin\beta\,(x-h)}{\sin(\alpha-\beta)}=\frac{\sin\beta\,.\,x}{\sin(\alpha-\beta)}-\frac{\sin\beta\,.\,h}{\sin(\alpha-\beta)}\;\ldots\ldots\ldots\ldots\;(3).$$

The general form of the equation, therefore, of a straight line is of the *first degree* in terms of both variables; and for brevity is usually written of the form

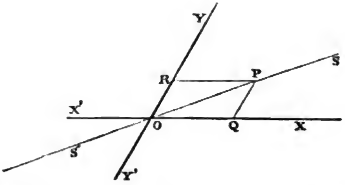
$$y=ax+b\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(4).$$

It is also often written $ay+bx+c=0$, for the purpose of avoiding fractional co-efficients; and likewise in the form $\frac{y}{a}+\frac{x}{b}=1$. To this latter form we shall again recur (see *prop. v. eq.* 16).

When the line passes through the origin O, then $h=0$, and the equation becomes

$$y=\frac{\sin\beta\,.\,x}{\sin(\alpha-\beta)}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(5).$$

Rectangular system. In this case $\alpha=\frac{1}{2}\pi$, and $\sin(\alpha-\beta)=\cos\beta$; whence $\frac{\sin\beta}{\sin(\alpha-\beta)}$ becomes $\frac{\sin\beta}{\cos\beta}=\tan\beta$; and the results corresponding to the preceding (3 and 5) become

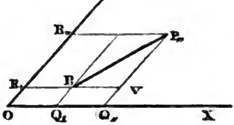


$$y=x\tan\beta-h\tan\beta\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(6).$$
$$y=x\tan\beta\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(7).$$

PROP. III.

Having given the equation of a straight line and the angle of ordination of the axes, to construct the line.

Let $y=ax+b$ be the given equation. Then for any assumed value of x, we can compute the corresponding value of y, we may construct any number of points in the following manner. Set off from scale the values of x assumed, upon the line OX, viz. OQ , OQ_2, *etc.* and the corresponding values of y upon OY, viz. OR_1, OR_2, *etc.* Complete the parallelograms OP_1, OP_2, *etc.*; and the points P_1, P_2, *etc.* will be points in the locus of the equation $y=ax+b$. As, however, a line is determined when *two points* in it are determined, it is obvious that the calculation of more than two values of y, and the subsequent laying down more than the two corresponding values of x and y, would be superfluous.

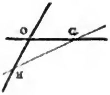


When $b=0$, the line passes through the *origin* O, in which case one point is already determined; and it is sufficient to find the position of *one point* P in addition to O, for the determination of the line (see *fig.* 1, above).

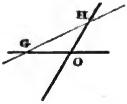
This method, though perfectly general, is unnecessarily tedious in execution;

and (except $b = 0$) one particular pair of cases furnishes simpler operations both for calculation and construction. These are making x and y separately equal to 0, in which case the corresponding values of y and x, viz. $y = b$, and $x = -\frac{b}{a}$ are to be set off upon OY and OX. The two points thus determined are in the line sought, and therefore in construction determine it.

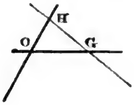
If we consider a and b to be any absolute and positive numbers, and affected with the signs + or —, according as they are prefixed, every possible variety of case in which no term is absent will be included in the general form $y = \pm ax \pm b$; and it will at once be seen that the figures which they represent are as below, the dark lines denoting the axes :—



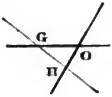
$y = ax + b$



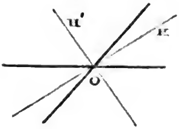
$y = ax - b$



$y = - ax + b$



$y = - ax - b.$



When $b = 0$, or the absolute term is absent, there are but two cases, the first and second of the above coinciding, and the third and fourth coinciding. They will be as in the margin, where OH denotes $y = ax$, and OH′ denotes $y = - ax$.

The only two cases yet to be noticed are, $x = \pm d_1$, and $y = \pm d_1$; viz. those where, in the general form $ax + by = c$, we have separately $b = 0$, or $a = 0$. These represent lines *parallel to the axes* of y and x respectively, and at the distances and on the side of O denoted by $\pm d$ and $\pm d_1$. For example; if $b = 0$, then, whatever value be given to y, the value of x is unaltered, and constantly equal to $\frac{c}{a}$; and, similarly, when $a = 0$, no change in the value of x can alter that of y. The lines represented then are $P_1 P_2$ by $y = + d$, $P_3 P_4$ by $y = - d$; $P_4 P_1$ by $x = + d_1$, and $P_2 P_3$ by $x = - d_1$. See *fig. p.* 248.

PROP. IV.

The equation of the straight line, subject to the condition of passing through one given point, is required.

This will, of course, leave one co-efficient indeterminate in the general equation of the line

$$y = ax + b.$$

But as it passes through the point (x_1, y_1) we have

$$y_1 = ax_1 + b.$$

Subtract the second from the first, to eliminate b; then

$$y - y_1 = a(x - x_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8).$$

This form is usually found to be the most convenient for subsequent use; but still, in some cases, the following, from which a is eliminated, is useful.

By transposition and division,

$$\frac{y-b}{y_1-b}=\frac{ax}{ax_1};\ \text{or } (y-b)\,x_1=(y_1-b)\,x \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (9).$$

Corollary.

In many inquiries relating to the conic sections, it is desirable to express in one equation the conjoint equations of two straight lines which pass through the same point. This will be readily effected by the following equation, $x_1\,y_1$ being the point through which they pass,

$$(y-y_1)^2+\mathrm{A}\,(y-y_1)\,(x-x_1)+\mathrm{B}\,(x-x_1)^2=0\quad\ldots\ldots\ldots\ (9a).$$

For there are two indeterminate quantities, A and B, both of which will be rendered determinate by the assignment of the conditions which fix the other two points, one in each line. An instance of its use will be given in the chapter on *Lines of the second order,* in this volume.

PROP. V.

To find the equation of a line passing through two given points.

SUBSTITUTING the co-ordinates of the points in the general equation, we have, collectively, the three equations,

$$\begin{aligned} y &= ax+b\\ y_1 &= ax_1+b\\ y_2 &= ax_2+b. \end{aligned}$$

By subtraction, as indicated on the left sides of the results, we have

$$\begin{aligned} y-y_1 &= a\,(x-x_1)\\ y_1-y_2 &= a\,(x_1-x_2). \end{aligned}$$

Eliminate a by cross multiplication; and we get

$$\left.\begin{aligned} &(y-y_1)\,(x_1-x_2)=(x-x_1)\,(y_1-y_2)\\ \text{or similarly, }&(y-y_2)\,(x_2-x_1)=(x-x_2)\,(y_2-y_1) \end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ (10).$$

If we reduce either of these to the ordinary general form of the equation of the straight line, we find it to be

$$y=\frac{y_1-y_2}{x_1-x_2}\,x+\frac{x_1\,y_2-x_2\,y_1}{x_1-x_2}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (11).$$

By comparison of which with that form, we have

$$a=\frac{y_1-y_2}{x_1-x_2},\ \text{and } b=\frac{x_1\,y_2-x_2\,y_1}{x_1-x_2}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ (12).$$

In many cases, it will be found convenient to refer it to the form in equation (3): in which case

$$a=\frac{y_1-y_2}{x_1-x^2}=\frac{\sin\beta}{\sin(\alpha-\beta)},\ \text{and } b=\frac{x_1\,y_2-x_2\,y_1}{x_1-x_2}=-\frac{h\sin\beta}{\sin(\alpha-\beta)}\quad\ldots\ (13).$$

To find β and h from these, we have

$$\frac{x_1-x_2}{y_1-y_2}=\frac{\sin(\alpha-\beta)}{\sin\beta}=\sin\alpha\cot\beta-\cos\alpha;\ \text{whence,}$$

$$\tan\beta=\frac{(y_1-y_2)\sin\alpha}{(x_1-x_2)+(y_1-y_2)\cos\alpha}\quad\ldots\ldots\ (14).$$

Also, $h = -\frac{x_1 y_2 - x_2 y_1}{x_1 - x_2} \cdot \frac{\sin(\alpha - \beta)}{\sin \beta}$; and from the preceding equation (13),

$$h = -\frac{x_1 y_2 - x_2 y_1}{y_1 - y_2} \text{. .... (15).}$$

This last is independent of the angle of ordination, as a moment's reflection would lead us to perceive that it should be.

When the given points are *in* the axes, this expression takes a simpler form. Put in that case $x_1 = h$, $y_1 = 0$, $x_2 = 0$, $y_2 = k$: then (11) becomes simply, after slight reduction,

$$hy + kx = hk, \text{ or } \frac{y}{k} + \frac{x}{h} = 1 \text{ .... (16)*.}$$

This form, however, is one into which every equation of the straight line may be changed; and is that referred to in *prop. ii.*

For the purpose of strict analogy, take $y = -ax + b$, as corresponding with k, a positive quantity: then this becomes $\frac{y}{b} + \frac{ax}{b} = 1$, or $\frac{y}{b} + \frac{x}{\frac{b}{a}} = 1$. Whence $k = b$, and $h = \frac{b}{a}$, will change the form as required. This form of the equation, which is often useful, is called the ***symmetrical equation.***

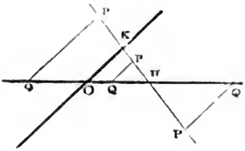
Ex. 1. Find the equation of a line drawn through the points (5, 6) and (2, − 3).

Here $y = ax + b$ becomes, in the two cases, by substituting the given values,

$$6 = 5a + b, \text{ and } -3 = 2a + b \text{ respectively.}$$

By subtraction, $9 = 3a$, or $a = 3$. Also $b = 6 - 5a = 6 - 15 = -9$. The equation, therefore, is

$$y = 3x - 9.$$



[*] It will of course be understood that h and k are to be affected with the proper signs of position; they, in the present case, being both positive.

The formula in this case is readily and easily deduced from the figure which represents the circumstances, by geometrical considerations only.

For let $OH = h$, $OK = k$, and P be *any* point in KH; and draw PQ parallel to OK; then, by similar triangles,

$$\text{OH : HQ :: OK : PQ, or}$$
$$h : h - x :: k : y, \text{ or}$$
$$hy + kx = hk.$$

The two lines, $hy + kx = hk$, and $h_1 y + k_1 x = h_1 k_1$, in order to their being *parallel*, must have

$$\frac{h}{k} = \frac{h_1}{k_1} \text{ or } hk_1 = h_1 k \text{ .... (16a)}$$

That they may be *perpendicular* to each other, the axes KO, OH, being rectangular, is determined by

$$\frac{h}{k} = -\frac{k_1}{h_1} \text{ or } hh_1 = -kk_1 \text{ .... (16b).}$$

That they may be perpendicular, the axes being inclined in an angle α, we shall have

$$-\frac{k_1}{h_1} = -\frac{1 - \frac{k}{h}\cos\alpha}{-\frac{k}{h} + \cos\alpha} = \frac{h - k\cos\alpha}{k - h\cos\alpha} \text{ .... (16c).}$$

The system of equations in note on *prop. (vi.)* also is of great utility in the class of researches, to which the *symmetrical equation* is applicable.

We might, however, instead of resolving the equations *ab initio*, have employed the formula (11), as in the following.

Ex. 2. The points are (— 5, 6) and (2, 3): what is the equation of the line?

$$y = \frac{y_1 - y_2}{x_1 - x_2} . x + \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}$$

$$= \frac{6 - 3}{-5 - 2} . x + \frac{+3(-5) - 6 . 2}{-5 - 2}$$

$$= -\frac{3x}{7} + \frac{27}{7}; \text{ or}$$

$$7y + 3x = 27.$$

Rectangular system. The whole of the preceding expressions, except those which determine or involve β, are independent of the angle of ordination, a. When this angle is a right one, the value of tan β in (14) becomes simply $\frac{y_1 - y_2}{x_1 - x_2}$; or the equation (11) becomes

$$y = x \tan \beta + \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (17).$$

PROP. VI.

The equations of two straight lines, referred to the same system of co-ordinates, are given, to determine their point of intersection.

THE co-ordinates x, y, of both lines will be identical [*] at the point of inter-

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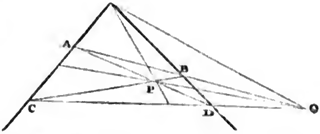
[*] Not only will the values of x and y at the point of section be equal, but it will be the case with the x and y of *every equation of a line or other locus that can be formed out of any combination of the given lines.* This is an evident and simple principle; but it is one of the utmost importance in the method of co-ordinates.

The following instance, as one amongst many, may be chosen for illustration, inasmuch as it will, in a subsequent page, be specially applied. It must be understood, like the proposition in the text, to be applicable to oblique as well as rectangular co-ordinates.



Given the four points A, B, C, D, situated in the lines OC, OD, to find the equations of the lines OP, OQ, QP, drawn to the points P, Q, as in the figure.

Put AO $= a$. BO $= b$, CO $= c$, and DO $= d$; then the equations of the lines AB, CD, AD, BC, are respectively as follows, by note on *prop. v.*

$$\text{(AB) is } ay + bx = ab \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$\text{(CD)} \ldots cy + dx = cd \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

$$\text{(AD)} \ldots ay + dx = ad \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

$$\text{(BC)} \ldots cy + bx = bc \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

Cross-multiply to eliminate the absolute term from (1, 2), then

$$y = -\frac{bd}{ac} \cdot \frac{c - a}{d - b} \cdot x \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

This is the equation of the line through Q, the intersection of AB, CD, and the origin O.

Again, treat (3, 4) similarly, and we obtain

$$y = +\frac{bd}{ac} \cdot \frac{c - a}{d - b} \cdot x \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$$

which expresses OP, and differs from that of OQ only in the sign.

Again, for the equation PQ we may find the co-ordinates of P and Q, and substitute them in section; and hence the two equations will be simultaneous. Let then the equations be

$$y = a_1x + b_1, \text{ and } y = a_2x + b_2; \text{ then}$$

$$x = -\frac{b_1 - b_2}{a_1 - a_2}, \text{ and } y = \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2} \quad\ldots\ldots\ldots\ldots\ldots\ldots (18).$$

If, however, the equation be of the form

$$a_1x + b_1y + c_1 = 0, \text{ and } a_2x + b_2y + c_2 = 0,$$

then the solution is

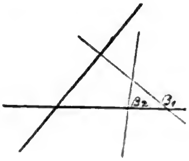
$$x = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \text{ and } y = -\frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1} \quad\ldots\ldots\ldots\ldots\ldots\ldots (19).$$

Ex. Required the co-ordinates of the point of intersection of the lines $y = 3x - 9$, and $7y + 3x = 27$.

Resolving these either by actual work, or by either system of equations (18) or (19), we readily find $x = \frac{15}{4}$ and $y = \frac{9}{4}$.

PROP. VII.

To determine the angle κ under which two lines of given equations intersect.



Let the line $y = a_1x + b_1$ make the angle β_1 with the axis of x, and $y = a_2x + b_2$ make the angle β_2 with it. Then the angle contained by the two lines is $\kappa = \beta_1 - \beta_2$; and if a be, as before, the angle of ordination, we shall have from (13)

$$a_1 = \frac{\sin \beta_1}{\sin (a - \beta_1)}, \text{ and } a_2 = \frac{\sin \beta_2}{\sin (a - \beta_2)}$$

From these, by ordinary methods, we readily obtain

$$\tan \beta_1 = \frac{a_1 \sin a}{1 + a_1 \cos a}; \quad \tan \beta_2 = \frac{a_2 \sin a}{1 + a_2 \cos a}.$$

But $\kappa = \beta_1 - \beta_2$ is the angle of inclination; and hence, substituting and reducing,

$$\tan \kappa = \frac{\tan \beta_1 - \tan \beta_2}{1 + \tan \beta_1 \tan \beta_2} = \frac{(a_1 - a_2) \sin a}{1 + (a_1 + a_2) \cos a + a_1 a_2} \quad\ldots\ldots\ldots\ldots (20).$$

This is the general relation which must subsist amongst the three quantities, a_1, a_2, and κ: and may be resolved for either a_1 or a_2, as (20) is expressive of the third κ in terms of the two others—this being the proposed object of the problem. Resolved for a_1 and a_2 respectively, it becomes

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the formula of *eq.* (11). Such process, however, would be unnecessarily tedious, as it may be avoided in the following manner :—

Cross multiply (1, 2) by the absolute terms, and *add*; then,

$$ac\,(b + d)\,y + bd\,(a + c)\,x = 2abcd \quad\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Do the like with (3, 4), and there results

$$ac\,(b + d)\,y + bd\,(a + c)\,x = 2abcd \quad\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

For the preceding reasons, (7) passes through Q and (8) through P: and the equations are identical, therefore, each indicates a line passing through both P and Q; that is, the equation (7) or (8) represents the line PQ.

$$a_1 = -\frac{a_2 \sin(\kappa + \alpha) + \sin\kappa}{\sin(\kappa - \alpha) + a_2 \sin\kappa} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21).$$

$$a_2 = -\frac{a_1 \sin(\kappa - \alpha) + \sin\kappa}{\sin(\kappa + \alpha) + a_1 \sin\kappa} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (22).$$

When $\kappa = \frac{\pi}{2}$, the preceding equation becomes

$$\left.\begin{aligned} a_2 &= -\frac{1 + a_1 \cos\alpha}{a_1 + \cos\alpha}, \\ \text{or, } 1 + (a_1 + a_2)\cos\alpha + a_1 a_2 &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots \quad (23),$$

which is the condition of relation amongst the co-efficients a_1, a_2, of the equations of the two lines, that implies the two lines being at right angles to each other.

Rectangular system. In this case $\alpha = \frac{\pi}{2}$; and then $\sin\alpha = 1$, and $\cos\alpha = 0$, being substituted in (20), give

$$\tan\kappa = \tan(\beta_1 - \beta_2) = \frac{a_1 - a_2}{1 + a_1 a_2} \quad \ldots\ldots\ldots\ldots\ldots \quad (24);$$

and if the inclination of the lines, κ, be $\frac{\pi}{2}$, this becomes simply

$$1 + a_1 a_2 = 0, \text{ or } a_2 = -\frac{1}{a_1} \ldots\ldots\ldots\ldots\ldots \quad (25).$$

PROP. VIII.

To find the equation of a line which passes through a given point and makes a given angle with a given line.

PUT $y = a_1 x + b_1$, the given line; and
$y = a_2 x + b_2$, the line sought;
$(x_1\ y_1)$ the given point; and
$\kappa =$ the given angle.

Then, since the line sought passes through $(x_1\ y_1)$, we have, equation (8),

$$y - y_1 = a_2 (x - x_1);$$

in which, inserting from (15) the value of a_2 in terms of a_1, α, and κ, we have

$$y - y_1 = -\frac{a_1 \sin(\kappa - \alpha) + \sin\kappa}{\sin(\kappa + \alpha) + a_1 \sin\kappa}(x - x_1) \quad \ldots\ldots\ldots\ldots \quad (26).$$

When the lines are to be parallel, $\kappa = 0$, and this becomes simply, as we might expect,

$$\begin{aligned} y - y_1 &= -\frac{-a_1 \sin\alpha + 0}{\sin\alpha + a_1 0}(x - x_1) \\ &= a_1 (x - x_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (27). \end{aligned}$$

When the lines are to be at right angles, then $\kappa = \frac{\pi}{2}$, and we have (26, 27) changed into

$$y - y_1 = -\frac{1 + a_1 \cos\alpha}{a_1 + \cos\alpha}(x - x_1) \quad \ldots\ldots\ldots\ldots\ldots \quad (28).$$

Rectangular system. When $\alpha = \frac{\pi}{2}$, the equations (26, 27, 28) are converted into the following:

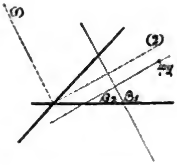
$$y - y_1 = \frac{a_1 - \tan \kappa}{1 + a_1 \tan \kappa} (x - x_1) \quad \ldots\ldots\ldots \quad (29).$$

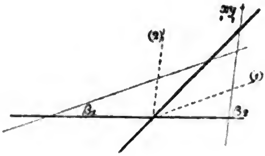
$$y - y_1 = a_1 (x - x_1) \quad \ldots\ldots\ldots \quad (30).$$

$$y - y_1 = - \frac{1}{a_1} (x - x_1) \quad \ldots\ldots\ldots \quad (31).$$

Scholium.

Some degree of care is required in securing precision both of description and of algebraic expression, when speaking of the angles formed by straight lines referred to co-ordinate axes. In the preceding investigation, the angle κ is taken as in (*fig.* 1), where β_1 is *greater* than β_2 by the given angle κ: but it may also be given so that the contrary may take place, and β_1 be *less* than β_2 by the given angle of inclination κ. In this latter case, the formula must be modified by changing the sign of κ, when the expression will become that of *fig.* 2.





The proper distinction between the conditions as to the position of the new line, will be obvious if we conceive a line to be drawn through the origin parallel to the given line (as the dotted line "(1)" in the figures); and to assign whether the parallel (2) to the required line through the origin makes with the axis of x taken positively, an angle greater or less than that made by (1). When this is not determinate, the solution is double, as in the corresponding geometrical problem.

Ex. A line passing through a point whose co-ordinates are $x_1 = -5$, and $y_1 = -6$, is drawn perpendicular to a line whose equation is $5y + 4x - 6 = 0$: what is the equation of this perpendicular, and where does it cut the axes of co-ordinates?

Denoting the two lines by

$$y = ax + b = -\frac{4}{5} x + \frac{6}{5}, \text{ and } y = a'x + b',$$

$$\text{we have } a' = -\frac{1}{a} = \frac{5}{4}.$$

$$\text{Also } -6 = -5a' + b' = 0 = -\frac{25}{4} + b', \text{ or } b' = \frac{1}{4}.$$

Whence the equation of the perpendicular is reduced to

$$y = \frac{5x}{4} + \frac{1}{4}.$$

$$\text{Also, if } y = 0, \text{ then } x = -\frac{1}{5}; \text{ and if } x = 0, \text{ then } y = \frac{1}{4};$$

which give the distances of the intersections respectively of the line with the axes of x and y from the origin of co-ordinates.

PROP. IX.

To find the equation of a line, which, passing through a given point, shall make equal angles with two given lines.

DENOTE the two given lines and that sought, respectively, by

$$y = a_1 x + b_1,$$
$$y = a_2 x + b_2,$$
$$y = ax + b.$$

Then, that the third line may form *equal geometrical angles* with the other two, it must form *supplementary co-ordinate angles* with them. Let them, then, be κ and $\pi - \kappa$; and we shall have by (20),

$$\tan \kappa = \frac{(a - a_1) \sin \alpha}{1 + (a + a_1) \cos \alpha + aa_1},$$

$$\tan \kappa = - \tan (\pi - \kappa) = - \frac{(a - a_2) \sin \alpha}{1 + (a + a_2) \cos \alpha + aa_2}.$$

Equate these values of $\tan \kappa$, and reduce the resulting equation according to the powers of a; then there results

$$a^2 \left\{ a_1 + 2 \cos \alpha + a_2 \right\} + 2a \left\{ 1 - a_1 a_2 \right\} = a_1 + 2a_1 a_2 \cos \alpha + a_2 \ldots (32).$$

Let a', a'', be the values of a in this equation; and let $x_{,}$ $y_{,}$ be the point through which the line is to pass. Then

$$y - y_{,} = a' (x - x_{,}), \text{ and } y - y_{,} = a'' (x - x_{,})$$

are the equations of the two lines, either of which fulfils the conditions of the problem.

Moreover, if we insert the values of a', a'', deduced from (31) in the criterion (23), we shall find it fulfilled; and hence we learn that the two lines just determined are at right angles to each other, as, geometrically, we know should be the case.

Rectangular equation. In this case $\cos \alpha = 0$, and the condition (32) is simplified to

$$a^2 + 2a . \frac{1 - a_1 a_2}{a_1 + a_2} - 1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (33).$$

PROP. X.

To find the length of the perpendicular from a given point to a given line.

LET $y = ax + b$; or $y - y_1 = ax + b - y_1$ be the given line; $(x_1\ y_1)$ being the given point; then the perpendicular to it from $(x_1\ y_1)$ is expressed by the equation (26),

$$y - y_1 = - \frac{1 + a \cos \alpha}{a + \cos \alpha} (x - x_1).$$

Resolving these two equations for the determination of their point of intersection, we have

$$y - y_1 = - \frac{(1 + a \cos \alpha)(y_1 - ax_1 - b)}{1 + 2a \cos \alpha + a^2}.$$

$$x - x_1 = \frac{(a + \cos \alpha)(y_1 - ax_1 - b)}{1 + 2a \cos \alpha + a^2},$$

Whence from (1) we have the distance of $(x_1\, y_1)$ and $(x\, y)$ expressed by

$$\sqrt{(x-x_1)^2+2(x-x_1)(y-y_1)\cos\alpha+(y-y_1)^2}=\frac{\pm(y_1-ax_1-b)\sin\alpha}{\sqrt{1+2a\cos\alpha+a^2}} \quad \ldots\ldots\ldots \quad (34).$$

The double sign expressing that the line may be measured from either extremity towards the other.

Rectangular system. In this case $\alpha=\frac{\pi}{2}$, and the value of the perpendicular is

$$\frac{\pm(y_1-ax_1-b)}{\sqrt{1+a^2}} \quad \ldots\ldots\ldots \quad (35).$$

PROP. XI.

To assign the relation amongst the co-ordinates of three points, that they may be in one line.

This will be fulfilled if lines drawn through any two of the three also pass through the third. Now the equation of the line $(x_1\, y_1)\,(x_2\, y_2)$ is, (*eq.* 10),

$$(y-y_1)(x_1-x_2)=(x-x_1)(y_1-y_2);$$

and that this shall pass through $(x_3\, y_3)$, it must be fulfilled when these are substituted for (xy); which gives

$$(y_3-y_1)(x_1-x_2)=(x_3-x_1)(y_1-y_2), \text{ or}$$

$$(x_1 y_2-x_2 y_1)+(x_2 y_3-x_3 y_2)+(x_3 y_1-x_1 y_3)=0 \quad \ldots\ldots \quad (36).$$

This solution may be put in a different form: for since the three points are in one line, the line through each two of them must be in a line which gives the same value of $\frac{\sin\beta}{\sin(\alpha-\beta)}$. This will imply the simultaneous equality of the fractions

$$\frac{y_1-y_2}{x_1-x_2}=\frac{y_2-y_3}{x_2-x_3}=\frac{y_3-y_1}{x_3-x_1} \quad \ldots\ldots\ldots \quad (37).$$

Corollary 1.

If the point $(x_3\, y_3)$ be in one of the axes, as that of x; then the condition is reduced to $y_3=0$, and hence

$$y_2 x_1-y_1 x_2+x_3(y_1-y_2)=0 \quad \ldots\ldots\ldots \quad (37a);$$

or if in the axis of y, then $x_3=0$, and the condition becomes

$$y_2 x_1-y_1 x_2+y_3(x_2-x_1)=0 \quad \ldots\ldots\ldots \quad (37b).$$

Corollary 2.

If $(x_3\, y_3)$ be the origin; then the condition is simply

$$y_1 x_2-y_2 x_1=0 \quad \ldots\ldots\ldots \quad (37c).$$

PROP. XII.

To assign the relation amongst the co-efficients of three equations, so that the lines represented by them shall pass through one point.

Let $\left.\begin{cases} y=a_1x+b_1 \\ y=a_2x+b_2 \\ y=a_3x+b_3 \end{cases}\right\}$ be the three equations.

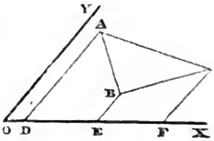
The problem evidently requires that the values of $(x\, y)$ shall be the same when formed from each pair of these three equations: or in other words, it will be fulfilled if we eliminate x and y from the three given equations. This elimination being performed, gives the final equation of condition:

$$(a_1 b_2 - a_2 b_1) + (a_2 b_3 - a_3 b_2) + (a_3 b_1 - a_1 b_3) = 0 \; \ldots\ldots\ldots\ldots\ldots (38).$$

The similarity of this equation to the preceding (the one being that of the given co-ordinates, and the other that of the given co-efficients), though very remarkable, does not seem to have been noticed by elementary writers.

PROP. XIII.

To express the area of a triangle in terms of the rectilinear co-ordinates of the three angles.



Let the co-ordinates of the three angles A, B, C, be $(x_1\, y_1)$, $(x_2\, y_2)$, and $(x_3\, y_3)$; and draw the ordinates y_1, y_2, y_3, of the points, viz. AD, BE, CF. Also, let a be the angle of ordination. Then, *vol. i. p.* 472, *art.* 3, we have

$$\text{ADEB} = \tfrac{1}{2}\, \text{DE}\, \{\text{AD} + \text{BE}\} \sin \text{ADE} = \tfrac{1}{2}(x_2 - x_1)(y_2 + y_1) \sin a,$$
$$\text{BEFC} = \tfrac{1}{2}\, \text{EF}\, \{\text{BE} + \text{FC}\} \sin \text{BEF} = \tfrac{1}{2}(x_3 - x_2)(y_3 + y_2) \sin a,$$
$$\text{ADFC} = \tfrac{1}{2}\, \text{DF}\, \{\text{AD} + \text{CF}\} \sin \text{CFD} = \tfrac{1}{2}(x_3 - x_1)(y_3 + y_1) \sin a.$$

Hence $\triangle\text{ABC} = \text{ADFC} - \text{ADEB} - \text{BEFC}$,

$$= \tfrac{1}{2}\{(y_2 x_1 - y_1 x_2) + (y_3 x_2 - y_2 x_3) + (y_1 x_3 - y_3 x_1)\} \sin a \, .. \, (39).$$

Corollary.

When the three points are in one straight line, the area of the triangle becomes zero; and in this case we shall have,

$$(y_1 x_2 - y_2 x_1) + (y_2 x_3 - y_3 x_2) + (y_3 x_1 - y_1 x_3) = 0:$$

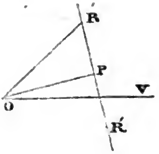
which was the equation of condition found in (37), when the three points are situated in one straight line.

POLAR EQUATIONS OF THE STRAIGHT LINE.

These equations might be deduced by means of the equations of transformation hereafter to be given, from the corresponding rectilinear ones found in the preceding chapter. It is, however, not only more elegant, but also much more simple, to form the polar equations from a fundamental and uniform process.

PROP. I.

To find the general forms of the polar equation of a straight line.



Let OV be the origin of angular, and O that of radial ordinates. Draw the perpendicular OP to the line RR′; and let VOP $= \beta$, OP $= p$, ROV $= \theta$, and OR $= r$. Then we have at once,

$$r = p \sec(\theta - \beta), \text{ or } p = r\cos(\theta - \beta) \; \ldots . (1, 2).$$

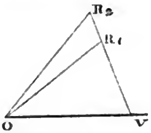
It may also be put under the form

$$\frac{r\cos\theta}{p\sec\beta} + \frac{r\sin\theta}{p\,\text{cosec}\,\beta} = 1 \; \ldots\ldots\ldots . (3),$$

which has a considerable analogy to one of the forms of the rectilinear equations of a straight line. Chap. ii.

PROP. II.

To express the distance between two points in reference to polar co-ordinates.



Let R_1, R_2, be the points; and their polar co-ordinates be $(r_1\ \theta_1)$, $(r_2\ \theta_2)$; then the angle R_1OR_2 is equal to $\theta_2 - \theta_1$, and we obviously (*vol. i. p.* 453) get the value sought, viz.:

$$R_1 R_2 = \pm\sqrt{r_1^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1) + r_2^2} \,.\,.\ (4).$$

PROP. III.

To assign the form of the polar equation of a straight line passing through one given point.

Here the general form (2), and the particular case of the line passing through a given point, furnish the equations

$$p = r\cos(\theta - \beta),$$
$$p = r_1 \cos(\theta_1 - \beta).$$

By eliminating p, and reducing, we get

$$(r\cos\theta - r_1\cos\theta_1)\cos\beta + (r\sin\theta - r_1\sin\theta_1)\sin\beta = 0 \ \ldots\ldots\ (5),$$

$$\text{or } \tan\beta = -\frac{r\cos\theta - r_1\cos\theta_1}{r\sin\theta - r_1\sin\theta_1} \ \ldots\ldots\ldots\ldots\ldots\ldots\ (6),$$

in which β is indeterminate.

We might, however, have eliminated β, and given a result containing the indeterminate quantity p. For of the two equations take the difference of the inverse functions; then

$$\cos^{-1}\frac{p}{r} - \cos^{-1}\frac{p}{r_1} = \theta - \theta_1.$$

Take the cosines of both sides; then transpose, square, *etc.*, whence

$$\frac{p}{r}\cdot\frac{p}{r_1} + \sqrt{\left(1 - \frac{p^2}{r^2}\right)\left(1 - \frac{p^2}{r_1^2}\right)} = \cos(\theta - \theta_1),$$

$$\text{or } p^2\left\{r^2 - 2rr_1\cos(\theta - \theta_1) + r_1^2\right\} = r^2 r_1^2 \sin^2(\theta - \theta_1) \ \ldots\ldots\ldots\ (7),$$

$$\text{or } p = \frac{\pm\, rr_1 \sin(\theta - \theta_1)}{\sqrt{r^2 - 2rr_1\cos(\theta - \theta_1) + r_1^2}} \ \ldots\ldots\ldots\ldots\ldots\ldots\ (8).$$

PROP. IV.

To find the polar equation of a line passing through two given points.

Let $\left\{\begin{array}{l} p = r\cos(\theta - \beta) \\ p = r_1\cos(\theta_1 - \beta) \\ p = r_2\cos(\theta_2 - \beta) \end{array}\right\}$ be the general form and two cases.

Eliminate p from the first and second, and from the first and third; then, as before,

$$(r\cos\theta - r_1\cos\theta_1) + (r\sin\theta - r_1\sin\theta_1)\tan\beta = 0,$$
$$(r\cos\theta - r_2\cos\theta_2) + (r\sin\theta - r_2\sin\theta_2)\tan\beta = 0,$$

from which eliminate $\tan\beta$, and there results the required equation,

$$rr_1 \sin(\theta - \theta_1) + r_1 r_2 \sin(\theta_1 - \theta_2) + r_2 r \sin(\theta_2 - \theta) = 0 \;\ldots\ldots\ldots (9),$$

$$\text{or } \frac{\sin(\theta - \theta_1)}{r_2} + \frac{\sin(\theta_1 - \theta_2)}{r} + \frac{\sin(\theta_2 - \theta)}{r_1} = 0 \;\ldots\ldots\ldots\ldots\; (10).$$

Each of these forms is elegant and symmetrical; but other forms may be given of perhaps greater utility in subsequent investigation. It may be well to remark, however, that the values of p and β may be found from (6) and (8) of the preceding proposition, by writing θ_2 for θ_1, and θ_1 for θ. This gives

$$\left.\begin{array}{c} \tan\beta = -\dfrac{r_1 \cos\theta_1 - r_2 \cos\theta_2}{r_1 \sin\theta_1 - r_2 \sin\theta_2} \\ \sin\beta = \dfrac{\pm(r_1 \cos\theta_1 - r_2 \cos\theta_2)}{\sqrt{r_1^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) + r_2^2}};\; \cos\beta = \dfrac{\mp(r_1 \sin\theta_1 - r_2 \sin\theta_2)}{\sqrt{r_1^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) + r_2^2}} \\ p = \dfrac{\pm r_1 r_2 \sin(\theta_1 - \theta_2)}{\sqrt{r_1^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) + r_2^2}} \end{array}\right\}$$

To proceed to the transformation:— .... (11).

$$r\{r_1 \sin(\theta - \theta_1) + r_2 \sin(\theta_2 - \theta)\} = - r_1 r_2 \sin(\theta_1 - \theta_2);$$

or, expanding the sines and collecting the co-efficients, we get, after a little obvious reduction, the following formulæ:—

$$(r \sin\theta - r_1 \sin\theta_1)(r_1 \cos\theta_1 - r_2 \cos\theta_2) = (r \cos\theta - r_1 \cos\theta_1)(r_1 \sin\theta_1 - r_2 \sin\theta_2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

$$r \sin\theta (r_1 \cos\theta_1 - r_2 \cos\theta_2) - r \cos\theta (r_1 \sin\theta_1 - r_2 \sin\theta_2) = r_1 r_2 (\cos\theta_1 \sin\theta_2 - \cos\theta_2 \sin\theta_1)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

These equations (12, 13) have a remarkable analogy to the corresponding rectilinear equations, *chap. ii. pr. iv. p.* 256, *eq.* (11).

PROP. V.

To assign the angle of inclination of two straight lines whose equations are given.

This is obviously the difference of the angles β_1, β_2 formed by the perpendiculars p_1, p_2, with the angular axis OV; that is, either $\kappa = \pm(\beta_1 - \beta_2)$, or $\kappa = \pi \mp (\beta_1 - \beta_2)$.

PROP. VI.

To find the polar co-ordinates of the point of intersection of two straight lines.

Let them be

$$p_1 = r \cos(\theta - \beta_1),$$
$$p_2 = r \cos(\theta - \beta_2).$$

Then by division,

$$\frac{p_1}{p_2} = \frac{\cos(\theta - \beta_1)}{\cos(\theta - \beta_2)} = \frac{\cos\beta_1 + \sin\beta_1 \tan\theta}{\cos\beta_2 + \sin\beta_2 \tan\theta}, \text{ or}$$

$$\tan\theta = -\frac{p_1 \cos\beta_2 - p_2 \cos\beta_1}{p_1 \sin\beta_2 - p_2 \sin\beta_1} \;\ldots\ldots\ldots\ldots\; (14);$$

and substituting the values of $\sin\theta$, $\cos\theta$ in the first form, we get

$$\frac{1}{r} = \frac{\cos(\theta - \beta_1)}{p_1} = \frac{\pm \sin(\beta_1 - \beta_2)}{\sqrt{p_1^2 - p2_1 p_2 \cos(\beta_1 - \beta_2) + p_2^2}} \;\ldots\ldots\; (15);$$

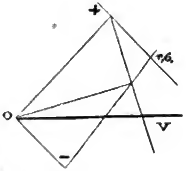
the co-ordinates are therefore determined.

Scholium.

If one of the lines pass through the pole, the expressions become simplified. The student should write them out under this condition.

PROP. VII.

To find the equation of a line which passes through a given point, and makes a given angle with a given line.



Let $p = r\cos(\theta - \beta)$ be the given line, and $r_1\ \theta_1$ the given point; then if κ be the given angle, the general form of the equation of the line sought becomes

$$p_1 = r\cos\{\theta - (\beta \mp \kappa)\}$$

and since it also passes through $r_1\ \theta_1$, we have

$$p_1 = r_1\cos\{\theta_1 - (\beta \mp \kappa)\}$$

eliminating p_1 from which, we obtain the equation sought, viz.

$$r_1\cos\{\theta_1 - (\beta \mp \kappa)\} = r\cos\{\theta - (\beta \mp \kappa)\} \ldots\ldots\ldots (16).$$

It will readily be distinguished when + or — is to be used; since it is easy, when the one of the two lines through $r_1\ \theta_1$ is chosen, to assign whether κ tends to increase or lessen β in the general equation. These are as marked +, —, in the figure.

When the new line is to be *parallel to the given one*, we have $\kappa = 0$, and the equation is reduced to

$$r_1\cos(\theta_1 - \beta) = r\cos(\theta - \beta)\ .\ \ldots\ldots\ldots\ldots\ldots (17).$$

When the new line *is to be perpendicular to the given one*, we get in (16), κ converted into $\frac{1}{2}\pi$, and hence

$$r_1\sin(\theta_1 - \beta) = r\sin(\theta - \beta) \ldots\ldots\ldots\ldots\ldots\ldots (18).$$

The general equation may also be reduced to a form analogous to the rectangular equation of a straight line under the same circumstances. For expanding we have,

$$r_1\cos\theta_1\cos(\beta\mp\kappa) + r_1\sin\theta_1\sin(\beta\mp\kappa) = r\cos\theta\cos(\beta\mp\kappa) + r\sin\theta\sin(\beta\mp\kappa),$$

or

$$r\sin\theta - r_1\sin\theta_1 = -\cot(\beta \mp \kappa)(r\cos\theta - r_1\cos\theta_1) \ldots\ldots (19).$$

PROP. VIII.

To find the length of the perpendicular from a given point to a given straight line.

Let $r_1\ \theta_1$ be the given point, and $p = r\cos(\theta - \beta)$ the given line. The equation of the perpendicular from $r_1\ \theta_1$ upon the line is (18),

$$r_1(\sin\theta_1 - \beta) = r\sin(\theta - \beta).$$

Eliminate r from the equations of these two lines, by division, which gives

$$\frac{r_1\sin(\theta_1 - \beta)}{p} = \frac{r\sin(\theta - \beta)}{r\cos(\theta - \beta)} = \tan(\theta - \beta),$$

$$\text{Whence } p\sec(\theta - \beta) = \pm\sqrt{p^2 + r_1^2\sin^2(\theta_1 - \beta)}.$$

But from the equation of the line we have

$$r = p\sec(\theta - \beta) = \pm\sqrt{p^2 + r_1^2\sin^2(\theta_1 - \beta)}.$$

The co-ordinates of the point of intersection are therefore determined. Denote then the length of the perpendicular by l; and we have

$$l^2 = r^2 - 2rr_1, \cos\{(\theta - \beta) - (\theta_1 - \beta)\} + r_1^2;$$

in which, inserting the values of r, and $\cos(\theta-\beta) = \frac{p}{\sqrt{p^2 + r_1^2 \sin^2(\theta_1 - \beta)}}$, we obtain

$$l = \pm \{p - r_1 \cos(\theta_1 - \beta)\} \ldots\ldots\ldots\ldots (18).$$

PROP. IX.

To find the equation of a line, which, drawn through a given point, shall make equal angles with two given lines.

Let $r_1\ \theta_1$ be the given point; and let the two given lines and the line sought be respectively expressed by

$$p_1 = r\cos(\theta - \beta_1),$$
$$p_2 = r\cos(\theta - \beta_2),$$
$$p = r\cos(\theta - \beta).$$

The angle between the first and third is $\beta - \beta_1$, and that between the second and third is $\beta_2 - \beta$; whence, since the line sought is equally inclined to these, we have $\beta = \frac{1}{2}(\beta_1 + \beta_2)$.

Also, since the line sought is to pass through $r_1\ \theta_1$, we have

$$p = r_1 \cos(\theta_1 - \beta).$$

These two results combined, then, give the equation sought,

$$r_1 \cos\{\theta_1 - \tfrac{1}{2}(\beta_1 + \beta_2)\} = r\cos\{\theta - \tfrac{1}{2}(\beta_1 + \beta_2)\} \ldots\ldots\ldots (19).$$

PROP. X.

To assign the relations amongst the co-ordinates of three points, that they may be situated in one line.

Let them be $(r_1\ \theta_1)$, $(r_2\ \theta_2)$, and $(r_3\ \theta_3)$; then the line through the first two of them is (9)

$$r_1 r_2 \sin(\theta_1 - \theta_2) + r_2 r \sin(\theta_2 - \theta) + r r_1 \sin(\theta - \theta_1) = 0;$$

and since $r_3\ \theta_3$ is to be in this line, we shall have the condition

$$r_1 r_2 \sin(\theta_1 - \theta_2) + r_2 r_3 \sin(\theta_2 - \theta_3) + r_3 r_1 \sin(\theta_3 - \theta_1) = 0 \ldots (20),$$

$$\text{or } \frac{\sin(\theta_1 - \theta_2)}{r_3} + \frac{\sin(\theta_2 - \theta_3)}{r_1} + \frac{\sin(\theta_3 - \theta_1)}{r_2} = 0 \ldots\ldots\ldots\ldots (21).$$

PROP. XI.

To assign the relation amongst the co-efficients of three equations, that the three lines may meet in one point.

Let them be

$$p_1 = r\cos(\theta - \beta_1),$$
$$p_2 = r\cos(\theta - \beta_2),$$
$$p_3 = r\cos(\theta - \beta_3).$$

Then, proceeding as in *prop. vi.* with the first and second and the first and third, we shall eliminate r and θ, and obtain the relation

$$\frac{p_1 \cos\beta_2 - p_2 \cos\beta_1}{p_1 \sin\beta_2 - p_2 \sin\beta_1} = -\tan\theta = \frac{p_1 \cos\beta_3 - p_3 \cos\beta_1}{p_1 \sin\beta_3 - p_3 \sin\beta_1};$$

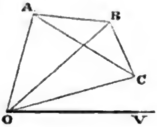
reducing which, there results, after slight transformation,

$$p_1 \sin(\beta_2 - \beta_3) + p_2 \sin(\beta_3 - \beta_1) + p_3 \sin(\beta_1 - \beta_2) = 0 \ \ldots\ (22);$$

a form remarkable for its similarity to that of the preceding proposition, and in this respect analogous to what takes place in rectilinear co-ordinates.

PROP. XII.

To express the area of a triangle in terms of the polar co-ordinates of the three angles.



Denote the co-ordinates of the angles A, B, C, by $(r_1\theta_1)$, $(r_2\theta_2)$, $(r_3\theta_3)$, in reference to pole O and angular axis OV.

Then, *vol. i. p.* 472, *art.* 2, *cor.* 3, we have

$$\Delta AOB = \tfrac{1}{2}\,AO\,.\,OB \sin AOB = \tfrac{1}{2}\,r_1\,r_2 \sin(\theta_1 - \theta_2),$$
$$\Delta BOC = \tfrac{1}{2}\,BO\,.\,OC \sin BOC = \tfrac{1}{2}\,r_2\,r_3 \sin(\theta_2 - \theta_3),$$
$$\Delta COA = \tfrac{1}{2}\,CO\,.\,OA \sin COA = -\tfrac{1}{2}\,r_3\,r_1 \sin(\theta_3 - \theta_1) :—$$

whence we get

$$\Delta ABC = \Delta AOB + \Delta BOC - \Delta COA$$
$$= \tfrac{1}{2}\left\{r_1\,r_2 \sin(\theta_1 - \theta_2) + r_2\,r_3 \sin(\theta_2 - \theta_3) + r_3\,r_1 \sin(\theta_3 - \theta_1)\right\} \;..\;(23),$$

which gives the area sought.

Corollary.

When the three points A, B, C, are in one line, the area of the triangle ABC is 0; and hence in this case,

$$r_1\,r_2 \sin(\theta_1 - \theta_2) + r_2\,r_3 \sin(\theta_2 - \theta_3) + r_3\,r_1 \sin(\theta_3 - \theta_1) = 0,$$

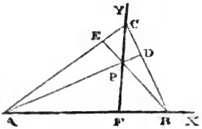
which is the result obtained in (20) as the *condition* of A, B, C, being in a straight line.

PROPOSITIONS ILLUSTRATIVE OF THE EQUATION OF THE STRAIGHT LINE IN A PLANE.

PROP. I.

Lines drawn from the angles of a triangle, perpendicular to their opposite sides, all intersect in the same point.

First solution *.



Let AD, BE, CF, be the three perpendiculars: and take FB, FC, as rectangular axes of x and y.

Express the co-ordinates of B by h, 0,

.... A by $-h_1$, 0,

.... C by 0, k.

Also, let BE, AD, respectively cut off from FY the segments k_1 and k_2. Then we shall have the following equations :—

(BC) is $hy + kx = hk$ .... (1).

(CA) is $-h_1y + kx = -h_1k$ ...(2).

(AD) is $-h_1y + k_2x = -h_1k_2$ ..(3).

(BE) is $hy + k_1x = hk_1$ ....(4).

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* It has been thought more conducive to the student's improvement, to give solutions of the same theorem by several different methods, than to exhibit those methods in their application to different theorems. It will be seen that some of the methods are more elegant in their application to this theorem than others are; and in making the experiment upon different theorems, the student will often find the facility of the applications reversed.

Then, since (1, 3) are perpendicular, we have (15, *note*)

$$h\,(-\,h_1) = -\,kk_2, \text{ or } hh_1 = kk_2 \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

And since (2, 4) are perpendicular, we have

$$(-\,h_1)\,h = -\,kk_1, \text{ or } hh_1 = kk_1 \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Comparing (5, 6), we see that $k_2 = k_1$, or both the perpendiculars AD, BE, cut equal lines from FY: that is, they pass through the same point in FC.

Second solution.

The equations (1, 2) are as in the preceding solution; and since AD, BE are perpendicular to BC, CA, their equations are

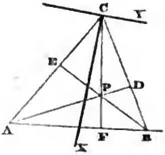
$$y = -\frac{h_1}{k}(x - h) \ldots\ldots (3). \qquad y = \frac{h}{k}(x + h_1) \ldots\ldots (4).$$

When each of these lines (3, 4) cuts the perpendicular CF, we shall have $x = 0$, and the co-ordinates y have in those cases respectively the values

$$y = \frac{h_1\,h}{k}, \text{ and } y = \frac{h\,h_1}{k}.$$

These being equal, the two perpendiculars, AD, BE, cut the third, CF, in the same point as, suppose, P.

Third solution.



Let lines bisecting the interior and exterior angles at C, be taken as the axes of x and y respectively. Denote the angle C by C, and the including sides AC, BC, by b and a.

Then the co-ordinates $x_1\,y_1$ and $x_2\,y_2$ of B and A are respectively

$$x_1 = a\cos\tfrac{1}{2}\mathrm{C}, \text{ and } y_1 = a\sin\tfrac{1}{2}\mathrm{C},$$
$$x_2 = b\cos\tfrac{1}{2}\mathrm{C}, \text{ and } y_2 = -\,b\sin\tfrac{1}{2}\mathrm{C}.$$

The equation of the line AB will therefore be, on insertion of these values of $x_1\,y_1$ and $x_2\,y_2$ in the usual formula (*p.* 256, *eq.* 11),

$$y = \frac{a+b}{a-b}\tan\tfrac{1}{2}\mathrm{C}\,.\,x - \frac{2ab\sin\frac{1}{2}\mathrm{C}}{a-b} \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the equation of CF, perpendicular to this from the origin, is

$$y = -\frac{a-b}{a+b}\cot\tfrac{1}{2}\mathrm{C}\,.\,x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The equations of the lines CB and CA are also respectively

$$y = x\tan\tfrac{1}{2}\mathrm{C} \ldots\ldots\ldots (3). \qquad y = -\,x\tan\tfrac{1}{2}\mathrm{C} \ldots\ldots\ldots\ldots\ldots (4).$$

The perpendicular from $x_1\,y_1$ upon (4) is readily found.

$$y - a\sin\tfrac{1}{2}\mathrm{C} = (x - a\cos\tfrac{1}{2}\mathrm{C})\cot\tfrac{1}{2}\mathrm{C}, \text{ or}$$
$$y = x\cot\tfrac{1}{2}\mathrm{C} - a\cos\mathrm{C}\operatorname{cosec}\tfrac{1}{2}\mathrm{C} \ldots\ldots\ldots\ldots\ldots (5).$$

And that from $x_2\,y_2$ upon (3) is, in like manner,

$$y + b\sin\tfrac{1}{2}\mathrm{C} = -\,(x - b\cos\tfrac{1}{2}\mathrm{C})\cot\tfrac{1}{2}\mathrm{C}, \text{ or}$$
$$y = -\,x\cot\tfrac{1}{2}\mathrm{C} + b\cos\mathrm{C}\operatorname{cosec}\tfrac{1}{2}\mathrm{C} \ldots\ldots\ldots\ldots\ldots (6).$$

Equate the value of y in (2) successively with those in (5) and (6), and resolve the resulting equations for x: then in both cases we find

$$x = \tfrac{1}{2}(a + b)\cos\mathrm{C}\operatorname{cosec}\tfrac{1}{2}\mathrm{C}.$$

As before, therefore, the three lines pass through the same point, P.

Fourth method.

[*Fig. to first solution.*]

Employing *oblique axes*, denote that of x by AB, and that of y by AC; also let CF, BE, AD, be the perpendiculars from C, B, and A, respectively, and let CF, BE, meet in P. Put

$$\mathrm{AB} = h,\ \mathrm{AC} = k,\ \mathrm{AF} = h_1 = k \cos \mathrm{A}, \text{ and } \mathrm{AE} = k_1 = h \cos \mathrm{A}.$$

Then the equations of BE, CF, become (*note, p.* 257),

$$hy + k_1 x = hk_1, \text{ or } y + x \cos \mathrm{A} = h \cos \mathrm{A} \ldots\ldots\ldots\ldots (1).$$

$$h_1 y + kx = h_1 k, \text{ or } y \cos \mathrm{A} + x = k \cos \mathrm{A} \ldots\ldots\ldots\ldots (2).$$

To find the line AP, which passes through the origin and the intersection of (1, 2), multiply (1) by k, and (2) by h, and subtract; then

$$y = \frac{h - k \cos \mathrm{A}}{k - h \cos \mathrm{A}} . x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

But the equation of BC is, (*eq.* 15, *note*),

$$hy + kx = hk, \text{ or } y = -\frac{k}{h}x + k; \text{ or } a_1 = -\frac{k}{h}.$$

Then this value, substituted in (*eq.* 21), gives, as the equation of the perpendicular from A to BC, the following:—

$$y = -\frac{1 - \frac{k}{h} \cos \mathrm{A}}{-\frac{k}{h} + \cos \mathrm{A}} . x = \frac{h - k \cos \mathrm{A}}{k - h \cos \mathrm{A}} . x \ldots\ldots\ldots\ldots (4).$$

The identity of (3, 4) shows that the line (4), which is perpendicular to BC, coincides with that drawn from A through the intersection, P, of the other two perpendiculars, and hence establishes the theorem.

Fifth method.

Employ (1, 2, 3) of the last method, which are the equations of the three perpendiculars, and subject the expression to the criterion of (*eq.* 35). The lines reduced to the form required are

$$y = - x \cos \mathrm{A} + h \cos \mathrm{A},$$

$$y = - x \sec \mathrm{A} + k,$$

$$y = \frac{h - k \cos \mathrm{A}}{k - h \cos \mathrm{A}} . x,$$

and the expression is reduced to

$$\{(-\cos \mathrm{A}.k) - \sec \mathrm{A}.h \cos \mathrm{A}\} + \left\{-\sec \mathrm{A}.0 - \frac{h - k \cos \mathrm{A}}{k - h \cos \mathrm{A}}\right\} + \frac{h - k \cos \mathrm{A}}{k - h \cos \mathrm{A}} . h \cos \mathrm{A} - \cos \mathrm{A}.0,$$

which upon reduction becomes successively,

$$- k \cos \mathrm{A} + h - \frac{h - k \cos \mathrm{A}}{k - h \cos \mathrm{A}}\{k - h \cos \mathrm{A}\} = \frac{(h - k \cos \mathrm{A}) - (h - k \cos \mathrm{A})}{k - h \cos \mathrm{A}} = 0,$$

and therefore the three lines pass through one point.

Sixth solution.

[*Fig. solution* 1.]

Take AD as origin of θ; denote DAC, DAB, by γ and $-\beta$, and AD by h: these conditions fix all the lines of the figure. Then

$$\mathrm{AF} = \mathrm{AC} \cos (\gamma + \beta) = h \sec \gamma \cos (\gamma + \beta),$$

and the equation of CF is therefore

$$h \sec \gamma \cos (\gamma + \beta) = r \cos (\theta + \beta) \ldots\ldots\ldots\ldots\ldots (1).$$

Similarly, that of BE is

$$h \sec (-\beta) \cos (-\gamma - \beta) = r \cos (\theta - \gamma) \ldots\ldots\ldots\ldots (2).$$

And when each of these cuts AD, we have $\theta = 0$, and the two results become

$$h \sec \gamma \cos (\gamma + \beta) = r \cos \beta, \text{ and } h \sec \beta \cos (\gamma + \beta) = r \cos \gamma.$$

These give

$$r = \frac{h \sec \gamma \cos (\beta + \gamma)}{\cos \beta} = h \sec \beta \sec \gamma \cos (\beta + \gamma),$$

$$\text{and } r = \frac{h \sec \beta \cos (\beta + \gamma)}{\cos \gamma} = h \sec \beta \sec \gamma \cos (\beta + \gamma);$$

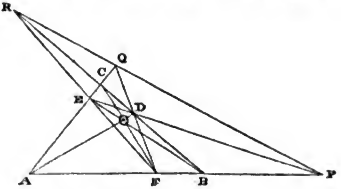
which being identical, the two perpendiculars, BE, CF, intersect the perpendicular AD in the same point, P.

Scholium.

These several methods of establishing the same proposition, are given in order to furnish examples to the student of almost every kind of treatment of which this class of theorems admits. As a still further exercise, the student is recommended to try the use of AB as axis of x, and a line through A at right angles to AB as axis of y. For this purpose it is re-proposed in *Ex.* 7 of the following exercises: and by methods having considerable analogy to one or other of these the other two theorems there enunciated will be proved.

PROP. II.

If straight lines, AD, BE, CF, be drawn through the angles A, B, C, of a triangle and any point O, to meet the opposite sides in D, E, F; and if EF, FD, DE, be drawn to meet the sides of the triangle in R, Q, P, these points P, Q, R, will be in the same straight line.



Take RB, RF, as oblique axes of x and y respectively; and put $RB = a$, $RC = a_1$, $RF = b$, $RE = b_1$, and $RD = c$. Then we have the several equations of the lines concerned, as below.

$$\text{(FB)} \ldots\ldots\ ay + bx = ab \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\text{(ED)} \ldots\ldots\ cy + b_1x = b_1c \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\text{(EC)} \ldots\ldots\ a_1y + b_1x = a_1b_1 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\text{(FD)} \ldots\ldots\ cy + bx = bc \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Cross-multiply (1, 2) by b_1c, ab, and subtract: then

$$ac(b_1 - b)\,y + bb_1(c - a)\,x = 0 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

which denotes the line RP, passing through the origin R and the intersection of AB, ED.

In like manner, from (3, 4) we obtain the equation of RQ,

$$a_1c(b - b_1)\,y + bb_1(c - a_1)\,x = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

It now only remains to show, that under the conditions of the question, (5) and (6) are identical; in order that RP and RQ may coalesce, or R, Q, P, be in one line.

As above, the equations of the lines EB, FC, are found to be:

$$\text{(EB)}\ldots\ldots\ ay + b_1x = ab_1 \ldots\ldots\ldots\ldots (7),$$

$$\text{(CF)}\ldots\ldots\ a_1y + bx = a_1b \ldots\ldots\ldots\ldots (8).$$

Cross-multiply and add; then we have (*note prop. vi.*) the equation of the line AO expressed by

$$aa_1\,(b + b_1)\,y + bb_1\,(a + a_1)\,x = 2aa_1\,bb_1 \ldots\ldots\ldots\ldots (9).$$

Now when AO cuts the axis RB of x, we have $y = 0$ and $x = c$; but in this case (9) gives

$$x = \frac{2aa_1}{a + a_1} = c, \text{ or } a(c - a_1) = a_1(a - c) \ldots\ldots\ldots\ldots (10).$$

Insert (10) in (6) and reduce; then there results

$$ac\,(b_1 - b)\,y + bb_1\,(c - a)\,x = 0,$$

which is identical with (5). The theorem is hence established.

Or thus:

Construct as before, and take QA, QF, for the axes of x and y.

Put $QA = a_1$, $QE = a_2$, $QC = a_3$,

$QF = \beta_1$, $QD = \beta_2$; then we have for the equations of AF, ED, DC, FE, AD, CF.

$$\begin{array}{l|l}
\text{(AF)}\ldots\ldots \dfrac{x}{a_1} + \dfrac{y}{\beta_1} = 1 \ldots\ldots (1) & \text{(FE)}\ldots\ldots \dfrac{x}{a_2} + \dfrac{y}{\beta_1} = 1 \ldots (4) \\
\text{(ED)}\ldots\ldots \dfrac{x}{a_2} + \dfrac{y}{\beta_2} = 1 \ldots\ldots (2) & \text{(AD)}\ldots\ldots \dfrac{x}{a_1} + \dfrac{y}{\beta_2} = 1 \ldots (5) \\
\text{(DC)}\ldots\ldots \dfrac{x}{a_3} + \dfrac{y}{\beta_2} = 1 \ldots\ldots (3) & \text{(CF)}\ldots\ldots \dfrac{x}{a_3} + \dfrac{y}{\beta_1} = 1 \ldots (6)
\end{array}$$

Subtracting (2) from (1) and (3) from (4), gives for the equations of QP and QR respectively,

$$\left(\frac{1}{a_1} - \frac{1}{a_2}\right) x + \left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) y = 0 \ldots\ldots\ldots\ldots (7)$$

$$\left(\frac{1}{a_2} - \frac{1}{a_3}\right) x + \left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) y = 0 \ldots\ldots\ldots\ldots (8).$$

Adding equations (5) and (6), or (1) and (3), and also (2) and (4), we get for the equation of BE,

$$\left(\frac{1}{a_1} + \frac{1}{a^3}\right) x + \left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right) y = 2 \ldots\ldots\ldots\ldots (9)$$

$$\text{or } \frac{2}{a_2}.\,x + \left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right) y = 2 \ldots\ldots\ldots\ldots (10).$$

$$\text{Wherefore, } \frac{1}{a_1} + \frac{1}{a_3} = \frac{2}{a_2}, \text{ or } \frac{1}{a_1} - \frac{1}{a_2} = \frac{1}{a_2} - \frac{1}{a_3}.$$

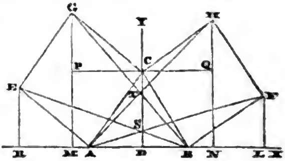
Consequently, equations (7) and (8) are identical, and therefore P, Q, R, are in one line.

Scholium.

This proposition is identical with that at *p.* 221, and is solved in that place by means of transversals. Most of the theorems relative to rectilinear transversals may be established in a similar manner; and the student is urged to attempt their demonstration by such means. He will thus be led gradually to perceive the intimate relation that subsists between different and apparently unconnected methods of investigation, and to appreciate the peculiar advantages and beauties of each.

PROP. III.

Squares AG, BH, are described on two adjacent sides, AC, CB, of a triangle ACB; and lines are drawn from each angle E, G, of the square on AC to the angle B of the triangle, and lines from F, H, of the square on BC to the angle A; E, F, being those angles of the squares contiguous to the side AB of the triangle, and G, H, those most remote from it: it is required, then, to prove (1) *that AF, BE, intersect in CD, the perpendicular from C to AB; and* (2), *that AH, BG, intersect at right angles.*



Draw the perpendicular CD, and take DB, DC, as the positive axes of x and y. Draw ER, GM, HN, FL, perpendicular, and PCQ parallel to AB. Let DA $= a$, DB $= b$, and DC $= c$. Then, by obvious geometrical properties of the figure, the triangles CDA, ARE, CPG, are equal in all respects, as are likewise CDB, BLF, CQH, equal in all respects. Whence we have the co-ordinates of the several points denoted thus:—

$-a, 0$ for A	$-(a+c), a$ for E	$-c, (a+c)$ for G,
$b, 0$ for B	$(b+c), b$ for F	$c, (b+c)$ for H.

The equations for the four lines specified in the theorem are hence found at once to be

$$(\text{AF})\ldots\ldots\ldots y(a+b+c) - bx = ab,$$
$$(\text{BE})\ldots\ldots\ldots y(a+b+c) + ax = ab,$$
$$(\text{AH})\ldots\ldots\ldots y(a+c) - x(b+c) = a(b+c),$$
$$(\text{BG})\ldots\ldots\ldots y(b+c) + x(a+c) = b(a+c).$$

When AF, BE, meet the perpendicular, we shall have in both cases $x = 0$; and this gives, as the distance at which each meets CD from D, the value

$$y = \frac{ab}{a+b+c}.$$

They intersect, therefore, in the same point S, as stated in the enunciation.

Again, put the equations of AH, BG, under the form,

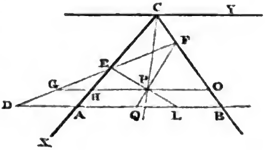
$$y = +\frac{b+c}{a+c}x + \frac{b+c}{a+c}.a,$$
$$y = -\frac{a+c}{b+c}x + \frac{a+c}{b+c}.b;$$

in which the co-efficients of x fulfil the criterion of perpendicularity.

The theorem is, therefore, proved.

PROP. IV.

Let ABC be a triangle, having three given points D, L, Q, in the side AB; through D draw any straight line DEF, cutting AC, BC, in E and F respectively, and through E, F, straight lines LE, QF, intersecting in P: determine the locus of P.



Take CY parallel to AB for the axis of y, CA being that of x. Denote the values of y for the three given points D, L, Q, by y_1, y_2, y_3, and the common value of x by x_1; also put for the equation of CB referred to the same system of co-ordinates,

$$y = \mathrm{A}x \text{ .... (1).}$$

Also, since the line DEF passes through the one given point $x_1\, y_1$, we have its equation, a being indeterminate,

$$y - y_1 = a\,(x - x_1) \text{ .... (2).}$$

The co-ordinates of E are found from combining (2) with $y = 0$, which gives

$$y = 0,\ x = -\frac{y_1 - ax_1}{a} \text{ .... (3);}$$

and similarly, those of F are found from (1, 2), giving

$$y = \frac{\mathrm{A}\,(y_1 - ax_1)}{\mathrm{A} - a},\ x = \frac{y_1 - ax_1}{\mathrm{A} - a} \text{ .... (4).}$$

The equation of QF through $(y_3\ x_1)$ and (4) is, therefore,

$$-(y - y_3)\,(\mathrm{A}x_1 - y_1) + \mathrm{A}\,(y_3 - y_1)\,(x - x_1) = a\,(y_3 - \mathrm{A}x_1)\,(x - x_1) \text{ .. (5).}$$

Similarly, the equation of EL through $(y_2\ x_1)$ and (3) is

$$y_1\,(y - y_2) = ay_2\,(x - x_1) \text{ .... (6).}$$

Eliminating the indeterminate quantity a from (5, 6) we find in the usual manner

$$\frac{y_1\,(y - y_2)}{y_2\,(x - x_1)} = a = \frac{\mathrm{A}\,(y_3 - y_1)}{y_3 - \mathrm{A}x_1} - \frac{(\mathrm{A}x_1 - y_1)\,(y - y_3)}{(x - x_1)\,(y_3 - \mathrm{A}x_1)};$$

or, on reduction and mutual cancels, it becomes simply,

$$\{\mathrm{A}x_1\,(y_2 - y_1) - y_1\,(y_2 - y_3)\}\ y - \mathrm{A}y_2\,(y_3 - y_1)\ x = 0 \text{ .... (7).}$$

This is obviously the equation of a line CP through the origin of co-ordinates.

Another solution.

The solution above given, involves no considerations but those which purely belong to the doctrine of co-ordinates. It will often happen, however, that the introduction of geometrical properties will simplify and shorten the solution. This assertion will be illustrated by the following solution. See the *Mathematical Repository, N. S. vol. ii. Ex.* 161.

Through P (*fig. to last solution*) draw GHO parallel to AB, meeting DF, CA, CB, in G, H, O.

Put DL $= a$, QB $= b$, DQ $= c$, AL $= d$, AB $= m$, AC $= n$, HC $= x$, and HP $= y$. Then, by similar triangles,

$$\mathrm{AL : PH :: LD : PG, \text{ or } } d : y :: a : \mathrm{PG} = \frac{ay}{d};$$

$$\mathrm{DQ : PG :: QB : PO, \text{ or } } c : \frac{ay}{d} :: b : \mathrm{PO} = \frac{aby}{cd};$$

$$\text{and hence, } \mathrm{HO} = \mathrm{HP} + \mathrm{PO} = y + \frac{aby}{cd} = \frac{ab + cd}{cd}.y; \text{ and finally,}$$

$$\mathrm{AB : AC :: HO : HC, \text{ or } } m : n :: \frac{ab + cd}{cd}.y : x, \text{ or}$$

$$mcdx = n\,(ab + cd)\,y,$$

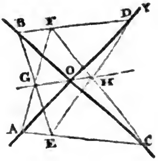
the equation of a line through the origin, as before.

Scholium.

By processes of this kind, a considerable number of the theorems relating to rectilinear transversals may be solved. Even where the propositions are determinate, they may be solved by finding the equations of two straight lines, in one of which, according to one set of the conditions, a point may be situated, and in the other of which, according to another set of the conditions (each set must contain $n - 1$ of the conditions, where the entire number is n): and their intersection furnishes the solution.

PROP. V.

Upon the opposite sides, AC, BD, of the quadrilateral figure ACBD, as bases, triangles are described, the vertices of each E and F being in the base AC, BD of the other triangle: it is required to prove, that if AD and BC intersect in O, AF and BE in G, and FC, ED, in H, the points G, O, H, will be in one straight line.



TAKE BC, AD, as axes of x and y, C, D, being on the positive branches; and let the notation of co-ordinates for the several points be as follows:

(A) ...... $(0, -a)$
(B) ...... $(-b, 0)$
(C) ...... $(c, 0)$
(D) ...... $(0, d)$
(F) ...... (x_1, y_1)
(E) ...... (x_2, y_2).

Then we have the equations of the several lines concerned as below:

(BD).... $-by + dx = -bd$ .... (1)
(AC)...... $cy - ax = -ac$ .... (2)
(AF)...... $x_1(y + a) = x(y_1 + a)$ .... (3)
(BE)...... $y(x_2 + b) = y_2(x + b)$ .... (4)
(FC)..... $y(x_1 - c) = y_1(x - c)$ .... (5)
(DE)...... $x_2(y - d) = x(y_2 - d)$ .... (6).

The intersection of AF, BE, gives, as the co-ordinates of G,

$$X_1 = \frac{(ax + by_2 + ab)\,x_1}{(x_2 + b)(y_1 + a) - x_1 y_2};\quad Y_1 = \frac{(ax_1 + by_1 + ab)\,y_2}{(x_2 + b)(y_1 + a) - x_1 y_2} \quad\ldots\ (7),$$

and that of FC, DE, give those of H,

$$X_2 = \frac{(dx_1 + cy_1 - cd)\,x_2}{x_2 y_1 - (x_1 - c)(y_2 - d)};\quad Y_2 = \frac{(dx_2 + cy_2 - cd)\,y_1}{x_2 y_1 - (x_1 - c)(y_2 - d)} \quad\ldots\ (8).$$

Again, to simplify these results, put D_1 and D_2 for the denominators of the values in (7, 8); and recollecting that $x_1\,y_1$ is in (1) and $(x_2\,y_2)$ in (2), we have at once

$$by_1 = dx_1 + bd \quad\ldots\ (9)$$
$$cy_2 = ax_2 - ac \quad\ldots\ (10).$$

Substituting these, we shall have the equations (7, 8) reduced to the forms

$$X_1 = \frac{a(b + c)\,x_1 x_2}{cD_1};\quad Y_1 = \frac{b(a + d)\,y_1 y_2}{dD_1} \quad\ldots\ (11)$$

$$X_2 = \frac{d(b + c)\,x_1 x_2}{bD_2};\quad Y_2 = \frac{c(a + d)\,y_1 y_2}{aD_2} \quad\ldots\ (12).$$

Now the criterion of GH passing through the origin O is easily applied to these expressions, viz.:

$$X_1 Y_2 - X_2 Y_1 = 0;$$

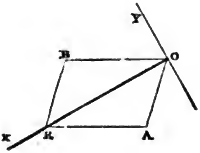
for it simply becomes

$$\frac{ac(a + d)(b + c)\,x_1 x_2 y_1 y_2}{ac\,D_1 D_2} - \frac{bd(b + c)(a + d)\,x_1 x_2 y_1 y_2}{bd\,D_1 D_2};$$

an expression obviously equal to zero, from the identity of its terms.

PROP. VI.

If from any point perpendiculars be drawn to two adjacent sides of a parallelogram and their co-terminal diagonal, the rectangle contained by the diagonal and its perpendicular will be equal to the sum of the rectangles of the two sides with their respective perpendiculars.



Refer the system to the diagonal OR and a perpendicular OY. Put

$$\text{BOR} = \alpha,\ \text{AOR} = \beta,\ \text{OR} = r.$$

Then the equations of OB, OA, OR, will be

$$(\text{OB}) \ldots\ y = + x \tan \alpha \ldots\ldots (1)$$
$$(\text{OA}) \ldots\ y = - x \tan \beta \ldots\ldots (2)$$
$$(\text{OR}) \ldots\ y = 0 \ldots\ldots (3).$$

The perpendiculars p_1, p_2, p_3, from the point $(x_1\, y_1)$ upon these are

$$p_1 = y_1 \cos \alpha - x_1 \sin \alpha \ldots\ldots (4)$$
$$p_2 = y_1 \cos \beta + x_1 \sin \beta \ldots\ldots (5)$$
$$p_3 = y_1 \ldots\ldots (6).$$

$$\text{Also OB} = \frac{r \sin \beta}{\sin (\alpha + \beta)},\ \text{and OA} = \frac{r \sin \alpha}{\sin (\alpha + \beta)}.$$

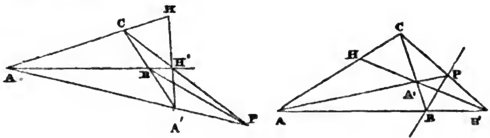
Whence,

$$\text{OB} \cdot p_1 + \text{OA} \cdot p_2 = \frac{r (y_1 \cos \alpha - x_1 \sin \alpha) \sin \beta}{\sin (\alpha + \beta)} + \frac{r (y_1 \cos \beta + x_1 \sin \beta) \sin \alpha}{\sin (\alpha + \beta)}$$
$$= \frac{r y_1 (\sin\beta \cos \alpha + \sin \alpha \cos \beta)}{\sin (\alpha + \beta)} - \frac{r x_1 (\sin \alpha \sin \beta - \sin \alpha \sin \beta)}{\sin (\alpha + \beta)}$$
$$= r y_1 = (by\ 6)\ \text{OR} \cdot p_3.$$

This theorem, which is of considerable use in the doctrine of statical moments, is generally established by means of geometry. Besides the proofs usually given by writers on mechanics, three very elegant ones may be seen in No. 5 of Professor Gill's *Math. Misc.*, New York, 1839-40. Another will be given in the chapter on *Statics*.

PROP. VII.

Let ABC be a triangle, and H a point in one of its sides, or its prolongation either way; through H draw lines, as HH′, intersecting BC, BA, in A′, H′. Join AA′ and CH′, and let these lines intersect in P; then the locus of P is a straight line passing through B.



Take AB, AC, for the axes of co-ordinates, and put AH′ $= \alpha$, AH $= \beta$, the sides opposite to A, B, C, being denoted, as usual, by a, b, c. Then the equations of the lines BC, HH′, and CH′, are respectively

$$\frac{y}{b}+\frac{x}{c}=1 \ldots\ldots\ldots\ldots\ldots\ldots \text{(1) equation of BC,}$$

$$\frac{y}{\beta}+\frac{x}{a}=1 \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)} \ldots\ldots \text{HH}'$$

$$\frac{y}{b}+\frac{x}{a}=1 \ldots\ldots\ldots\ldots\ldots\ldots \text{(3)} \ldots\ldots \text{CH}'.$$

Subtract (2) from (1) and we have for the equation of AA′,

$$\left(\frac{1}{b}-\frac{1}{\beta}\right)y+\left(\frac{1}{c}-\frac{1}{a}\right)x=0 \ldots\ldots\ldots\ldots \text{(4).}$$

Adding (4) and (3), we get for the equation of the locus of P,

$$\left(\frac{2}{b}-\frac{1}{\beta}\right)y+\frac{x}{c}=1 \ldots\ldots\ldots\ldots\ldots\ldots \text{(5).}$$

To find where this line cuts the axis of x, make $y=0$ in (5), and we get

$$x=c.$$

Consequently, the locus of P is a straight line passing through B.

EXERCISES ON THE STRAIGHT LINE.

1. Construct the following equations on the same co-ordinate axes ($a = 75^\circ$), and to the same scale :—

$-5x+4y-10=0$ | $5x-4y-10=0$

$-5x-4y-10=0$ | $5x+4y-10=0.$

2. Verify by the scale the calculated values of y, when $4x+3=0$; and find the inclination of each line to each of the axes of x and y.

3. Find the rectangular equations of the lines passing through each two of the four following points, and the distances from the origin at which they severally cut the axes, together with the area of the triangle formed by them.

$(-5, 4)$, $(5, -4)$, $(-5, -4)$, and $(5, 4)$.

4. Construct the lines $2y+5=0$, $4x-2=0$, and $x+y=0$; and find the equation of the line which cuts from the axes of x and y the segments a and b.

5. Two straight lines passing through the given point $(x_1\, y_1)$, and making a given angle κ, cut from the axes of x, segments which, reckoned from the origin, are in a given ratio $m:n$; find the equations of these lines, and the segments they cut from the axis of y.

6. Find the locus of the vertex of the triangle when the following parts are given:
   (1). The base and area.
   (2). The base, and the sides equal (or isosceles).
   (3). When the vertical angle and one of the angles at the base are given, find the locus of a point which divides the base in the given ratio $m:n$.

7. Show that each of the following triads of lines meet in one point :—
   (1). The lines from the angles bisecting the opposite sides.
   (2). The lines bisecting the vertical angles.
   (3). The line bisecting one interior angle and the other two exterior ones.
   (4). Lines from the angles to the points of contact of the inscribed circles.
   (5). Lines similarly drawn to the escribed points of contact.
   (6). Lines perpendicular to the sides from the angles.
   (7). Lines bisecting the sides perpendicularly.

8. If upon the sides of any triangle as diagonals, parallelograms be described

whose sides are parallel to two given lines, the other three diagonals will intersect in one point.

9. In the figure to *Euc.* 47, *i.*, the lines AL, BK, CM, intersect in one point; and the same holds true, when, instead of the squares BG, CH, BE, any similar rectangles are taken whose homologous sides are AB, BC, CA.

10. Transform the equation of a line passing through the given points $(x_1 y_1)$, $(x_2 y_2)$, into the forms:—

$$\frac{x}{a}+\frac{y}{b}=1, \text{ and } \frac{\cos\theta}{a_1}+\frac{\sin\theta}{b_1}=\frac{1}{r};$$

and assign a, a_1, b, b_1, in terms of x_1, x_2, y_1 y_2.

11. Lines are drawn from $2n$ points, A_1, A_2, A_3, ... A_{2n}, to meet in one point Z: and the sum of the squares of n of these lines is equal to the sum of the squares of the remaining n of them: it is required to show that Z is always situated in a straight line which may be found.

12. The three sides of a triangle and a point are given; and through the point a line is drawn to cut the sides of the triangle, and to be divided at the points of section in a given ratio: it is required to find the equation of that line.

13. If two triangles be described on the opposite sides of any quadrilateral as bases, each having its vertex in the opposite side: then the points in which the two sides of one triangle cut the two sides of the other, will be always in a line which passes through a fixed point. The solution to be by taking AB, CD, produced to meet as co-ordinate axes. See the solution given at *p.* 276.

14. If three sides of a triangle be cut by an arbitrary line, and a point be taken in it so that the three segments intercepted by the triangle and the point shall have the given ratios of m, n, p: then the locus of the point so taken is a straight line.

15. Upon the line CG any how divided in D, let squares CABD, DEFG, be described (on the same side if D lie between C and G, and on the contrary if D be in extension either way of CG), and join AG, CF: their intersection H will be in the line BED.

16. If exterior squares be described upon the three sides of a triangle, and the three pairs of contiguous corners of these squares be joined; then lines drawn from the angles of the triangle perpendicular to the lines which join the contiguous corners, will all pass through one point.

17. ABCD is a parallelogram; draw DE perpendicular to the diagonal BC, and from the points B and C, perpendiculars to AB, AC: these three perpendiculars shall meet in one point.

18. If the three following pairs of lines intersect in the points G, H, K, respectively, show that G, H, K, are in one line; viz. lines drawn from the angles at the base to bisect the sides, lines drawn from the same angles perpendicular to the sides, and lines bisecting the sides at right angles.

19. From a point, P, m, lines are drawn under specified angles, α_1, α_2 .. α_m, to terminate in given lines; and their sum is a given magnitude: show that the locus of P is a straight line, and assign the constants of its equation.

20. Assign the area of a polygon, the co-ordinates $(x_1 y_1)$, $(x_2 y_2)$ .... $(x_n y_n)$, of whose angular points are given.

21. Three straight lines, BA, AC, CB, are given by position, as likewise three points, D, E, F: it is required to describe through D, E, F, the sides HK, KG, GH, of a triangle, having its angular points G, H, K, situated respectively on the lines BC, CA, AB.

22. Point out the method of resolving the same problem when there are m lines and m points given.

23. Two lines, AB, AC, are given meeting in O, and A, B, are points in them: it is required to find the equation of the line which, passing through a given point P, cuts them in D and E respectively, so that the segments AC, BD, may have a given ratio, $m : n$.

24. To cut them so that AC, BD, may contain a given rectangle.

25. The line expressed by the equation $y = ax + b$ is referred to the co-ordinate angle α; show that when it is referred to a new origin hk, and to new axes making angles β, γ, with the original axis of x, the equation will become

$$y\{\sin\gamma - a\sin(\alpha-\gamma)\} + x\{\sin\beta - a\sin(\alpha-\beta)\} + (k - ah - b)\sin\alpha = 0.$$

26. Let a point xy, referred to axes having α for their angle of ordination, be transformed to a polar system by writing ra', rb', for x and y; then

$$a'^2 + 2a'b'\cos\alpha + b'^2 = 1.$$

27. Show that if a line pass through the points $(x_1\, y_1)$, $(x_2\, y_2)$, and another pass through $(x_3\, y_3)$, $(x_4\, y_4)$, these lines will intersect in a point whose co-ordinates are

$$X = -\frac{(x_1 y_2 - x_2 y_1)(x_3 - x_4) - (x_3 y_4 - x_4 y_3)(x_1 - x_2)}{(y_1 - y_2)(x_3 - x_4) - (y_3 - y_4)(x_1 - x_2)}$$

$$Y = +\frac{(x_1 y_2 - x_2 y_1)(y_3 - y_4) - (x_3 y_4 - x_4 y_3)(y_1 - y_2)}{(y_1 - y_2)(x_3 - x_4) - (y_3 - y_4)(x_1 - x_2)}.$$

28. The equation of a straight line through the points $x_1\, y_1$, $x_2\, y_2$, is

$$\frac{x}{\left(\dfrac{x_1 y_2 - x_2 y_1}{y_2 - y_1}\right)} + \frac{y}{\left(\dfrac{x_1 y_2 - x_2 y_1}{x_2 - x_1}\right)} = 1.$$

29. The point of intersection of the two lines whose equations are

$$\frac{x}{a} + \frac{y}{b} = 1, \text{ and } \frac{x}{a_1} + \frac{y}{b_1} = 1,$$

is that whose co-ordinates are

$$X = -\frac{aa_1(b - b_1)}{a_1 b - ab_1}, \text{ and } Y = \frac{bb_1(a - a_1)}{ab_1 - a_1 b}.$$

30. These lines intersect under an angle i such that

$$\cos i = \frac{aa_1 + bb_1}{\sqrt{(a^2 + b^2)(a_1^2 + b_1^2)}} = \frac{1 + \dfrac{bb_1}{aa_1}}{\sqrt{\left(1 + \dfrac{b^2}{a^2}\right)\left(1 + \dfrac{b_1^2}{a_1^2}\right)}}.$$

31. The perpendicular to the line $\frac{x}{b} + \frac{y}{b} = 1$, drawn through the point $x_1\, y_1$, is expressed by the equation

$$b(y - y_1) = a(x - x_1)$$

32. Perpendiculars through $x_1\, y_1$ and $x_2\, y_2$, to the lines $y = p_1 x + q_1$ and $y = p_2 x + q_2$, intersect in the point whose co-ordinates are

$$X = \frac{p_2(p_1 y_1 + x_1) - p_1(p_2 y_2 + x_2)}{p_2 - p_1};\ Y = -\frac{(p_1 y_1 + x_1) - (p_2 y_2 + x_2)}{p_2 - p_1}.$$

33. There are given m points, $x_1 y_2$, $x_2 y_2$, … $x_m y_m$, and another point $x_0 y_0$: it is required to find the equation of a line through $x_0 y_0$ such that if perpendiculars to it, p_1, p_2, … p_m, be drawn from the given points, and n_1, n_2, … n_m, be given numbers, then there shall subsist the relation (the perpendiculars on opposite sides of the line having contrary signs)

$$n_1 p_1 + n_2 p_2 + \dots n_m p_m = 0.$$

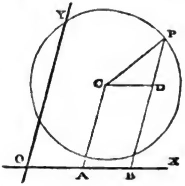
34. Show that for all possible positions of $x_0 y_0$, the lines drawn as in the preceding question will pass through one specific point; and assign its co-ordinates.

THE CIRCLE.

PROP. I.

To find the general form of the equation of the circle, referred to rectilinear co-ordinates.

The characteristic property of the circle, which serves as the definition of Euclid (*def.* 15, 16, *i.*) is well adapted to the solution of this problem.



Let the origin of co-ordinates be O, the angle of ordination XOY be a, P be a point in the circle whose centre is C: let $h\ k$ be the co-ordinates of the centre C, xy those of the variable point P in the circle, and ρ the radius CP of the circle. Then

$$\text{CD}^2 - 2\text{CD}\,.\,\text{DP}\cos\text{CDP} + \text{DP}^2 = \text{CP}^2, \text{ and}$$
$$\text{CDP} + \text{XOY} = \Pi,$$

we have, by substitution, the equation

$$(x-h)^2 + 2(x-h)\,(y-k)\cos a + (y-k)^2 = \rho^2 \ldots\ldots\ldots \text{(1)};$$

which is the general form of the equation, in terms of the radius, the angle of ordination, and the co-ordinates of the centre.

It may also by expansion be reduced to

$$x^2 + 2xy\cos a + y^2 - 2x\,(h + k\cos a) - 2y\,(k + h\cos a) + h^2 + 2hk\cos a + k^2 = \rho^2 \text{ (2)};$$

and which might be written

$$x^2 + 2xy\cos a + y^2 + ax + by + c = 0 \ldots\ldots\ldots\ldots$$

In these equations, it will be remarked that x^2 and y^2 have equal coefficients. This, in fact, is the test of the equation of the second degree being a circle.

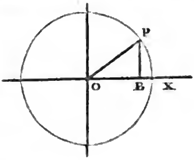
Again, if $a = \frac{1}{2}\pi$, we have $\cos a = 0$; and the *rectangular equation* is

$$x^2 + y^2 - 2hx - 2ky = \rho^2 - h^2 - k^2 \ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}.$$

Corollary.

There are two cases of position of the co-ordinates with respect to the circle referred to rectangular co-ordinates, which from their great simplicity, are very much used in investigating the properties of the circle. The first and most simple is, when the origin is at the centre; and the other when the origin is in the circumference, and one of the axes (as that of x) passes through the centre.

1. *The origin at the centre.*

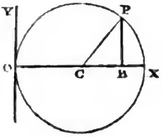


In this case $h = 0$, and $k = 0$; whence the equation is

$$x^2 + y^2 = \rho^2 \ldots\ldots\ldots\ldots \text{(4)};$$

which might have been obtained by an application of *Euc. i.* 47 to the triangle OBP; as the next might by the use of *Euc. iii.* 3, and 35.

2. *The centre on the axis of x, and origin on the curve.*



Here we have $h = \rho$, and $k = 0$; whence we get

$$\left.\begin{array}{r} y^2 + x^2 - 2\rho x = 0 \\ \text{or } y^2 = 2\rho x - x^2 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots \text{(5)}.$$

All other cases, whether referred to rectangular or oblique axes, are readily deducible in an equally simple manner.

PROP. II.

Having given the equation of a circle, to construct it in reference to given co-ordinates.

This requires the co-ordinates of the centre and the radius of the circle to be determined. Let the equation be

$$x^2 + y^2 + ax + by + c = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (6);$$

and this being compared with the general form in (3), we get

$$\left.\begin{array}{l} h = -\tfrac{1}{2}a,\ k = -\tfrac{1}{2}b, \\ \rho^2 = h^2 + k^2 - c \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

The centre can, therefore, be assigned, and the circle described.

In precisely the same way we may construct the problem in case of oblique axes of co-ordinates.

PROP. III.

To find the equation of a circle passing through two given points $(x_1\ y_1)$ *and* $(x_2\ y_2)$.

The general form of the equation is

$$x^2 + y^2 + ax + by + c = 0;$$

$$\text{and hence } x_1^2 + y_1^2 + ax_1 + by_1 + c = 0,$$

$$\text{and } x_2^2 + y_2^2 + ax_2 + by_2 + c = 0.$$

Subtract one of these last two results from the other; then,

$$(x_1 - x_2)\, a + (y_1 - y_2)\, b + (x_1^2 + y_1^2 - x_2^2 - y_2^2) = 0 \ldots\ldots\ldots (8).$$

Now as this contains two unknowns, a, b, (which are the doubles of the co-ordinates of the centre taken negatively, by *eq.* 7,) we cannot determine them absolutely. In fact, (8) is the equation of a straight line referred to co-ordinates a and b, and hence the centre of the circle through the two given points is always situated in a straight line parallel to that denoted by (8).

PROP. IV.

To assign the rectilineal equation of a circle described through three given points.

Denote them by (x_1y_1), (x_2y_2), (x_3y_3); and the circle as in equation (6). Then we shall have the four equations,

$$\begin{aligned} x^2 + y^2 + ax + by + c &= 0, \\ x_1^2 + y_1^2 + ax_1 + by_1 + c &= 0, \\ x_2^2 + y_2^2 + ax_2 + by_2 + c &= 0, \\ x_3^2 + y_3^2 + ax_3 + by_3 + c &= 0. \end{aligned}$$

Subtract these, each from the preceding; then we get

$$\begin{aligned} a(x - x_1) + b(y - y_1) + x^2 + y^2 - x_1^2 - y_1^2 &= 0, \\ a(x_1 - x_2) + b(y_1 - y_2) + x_1^2 + y_1^2 - x_2^2 - y_2^2 &= 0. \\ a(x_2 - x_3) + b(y_2 - y_3) + x_2^2 + y_2^2 - x_3^2 - y_3^2 &= 0. \end{aligned}$$

Cross-multiply, to eliminate a by subtraction in the usual manner; then there result the two equations,

$$\begin{aligned} &(x^2 - x_1^2 + y^2 - y_1^2)\,(x_1 - x_2) - (x_1^2 - x_2^2 + y_1^2 - y_2^2)\,(x - x_1), \\ &\qquad + b\{(y - y_1)\,(x_1 - x_2) - (y_1 - y_2)\,(x - x_1)\} = 0, \\ &(x_1^2 - x_2^2 + y_1^2 - y_2^2)\,(x_2 - x_3) - (x_2^2 - x_3^2 + y_2^2 - y_3^2)\,(x_1 - x_2), \\ &\qquad + b\{(y_1 - y_2)\,(x_2 - x_3) - (y_2 - y_3)\,(x_1 - x_2)\} = 0. \end{aligned}$$

Again, cross-multiply by the co-efficients of b; then there will result, after arranging the coefficients of the variables, the final equation:—

$$\left.\begin{array}{l}(x^2 + y^2)\,\{(y_1x_2 - y_2x_1) + (y_2x_3 - y_3x_2) + (y_3x_1 - y_1x_3)\} \\ -x\{(x_1{}^2 + y_1{}^2)(y_2 - y_3) + (x_2{}^2 + y_2{}^2)(y_3 - y_1) + (x_3{}^2 + y_3{}^2)(y_1 - y_2)\} \\ -y\{(x_1{}^2 + y_1{}^2)(x_2 - x_3) + (x_2{}^2 + y_2{}^2)(x_3 - x_1) + (x_3{}^2 + y_3{}^2)(x_1 - x_2)\}\end{array}\right\{$$

$$= (x_1{}^2 + y_1{}^2)(y_2x_3 - y_3x_2) + (x_2{}^2 + y_2{}^2)(y_3x_1 - y_1x_3) + (x_3{}^2 + y_3{}^2)(y_1x_2 - y_2x_1) \;\ldots\ldots\ldots (9).$$

This may have the arrangements of its coefficients changed; but the form here given, from the remarkable symmetry of its component parts, appears the most elegant that the result admits of.

PROP. V.

To find the co-ordinates of the intersections of a given straight line with a given circle.

Let them be

$$y = a_1x + b_1 \;\ldots\ldots\ldots\ldots (10),$$

$$x^2 + y^2 + ax + by + c = 0 \;\ldots\ldots\ldots\ldots (11).$$

Then, inserting the value of y from (1) in (2), we have

$$(1 + a_1{}^2)\,x^2 + (2a_1b_1 + a_1b + a)\,x + b_1{}^2 + bb_1 + c = 0 \;\ldots\ldots (12).$$

From which there result two values of x; and by means of (10), two corresponding values of y. The co-ordinates of the points of section are, therefore, known.

It is clear from this, that a circle can be cut in only two points by a straight line.

Scholium.

The process is precisely the same if the system of co-ordinates be oblique-angled; and no special remark is necessary on this more general case.

It is also obvious, that if the roots of (12) should be imaginary, the equations are those of a line and circle which have no one point common; or in other words, they do not meet at all. Moreover, when the two roots are equal, the line *touches* the circle, or the two points of section coalesce in the point of contact.

PROP. VI.

To find the equation of a line which, passing through a given point, (x_1y_1), touches a given circle,

$$(x - h)^2 + (y - k)^2 = \rho^2.$$

The equation of the line through (x_1y_1), p being at present undetermined, is

$$y - y_1 = p(x - x_1), \text{ or}$$

$$y - k = p(x - h) - p(x_1 - h) + (y_1 - k).$$

Substitute this value of $y - k$ in the equation of the circle; then, as in the preceding problem, we shall obtain the co-ordinates x of intersection of the arbitrary line and the circle. This gives

$$(1 + p^2)(x - h)^2 - 2p\{p(x_1 - h) - (y_1 - k)\}(x - h) + \{p(x_1 - h) - (y_1 - k)\}^2 = \rho^2;$$

from which the intersections can be found. But, that this line may represent a *tangent* to the circle, the two roots of this must be equal; or the second side of the equation, when the first is rendered a complete square, must be zero. This being done, that condition gives, after slight and obvious reductions,

$$p^2\{(x_1 - h)^2 - \rho^2\} + 2p(x_1 - h)(y_1 - k) + \{(y_1 - k)^2 - \rho^2\} = 0,$$

$$\text{or } p = \frac{(x_1 - h)(y_1 - k) \pm \rho\sqrt{(y_1 - k)^2 + (x_1 - h)^2 - \rho^2}}{(x_1 - h)^2 - \rho^2} \;\ldots\ldots\ldots (13),$$

which value of p, inserted in the equation of the line $y - y_1 = p(x - x_1)$ will be that of the tangent sought through $(x_1 y_1)$.

From this result we learn, that from the point $(x_1 y_1)$ there can *generally* be drawn two tangents to the same circle, the inclinations of which to the axis of x are designated by $\tan^{-1} p$, p being expounded by the two values found in (13).

These values will become equal when the radical part of the value of p becomes zero, and then only; that is, when

$$(y_1 - k)^2 + (x_1 - h)^2 = \rho^2,$$

or when $(x_1 y_1)$ is situated in the circumference of the circle.

Moreover, when the point $(x_1 y_1)$ is within the circle, we shall have $(y_1 - k)^2 + (x_1 - h)^2 - \rho^2$ *negative*, and the value of p therefore *imaginary*. In this case, then, no tangent can be drawn to the circle.

Both these results we know, from geometrical considerations, are true.

Corollaries.

1. The particular case where $(x_1 y_1)$ is in the circumference is often required, and the equation in this case assumes a very elegant and simple form. For, when the radical in p vanishes, the equation is at once changed to

$$(y - y_1)\{(x_1 - h)^2 - \rho^2\} = (x - x_1)(x_1 - h)(y_1 - k);$$

and since in this case $(x_1 - h)^2 - \rho^2 = -(y_1 - k)^2$, this becomes

$$(y - y_1)(y_1 - k) + (x - x_1)(x_1 - h) = 0 \ldots\ldots\ldots\ldots\ldots (14).$$

This formula is, in fact, the equation of a line through $(x_1 y_1)$ perpendicular to the line drawn through $(x_1 y_1)$ and (hk); which we know would be the tangent.

2. The same equation may also be put in a form having great resemblance to the equation of the circle, and easy to be recalled to memory, as follows:

Since $y - y_1 = (y - k) - (y_1 - k)$, and $x - x_1 = (x - h) - (x_1 - h)$, we have by substitution in the preceding equation and transposition, and substitution again,

$$(y_1 - k)(y - k) + (x_1 - h)(x - h) = (y_1 - k)^2 + (x_1 - h)^2 = \rho^2 \ldots\ldots (15).$$

3. When h and k are zero, or the origin at the centre of the circle, either of the two last equations reduces to

$$y_1 y + x_1 x = x_1^2 + y_1^2 = \rho^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (16);$$

a formula of great simplicity, elegance, and usefulness.

Scholium.

Had the equation of the circle been given in the form, the process of solution would have been the same, and the results, making allowance for difference of notation, would have been identical with those above. Moreover, had the given equation of the circle been referred to an *oblique angle of ordination*, the process would have been similar, though the results would have been of more complex form.

We shall, however, only annex the two forms which result from supposing $(x_1 y_1)$ in the circumference:

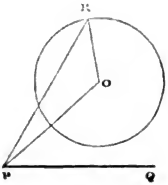
$$(y - y_1)\{(y_1 - k) + (x_1 - h)\cos a\} + (x - x_1)\{(x_1 - h) + (y_1 - k)\cos a\} = 0 \ldots\ldots\ldots\ldots (17),$$

$$(y_1 - k)(y - k) + (x_1 - h)(x - h) + \{(y_1 - k)(x - h) + (x_1 - h)(y - k)\}\cos a = \rho^2 \ldots\ldots\ldots\ldots (18);$$

the equation of the circle itself being as in (1).

PROP. VII.

To find the general form of the equation of the circle, referred to polar co-ordinates.



Suppose the circle defined by means of its radius and the co-ordinates of the centre (as in *prop.* i.), but by means of polar co-ordinates.

Let P be the pole, and PO the origin of polar angles: denote the co-ordinates of the centre O as follows: PO by h, and QPO by κ; and let R, a point in the circle, be denoted, PR by r, and RPQ by θ; and finally, the radius OR by ρ. Then we have

$$\text{RO}^2 = \text{RP}^2 - 2\text{RP}\,.\,\text{PO}\cos\text{RPO} + \text{PO}^2, \text{ or}$$

$$\rho^2 = r^2 - 2rh\cos(\theta - \kappa) + h^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (19).$$

This equation being resolved either for r or θ, gives for each value of one of them, two values of the other; and the general properties of the figure may often be deduced with even greater facility from this form of equation than from that given in *prop. i.*

Corollaries.

1. When $h = 0$, or the pole is at the centre of the circle, we have $\rho^2 = r^2$, or $r = \pm\,\rho$, for all values of θ, and independently also of κ.

2. When $h = \rho$, we have $r^2 - 2r\,\rho\cos(\theta - \kappa) = 0$, giving two values of r, viz. $r = 0$, and $r = 2\rho\cos(\theta - \kappa)$ as the equations.

These are the most usual forms employed when the conditions allow us to dispense with the utmost generality.

Scholium.

It has been customary to obtain the polar equation from the rectangular one by transformation of co-ordinates; but that process is both tedious and inelegant.

PROP. VIII.

To find the polar equation of the tangent to a given circle drawn through a given point.

Let the circle and its tangent through $(r_1\theta_1)$ be denoted by

$$\rho^2 = r^2 - 2rh\cos(\theta - \kappa) + h^2, \text{ and } r_1\cos(\theta_1 - \beta) = r\cos(\theta - \beta).$$

From the latter of these we get successively

$$\cos(\theta - \beta) = \frac{r_1\cos(\theta_1 - \beta)}{r},\; \theta - \kappa = (\beta - \kappa) + \cos^{-1}\frac{r_1\cos(\theta_1 - \beta)}{r},$$

$$\cos(\theta - \kappa) = \cos(\beta - \kappa)\,.\,\frac{r_1\cos(\theta_1 - \beta)}{r} - \sin(\beta - \kappa)\,.\,\frac{\sqrt{r^2 - r_1^2\cos^2(\theta_1 - \beta)}}{r}:$$

and this substituted in the equation of the circle, gives

$$\{r^2 - r_1^2\cos^2(\theta_1 - \beta)\} + 2h\sin(\beta - \kappa)\sqrt{r^2 - r_1^2\cos^2(\theta_1 - \beta)} + h^2\sin^2(\beta - \kappa)$$
$$= \rho^2 - \{h\cos(\beta - \kappa) - r_1\cos(\theta_1 - \beta)\}^2.$$

Now that the two values of $\sqrt{r^2 - r_1^2\cos^2(\theta_1 - \beta)}$, and therefore the two values of r may become equal, or the line itself become a tangent to the circle, we must have

$$\rho^2 - \{h \cos(\beta - \kappa) - r_1 \cos(\theta_1 - \beta)\}^2 = 0,$$

and β determined from this will give the equation of the tangent sought. Reducing the equation to the form

$$\rho^2 = \{\cos\beta\,(h \cos\kappa - r_1 \cos\theta_1) + \sin\beta\,(h \sin\kappa - r_1 \sin\theta_1\}^2,$$

and performing the multiplication, dividing by $\cos^2\beta$, *etc.*, we have

$$\tan^2\beta\,\{(h \sin\kappa - r_1 \sin\theta_1)^2 - \rho^2\} + 2\,(h \cos\kappa - r_1 \cos\theta_1)(h \sin\kappa - r_1 \sin\theta_1)\tan\beta$$
$$= \rho^2 - (h \cos\kappa - r_1 \cos\theta_1)^2,$$

the roots of which are

$$\tan\beta = \frac{-(h\cos\kappa - r_1\cos\theta_1)(h\sin\kappa - r_1\sin\theta_1) \pm \rho\sqrt{h^2 - 2hr_1\cos(\theta_1 - \kappa) + r_1{}^2 - \rho^2}}{(h \sin\kappa - r_1 \sin\theta_1)^2 - \rho^2.}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (20).$$

This shows, like equation (13), that generally there are two tangents possible to be drawn from the point $(r_1\theta_1)$ to the given circle: that when these are equal, the point $(r_1\theta_1)$ is in the circle, since then that the radical may be 0, we have the relation $\rho^2 = h^2 - 2r_1h\cos(\theta_1 - \kappa) + r_1{}^2$ between the co-ordinates of the point: and, lastly, that when ρ^2 is greater than $h^2 - 2r_1h\cos(\theta_1 - \kappa) + r_1{}^2$, or the point is within the circle, the tangent cannot be drawn. All this accords with what has before been shown.

Corollary 1.

When $(r_1\theta_1)$ is in the circumference, the radical vanishes, and we have, since $(h \sin\kappa - r_1 \sin\theta_1)^2 - \rho^2 = -(h \cos\kappa - r_1 \cos\theta_1)^2$,

$$(h \cos\kappa - r_1 \cos\theta_1)(h \sin\kappa - r_1 \sin\theta_1) + (h \cos\kappa - r_1 \cos\theta_1)^2 \tan\beta = 0,$$

$$\text{or } \tan\beta = -\frac{h \sin\kappa - r_1 \sin\theta_1}{h \cos\kappa - r_1 \cos\theta_1}.$$

Moreover, the equation of the tangent, on account of its passing through $(r_1\theta_1)$ is $r\cos(\theta - \beta) = r_1\cos(\theta_1 - \beta)$, or expanded and arranged

$$\tan\beta = -\frac{r \cos\theta - r_1 \cos\theta_1}{r \sin\theta - r_1 \sin\theta_1}.$$

Equating these values of $\tan\beta$, and reducing, we finally get

$$(r \sin\theta - r_1 \sin\theta_1)(h \sin\kappa - r_1 \sin\theta_1) + (r \cos\theta - r_1 \cos\theta_1)(h \cos\kappa - r_1 \cos\theta_1) = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (21).$$

a form perfectly analogous to that in (14), and in fact like it, the equation of a line through $(r_1\theta_1)$, perpendicular to that through $(r_1\theta_1)$ and $(h\kappa)$. It admits, however, of another transformation as follows:—

Corollary 2.

Since $r \sin\theta - r_1 \sin\theta_1 = (r \sin\theta - h \sin\kappa) - (r_1 \sin\theta_1 - h \sin\kappa)$,
and $r \cos\theta - r_1 \cos\theta_1 = (r \cos\theta - h \cos\kappa) - (r_1 \cos\theta_1 - h \cos\kappa)$.

Insert these in (20), and reduce: then,

$$\left.\begin{array}{l}(r \sin\theta - h \sin\kappa)(r_1 \sin\theta_1 - h \sin\kappa) - (r_1 \sin\theta_1 - h \sin\kappa)(r_1 \sin\theta_1 - h \sin\kappa) \\ +(r \cos\theta - h \cos\kappa)(r_1 \cos\theta_1 - h \cos\kappa) - (r_1 \cos\theta_1 - h \cos\kappa)(r_1 \cos\theta_1 - h \cos\kappa)\end{array}\right\} = 0,$$

or on reduction,

$$(r \sin\theta - h \sin\kappa)(r_1 \sin\theta_1 - h \sin\kappa) + (r \cos\theta - h \cos\kappa)(r_1 \cos\theta_1 - h \cos\kappa) = \rho^2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (22).$$

Having a remarkable analogy to the result of (15), as the following has to that of (16).

Corollary 3.

Let $h = 0$: then we have (21) changed to

$$\left.\begin{aligned} rr_1\,\{\sin\theta\sin\theta_1 + \cos\theta\cos\theta_1\} &= \rho^2 \\ \text{or, again, } rr_1\cos(\theta - \theta_1) &= \rho^2 \end{aligned}\right\}\ldots\ldots\ldots\ldots \quad (23).$$

Scholium.

The analogies between these several results and the corresponding ones (14, 15, 16), suggest that for some purposes the corresponding form of the polar equation of the circle might be useful. It is generally

$$\{r\sin\theta - h\sin\kappa\}^2 + \{r\cos\theta - h\cos\kappa\}^2 = \rho^2\ldots\ldots\ldots \quad (24).$$

PROP. IX.

To find the polar equation of the circle described through three given points.

Let $(r_1\theta_1)$, $(r_2\theta_2)$, and $(r_3\theta_3)$ be the three given points: then in virtue of the equation of the circle (*prop. vii. eq.* 18), and its passing through the fore-named points, we shall have

$$\begin{aligned} \rho^2 &= r^2 - 2rh\cos(\theta - \kappa) + h^2 \\ \rho^2 &= r_1^2 - 2r_1h\cos(\theta_1 - \kappa) + h^2 \\ \rho^2 &= r_2^2 - 2r_2h\cos(\theta_2 - \kappa) + h^2 \\ \rho^2 &= r_3^2 - 2r_3h\cos(\theta_3 - \kappa) + h^2. \end{aligned}$$

Subtract these each from the preceding, in succession: then,

$$\begin{aligned} r^2 - r_1^2 &= 2\{r\cos(\theta - \kappa) - r_1\cos(\theta_1 - \kappa)\}\,h, \\ r_1^2 - r_2^2 &= 2\{r_1\cos(\theta_1 - \kappa) - r_2\cos(\theta_2 - \kappa)\}\,h, \\ r_2^2 - r_3^2 &= 2\{r_2\cos(\theta_2 - \kappa) - r_3\cos(\theta_3 - \kappa)\}\,h. \end{aligned}$$

Dividing successively, we get

$$\frac{r^2 - r_1^2}{r_1^2 - r_2^2} = \frac{r\cos(\theta - \kappa) - r_1\cos(\theta - \kappa)}{r_1\cos(\theta_1 - \kappa) - r_2\cos(\theta_2 - \kappa)}, \text{ and expanding,}$$

$$= \frac{\{r\cos\theta - r_1\cos\theta_1\} + \{r\sin\theta - r_1\sin\theta_1\}\tan\kappa}{\{r_1\cos\theta_1 - r_2\cos\theta_2\} + \{r_1\sin\theta_1 - r_2\sin\theta_2\}\tan\kappa};$$

$$\frac{r_1^2 - r_2^2}{r_2^2 - r_3^2} = \frac{\{r_1\cos\theta_1 - r_2\cos\theta_2\} + \{r_1\sin\theta_1 - r_2\sin\theta_2\}\tan\kappa}{\{r_2\cos\theta_2 - r_3\cos\theta_3\} + \{r_2\sin\theta_2 - r_3\sin\theta_3\}\tan\kappa}.$$

These equations furnish two values of $\tan\kappa$; *viz.*

$$\tan\kappa = -\frac{(r^2 - r_1^2)\{r_1\cos\theta_1 - r_2\cos\theta_2\} - (r_1^2 - r_2^2)\{r\cos\theta - r_1\cos\theta_1\}}{(r^2 - r_1^2)\{r_1\sin\theta_1 - r_2\sin\theta_2\} - (r_1^2 - r_2^2)\{r\sin\theta - r_1\sin\theta_1\}},$$

$$\tan\kappa = -\frac{(r_1^2 - r_2^2)\{r_2\cos\theta_2 - r_3\cos\theta_3\} - (r_2^2 - r_3^2)\{r_1\cos\theta_1 - r_2\cos\theta_2\}}{(r_1^2 - r_2^2)\{r_2\sin\theta_2 - r_3\sin\theta_3\} - (r_2^2 - r_3^2)\{r_1\sin\theta_1 - r_2\sin\theta_2\}}.$$

Equate these two values of $\tan\kappa$, and reduce the result to a symmetrical form; then,

$$\left.\begin{aligned} &r^2\{r_1r_2\sin(\theta_1 - \theta_2) + r_2r_3\sin(\theta_2 - \theta_3) + r_3r_1\sin(\theta_3 - \theta_1)\} \\ &- r\cos\theta\{r_1(r_2^2 - r_3^2)\sin\theta_1 + r_2(r_3^2 - r_1^2)\sin\theta_2 + r_3(r_1^2 - r_2^2)\sin\theta_3\} \\ &+ r\sin\theta\{r_1(r_2^2 - r_3^2)\cos\theta_1 + r_2(r_3^2 - r_1^2)\cos\theta_2 + r_3(r_1^2 - r_2^2)\cos\theta_3\} \\ &- r_1r_2r_3\{r_1\sin(\theta_2 - \theta_3) + r_2\sin(\theta_3 - \theta_1) + r_3\sin(\theta_1 - \theta_2)\} \end{aligned}\right\} = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (25).$$

When the circle passes through the *origin of co-ordinates*, we have, by *cor.* 2, *prop. vii.*, the absolute term equal to zero; that is,

$$r_1\sin(\theta_2 - \theta_3) + r_2\sin(\theta_3 - \theta_1) + r_3\sin(\theta_1 - \theta_2) = 0\ldots\ldots \quad (26),$$

which is the criterion of this circumstance.

PROP. X.

Def. The radical axis of two circles is the straight line, which is similarly related to both of them. Its equation is found by subtracting the equation of one circle from that of the other.

The radical axes of three circles, taken two and two, meet in one point, the radical centre of the three circles.

Let the equations of the three circles be

$$(y - y_1)^2 + (x - x_1)^2 = r_1^2,$$
$$(y - y_1)^2 + (x - x_1)^2 = r_2^2,$$
$$(y - y_3)^2 + (x - x_3)^2 = r_3^2;$$

then the radical axes of (1, 2), (2, 3), and (3, 1), are

$$2y\,(y_1 - y_2) + 2x\,(x_1 - x_2) = -(r_1^2 - r_2^2) + (x_1^2 + y_1^2) - (x_2^2 + y_1^2),$$
$$2y\,(y_2 - y_3) + 2x\,(x_2 - x_3) = -(r_2^2 - r_3^2) + (x_2^2 + y_2^2) - (x_3^2 + y_3^2),$$
$$2y\,(y_3 - y_1) + 2x\,(x_3 - x_1) = -(r_3^2 - r_1^2) + (x_3^2 + y_3^2) - (x_1^2 + y_1^2).$$

Now the first and second of these being added, give that of a line having a common intersection with them. But this sum is identical with the third equation: wherefore the line represented by the third equation passes through the intersection of the other two: that is, the three radical axes meet in one point.

Corollary.

When the two circles cut each other, the radical axis is the chord of contact; when they touch, it is the common tangent; when they are each wholly without the other, or one wholly within the other, it is (as will appear from the corollary to the next proposition) the line from any point of which tangents being drawn to the two circles, they will be equal to one another. Also, if the radical centre be without them all, the tangents drawn from it to the three circles will be equal.

PROP. XI.

If from any point in the radical axis of two circles secants be drawn to the two circles, the rectangle under the segments of the one is equal to the rectangle under the segments of the other.

Let (xy) be any point in the plane of the circles fulfilling the condition, and let the secants be those passing through the centres of the circles. Then, the lengths of the lines from xy to those centres will be

$$\sqrt{(y - y_1)^2 + (x - x_1)^2}, \text{ and } \sqrt{(y - y_2)^2 + (x - x_2)^2};$$

wherefore the rectangles in question are

$$\left\{\sqrt{(y - y_1)^2 + (x - x_1)^2} - r_1\right\}\left\{\sqrt{(y - y_1)^2 + (x - x_1)^2} + r_1\right\}, \text{ and}$$
$$\left\{\sqrt{(y - y_1)^2 + (x - x_2)^2} - r_2\right\}\left\{\sqrt{(y - y_2)^2 + (x - x_2)^2} + r_2\right\}; \text{ or}$$
$$(y - y_1)^2 + (x - x_1)^2 - r_1^2, \text{ and}$$
$$(y - y_2)^2 + (x - x_2)^2 - r_2^2.$$

Equating these two values we have

$$2y\,(y_1 - y_2) + 2x\,(x_1 - x_2) = -(r_1^2 - r_2^2) + (x_1^2 + y_1^2) - (x_2^2 + y_2^2),$$

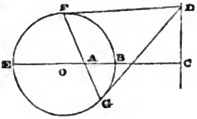
which is the equation of the radical axis.

Corollary.

If the point (xy) be without the two circles, the rectangles in question are equal to the squares of the tangents, and hence the tangents to the two circles from any point of the radical axis are equal to one another.

PROP. XII.

A line passes through a given point, and meets a given circle in two points from which tangents are drawn to intersect; what is the locus of their point of intersection?



Let the diameter EB through the given point A be taken as axis of x, the centre O being the origin; and let the line through A meet the circle in two points, F and G, denoted by (x_1y_1) and (x_2y_2): then we have the equations as follow, FD, GD, being tangents at F and G:

$$(\text{FG})\ldots\ldots y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\ldots\ldots\ldots\ldots (1),$$

$$(\text{FD})\ldots\ldots xx_1 + yy_1 = r^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$(\text{GD})\ldots\ldots xx_2 + yy_2 = r^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

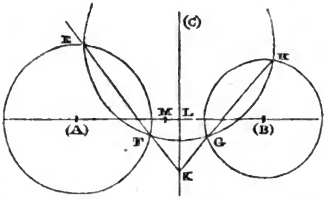
$$\text{From (2, 3) we get } \text{X}_2 = \frac{r^2(y_2 - y_1)}{y_2x_1 - y_1x_2},$$

$$\text{and } y = 0 \text{ in (1) gives } \text{X}_1 = \frac{y_2x_1 - y_1x_2}{y_2 - y_1}.$$

Whence $\text{X}_1\text{X}_2 = r^2$; or the point A being given, the abscissa X_2 of the point D is constant, and the locus of D is a straight line DC perpendicular to the diameter EB.

PROP. XIII.

Two given circles, (A), (B), are cut arbitrarily by a third circle (C), and the chords EF, GH, joining the points of intersection in (A) and (B), are produced to meet in K: it is required to prove that the locus of K is a straight line.



Take M, the middle of AB, for origin of rectangular co-ordinates, that of x coinciding with MB. Let $\text{AM} = \text{MB} = a$, and the radii of (A), (B), be r_1 and r_2 respectively. Also, let h, k, be co-ordinates of C, and r the radius of (C): then we have the following equations:

$$(\text{A})\ldots\ldots y^2 + x^2 + 2ax = r_1^2 - a^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$(\text{B})\ldots\ldots y^2 + x^2 - 2ax = r_2^2 - a^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$(\text{C})\ldots\ldots y^2 + x^2 - 2hx - 2ky = r^2 - h^2 - k^2\ldots\ldots\ldots\ldots (3).$$

Subtract (3) from (1) and (2), and the results give the equations of the chords EF, GH, which are

$$(\text{EF})\ldots\ldots 2ax + 2hx + 2ky = (r_1^2 - a^2) - (r^2 - h^2 - k^2)\ldots\ldots (4),$$

$$(\text{GH})\ldots -2ax + 2hx + 2ky = (r_2^2 - a^2) - (r^2 - h^2 - k^2)\ldots\ldots (5).$$

Subtract (5) from (4), and there results the equation of a line passing through K, the intersection of EF, GH. This is

$$4ax = r_1^2 - r_2^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

The line (6) is therefore independent of the value of y, and is, moreover, the same, whatever be the values of h, k, and r, which vanish from the equation. Whence the locus of K is the right line KL perpendicular to AB, and situated with respect to the middle M, so that

$$\mathrm{ML} = x = \frac{r_1^2 - r_2^2}{4a}.$$

This line has before been found (*p.* 288), and is, in fact, the radical axis of the circles (A), (B). By continuing this process of inquiry, all the other properties of this line may be developed.

EXERCISES ON THE CIRCLE.

1. Calculate the radii and co-ordinates of the centres of the following circles:—

(1). $x^2 + y^2 = 100.$
(2). $2y^2 + 2x^2 - 4(x + y) - 1 = 0.$
(3). $y^2 + x^2 - 6x + 4y = 3.$
(4). $(6y - 21)y + (6x - 8)x + 14 = 0.$
(5). $y^2 + x^2 + 4y - 3x = 0.$
(6). $(x + 2)x + (y - 4)y = 0.$
(7). $y^2 + x^2 = 4y.$
(8). $y^2 + 8 = -(x + 6)x.$
(9). $y^2 - 8y + x^2 - 12x + 52 = 0.$
(10). $(y - 4)y + (x + 2)x + 9 = 0$

2. Find the equations of the circles which respectively pass through the following triads of points:—

(1). $(-2, 5), (4, -6), (-2, -6).$
(2). $(-6, -1), (0, 0), (0, -1).$
(3). $(-1, 1), (1, -1), (1, 1).$
(4). $(0, 1), (1, 0), (0, 0).$
(5). $(-4, -1), (4, 1), (-1, 4).$
(6). $\left(\frac{1}{0}, 0\right), (1, 1), \left(-\frac{1}{2}, \frac{1}{2}\right).$
(7). $(0, 0), (1, 1), (2, 2).$

3. Find the locus of the vertex of the triangle in the following cases:—

(1). Where the base and vertical angle are given.
(2). Where the base and ratio of the sides are given.

4. Given the base and vertical angle, to find the loci of the centres of the inscribed and three escribed circles.

5. Ascertain whether the four following points be in one circle, (0, 0), (0, 4), (1, 1), and (1, — 1): and if not, whether (2, 6) be in the same circle with any three of them.

6. Let A, B, C, ... N be n given points, and Z a point so taken that $p_1 . \mathrm{AN}^2 + p_2 . \mathrm{BN}^2 + p_3 . \mathrm{CZ}^2 + \ldots . p_n . \mathrm{NZ}^2$ shall be equal to a given space: then the locus of Z will be a circle, the centre and radius of which it is required to determine.

7. The co-ordinates of the centre of a circle are (— 3, 4), and its radius is 5; also the abscissas of points at which tangents to the circle are drawn are 1, — 1, and 2: it is required to determine the angles of the triangle formed by those tangents, and the centre and radius of the circle which circumscribes that triangle.

8. Given the co-ordinates $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$ of the angles of a triangle, to show the process for finding the centres and radii of the inscribed and circumscribed circles.

9. Solve the same problem when the points $(r_1 \theta_1)$, $(r_2 \theta_2)$, $(r_3 \theta_3)$ are given.

10. A quadrilateral is inscriptible in a circle, and the positions of three of its angles are (1, 2), (— 1, — 2), (3, — 3), and the abscissa of the fourth is 2·5: what is the ordinate of that point?

11. Show that if tangents be drawn to touch three circles, two and two, the three points of intersection of each pair of tangents are in one line.

12. Equal lines are drawn from P to touch two circles (A), (B): what is the locus of P?

13. Find the centre and radius of a circle which touches three given circles.

14. A circle whose radius is r rolls upon the *inside* of another whose radius is $2r$, any point in the rolling one will trace out a diameter of the quiescent one.

15. Let AB, AC be two straight lines given in position, and DE be a line of given length terminated by them at D and E; and let the perpendiculars DV, EV be drawn to AB, AC, and meet in V: the locus of V is a circle of which A is the centre.

16. If all lines drawn from a given point to meet a circle be divided in the same given ratio: the locus of the points is defined by a polar equation which it is required to assign.

17. The point of contact of the base of a triangle with its inscribed circle, the radius of that circle, and the difference of the sides of the triangle being given; the locus of the vertex is a straight line.

18. Find the locus of the points on a plane from which two given circles subtend equal angles.

19. A circle passes through the points (1, 2), (3, 2), (4, 3), and a line through (— 1, 3) and (2, 6): how far from the centre of the circle does the line pass?

20. Let AB, AC be two straight lines given by position, and let DE, a line of given length, resting upon them in D and E; and from D, E draw perpendiculars to AB, AC, meeting in V: show that the locus of V is a circle passing through A.

21. The locus will be a circle under the following conditions:—
  (1). Given the base and sum of the squares of the sides of a triangle to find the locus of the vertex.
  (2). Given the base and vertical angle of a triangle, to find the locus of
    (*a*) the intersection of the perpendiculars from the angles upon the opposite sides:
    (*b*) the centre of the inscribed circle:
    (*c*) the intersection of lines from the angular points to bisect the opposite sides.

22. If the opposite pairs of sides of a salient hexagon inscribed in a circle be produced to meet, then of the three angles thus formed the middle one will be equal to the sum of the other two.

23. If we have the equations simultaneously given

$$\frac{t - a_1 u - b_1}{\sqrt{1 + a_1^2}} = \frac{t - a_2 u - b_2}{\sqrt{1 + a_2^2}} = \frac{t - a_3 u - b_3}{\sqrt{1 + a_3^2}};$$

show what they signify, and what problem is solved by means of them.

24. From any point in a circle which circumscribes a triangle draw perpendiculars to the sides of the triangle; then the feet of these perpendiculars will be in one line.

25. From each of the angles of a quadrilateral inscribed in a circle draw perpendiculars to the two sides containing the opposite angle: and draw the four lines through the four pairs of points of intersection of these perpendiculars

with the sides upon which they fall: then these four lines will all pass through the same point.

26. The locus of all the points from which tangents drawn to two given circles they shall have a given ratio, is a third circle, whose equation is required.

27. If there be given n points, a_1b_1, a_2b_2, ... a_nb_n, and we draw lines from each of them to another point, so taken that the sum of the squares of all these lines is equal to a given space q^2: then the locus of that point is a circle, whose equation is

$$y^2 + x^2 - \tfrac{2}{n}(a_1 + a_2 \ldots a_n)x - \tfrac{2}{n}(b_1 + b_2 \ldots b_n)y = \tfrac{1}{n}\{q^2 - (a_1^2 + b_1^2 \ldots a_n^2 + b_n^2)\}.$$

28. If from a point, assumed arbitrarily in the plane of a triangle, perpendiculars be drawn to the sides, and a circle be drawn through the three intersections; these will cut the sides again in three points, from which perpendiculars to those sides being drawn, they will all pass through the same point.

29. If the first-named three points of the preceding theorem be joined, and perpendiculars be drawn from the angles of the original triangle to the sides of the new one, these will all meet in one point.

30. If the perpendiculars be drawn from this new point upon the sides of the original triangle, they will intersect in three points situated in the circle which passes through the feet of the three former perpendiculars.

31. If one circle whose centre is $(a_1\ b_1)$ touch another internally whose centre s $(a_2\ b_2)$, what relation exists between their radii?

32. Show, generally, that if two circles touch each other, the straight line joining their centres passes through the point of contact: and if two circles cut each other, the line joining their points of section is perpendicular to that joining their centres.

33. Find the co-ordinates of the centres of similitude of two circles whose equations are given.

34. Show that the three exterior centres of similitude of three circles, considered two and two, are in one straight line; and that each of the exterior centres of similitude is in a straight line with two of the interior centres of similitude.

35. A quadrilateral ABDC, inscribed in a circle, has one side AB given in magnitude and position, and the opposite side CD given in magnitude only; also AC, BD, meet in E, and AD, BC, in F: show that the line EF always passes through a fixed point H, and give a method of finding that point by construction.

36. In the preceding construction assign the loci of E and F.

37. Two given circles are seen from points, at which they subtend equal angles: show that the locus of these points of view is a third circle, having its centre in the line joining the centres of the other two.

38. Find the co-ordinates of the point from which three given circles subtend equal angles.

39. Let the opposite sides AB, CD, and AD, BC, of a quadrilateral inscribed in the circle be produced to meet in E and F, and let circles ADF, CDE, ABF, CDF, be described; these will all intersect in one point L, situated in EF; and AD, BC, intersect in G, the line LG will be perpendicular to EF, and bisect the angles ALC, BLD.

40. Circles described on the three diagonals of a complete quadrilateral as diameters, will have the same radical axis.

41. Circles described as in (39), with respect to *any* quadrilateral, pass through one point; and it is such that perpendiculars from it to the four sides cut those sides respectively in four points, all situated in one straight line.

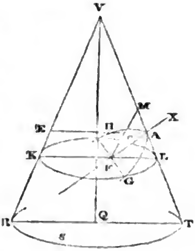
THE CONIC SECTIONS, OR LINES OF THE SECOND ORDER.

DEFINITION.

A RIGHT cone being cut by a plane, the common section of the conic surface and the plane is called a *conic section*. It receives different names from the relative positions of the cone and plane, which will be explained at the conclusion of *prop. i.* See also the Geometrical Treatise, *pp.* 104-7.

PROP. I.

Every conic section is a line of the second order.



LET RST, a circle whose centre is Q, be the base of the cone, and V its vertex; and let the cone be cut by any plane in a line HAG, this line is called a conic section, and it is to be shown that its equation is of the second degree.

For, draw any other section HGL parallel to the base RST, cutting the plane HAG in HFG; through the axis VQ of the cone draw a plane perpendicular to HAG cutting it in AF, the plane KGL in KL, and the cone in RV, VT; and, finally, draw AE parallel to KL or RT, and FM parallel to RV, meeting AE in C and AV in M.

Then, since the plane KGL is parallel to RST, it is a circle; and since RVT is drawn through the axis, the line RT is a diameter, and therefore KL is also a diameter of KGL. Again, since RVT is drawn through the axis it is perpendicular to the plane RST, and consequently to the plane KGL; and it is perpendicular to the plane HAG, by construction: whence it is perpendicular to their common section HFG, or the line HFG is at right angles to both the lines AF and KFL. Finally, since KL is the diameter of the circle KGL, and HFG a line cutting the circle in G and H, and the diameter KL at right angles in F, the line GH is bisected in F, and

$$\mathrm{FG}^2 = \mathrm{KF} \,.\, \mathrm{FL}.$$

Put $\mathrm{AF} = x$, $\mathrm{FG} = y$, $\mathrm{AV} = h$, $\mathrm{VAF} = \alpha$, and $\mathrm{AVQ} = \beta$: then by the triangles FAL, EVA, AFC, and AMC, we have

$$\mathrm{FL} = \frac{\mathrm{AF} \sin \mathrm{FAL}}{\sin \mathrm{FLA}} = \frac{x \sin (\pi - \alpha)}{\sin (\frac{1}{2}\pi - \beta)} = \frac{x \sin \alpha}{\cos \beta},$$

$$\mathrm{EA} = \frac{\mathrm{AV} \sin \mathrm{AVE}}{\sin \mathrm{VAE}} = \frac{h \sin 2\beta}{\sin (\frac{1}{2}\pi - \beta)} = \frac{h \sin 2\beta}{\cos \beta},$$

$$\mathrm{AC} = \frac{\mathrm{AF} \sin \mathrm{AFC}}{\sin \mathrm{ACF}} = \frac{x \sin \{\pi - (\alpha + 2\beta)\}}{\sin (\frac{1}{2}\pi + \beta)} = \frac{x \sin (\alpha + 2\beta)}{\cos \beta}.$$

But we have already seen that

$$\mathrm{FG}^2 = \mathrm{LF} \,.\, \mathrm{FK} = \mathrm{LF} \,.\, \mathrm{EC} = \mathrm{LF} \left\{ \mathrm{EA} - \mathrm{AC} \right\};$$

in which, inserting the preceding values of these lines, we get the equation

$$y^2 = \frac{x \sin \alpha}{\cos \beta} \left\{ \frac{h \sin 2\beta}{\cos \beta} - \frac{x \sin (\alpha + 2\beta)}{\cos \beta} \right\}, \text{ or}$$

$$y^2 \cos^2 \beta + x^2 \sin \alpha \sin (\alpha + 2\beta) = hx \sin \alpha \sin 2\beta. \ \ldots\ldots\ldots \ (1).$$

The equation of the conic section is therefore of the second degree, and the line itself of the second order.

Discussion.

The form of the equation deduced above is such, that the coefficients of y^2 and x are essentially positive. For $\cos^2\beta$ is positive, both because it is a square and because $\cos \beta$ is itself positive, since 2β is less than π, and hence β less than $\frac{1}{2}\pi$. Also, $\sin \alpha$ and $\sin 2\beta$ are each positive, both α and 2β being each less than π; and h is essentially positive from the mode of its introduction. The only changes, therefore, that can take place in the *general form of the equation*, and consequently in the *general features of the curve*, must hence arise from the coefficient $\sin \alpha \sin (\alpha + 2\beta)$, of which the former factor $\sin \alpha$ is positive, and therefore depends wholly on $\sin (\alpha + 2\beta)$. This may be *zero*, *positive*, or *negative;* indicating respectively that the plane HAG is parallel to the edge VR of the cone, meets it on the side R, or meets it in RV produced beyond V, being in all cases perpendicular to the plane RVT.

CASE I.

The parabola.

In this case $\alpha + 2\beta = \pi$, and the equation is reduced to

$$y^2 = \frac{hx \sin \alpha \sin 2\beta}{\cos^2 \beta} = 2hx \sin \alpha \tan \beta = 4hx \sin^2 \beta \ldots\ldots\ldots\ldots \ (2).$$

This is often written in one or other of the forms (or some analogous to them) below:

$$y^2 = px, \text{ or } y^2 = 4mx. \ \ldots\ldots\ldots\ldots\ldots\ldots \ (3).$$

CASE II.

The ellipse.

Here we have $\alpha + 2\beta$ less than π; and for the more complete discussion we shall consider separately those cases that have any variety.

1. Let $\alpha = 0$; then $y^2 \cos^2\beta = 0$, or $y = 0$, $\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (4), whatever value β may have. This indicates that the locus of the equation is a straight line coinciding with the axis of x, and which, in reference to the figure, is the line of contact of the plane HAG (now become a tangent-plane to the cone) with the surface of the cone.

2. Let $\alpha = \frac{1}{2}\pi - \beta$; then $\alpha + 2\beta = \frac{1}{2}\pi - \beta + 2\beta = \frac{1}{2}\pi + \beta$, and $\sin (\alpha + 2\beta) = \cos \beta$: whence the equation becomes

$$y^2 \cos^2\beta + x^2 \cos^2\beta = 2hx \sin \beta \cos^2\beta, \text{ or}$$

$$y^2 + x^2 = 2hx \sin \beta; \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (5)$$

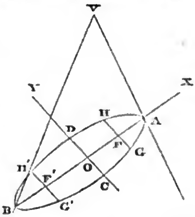
the equation of a circle whose origin is at A, and whose radius is $h \sin \beta$.

The condition above assumed, is the same as that the cutting-plane shall be parallel to the base of the cone.

3. Let $h = 0$, $\alpha + 2\beta$ being of any value less than π; then we have

$$y^2 \cos^2\beta + x^2 \sin \alpha \sin (\alpha + 2\beta) = 0; \ \ldots\ldots\ldots\ldots\ldots \ (6)$$

the sum of two squares, multiplied respectively by positive factors, equal to zero; indicating that the locus is reduced to a *point*, viz. the vertex V of the cone.



4. Let now the only condition be $\alpha + 2\beta$ less than π; or the lines VR, AF, make angles with AV, which are together less than two right angles. In this case they will meet in some point B. The ellipse is, therefore, limited to the sheet of the cone VRST, forming one continued line, commencing and terminating at the point A.

Again, for each value of x there are two equal and opposite values of y, as is evident from the equation

$$y = \pm \sec\beta \sqrt{hx \sin\alpha \sin 2\beta - x^2 \sin\alpha \sin(\alpha + 2\beta)}. \ \dots\dots\dots\dots\dots\dots \ (7)$$

This shows that AF divides the curve into two equal and similar branches, AHH′B and AGG′B, symmetrically disposed with respect to it.

Further, the preceding general investigation might have been conducted with regard to a plane cutting the side VR in B. In such case let VB $= h_1$, VBA $= \alpha_1$; and we get, as before,

$$y^2 \cos^2\beta + x^2 \sin\alpha_1 \sin(\alpha_1 + 2\beta) = h_1 x \sin\alpha_1 \sin 2\beta; \ \dots\dots \ (8).$$

in which $\alpha_1 = \pi - (\alpha + 2\beta)$, or $\sin\alpha_1 = \sin(\alpha + 2\beta)$; $\alpha_1 + 2\beta = \pi - \alpha$, or $\sin(\alpha_1 + 2\beta) = \sin\alpha$; and $h \sin\alpha = h_1 \sin\alpha_1 = h_1 \sin(\alpha + 2\beta)$.

The values of y, therefore, in the foregoing equation and in that just found, are the same for the same values of x, estimated in the former case from A, and in the latter from B; that is, if BF′ = AF, then F′G′ or F′H′ = FG or FH. The curve is therefore composed of symmetrical branches, considered with respect to the axis CD, bisecting AB at right angles. The ellipse, is, therefore, composed of *four* equal and similar branches or quadrants, symmetrically disposed in pairs when taken with respect to AB, CD, and the centre O.

5. The ***relation between the axes*** AB, CD, is thus obtained, as well as the fundamental property from which we may derive all the others.

Let OA = OB = a, OC = OD = b; then in the triangle AVB we get

$$h = \frac{2a \sin(\alpha + 2\beta)}{\sin 2\beta}; \ \dots\dots\dots\dots\dots\dots \ (9),$$

and hence the equation of the ellipse becomes

$$y^2 \cos^2\beta + x^2 \sin\alpha \sin(\alpha + 2\beta) = 2ax \sin\alpha \sin(\alpha + 2\beta); \ \dots\dots \ (10),$$

or again, referring the origin to the middle of AB by writing $x + a$ for x, we get at once

$$y^2 \cos^2\beta + x^2 \sin\alpha \sin(\alpha + 2\beta) = a^2 \sin\alpha \sin(\alpha + 2\beta). \ \dots\dots (11).$$

Now OC and OD are the values of y in this equation when $x = 0$; and hence they are found to be

$$b = \pm a \sec\beta \sqrt{\sin\alpha \sin(\alpha + 2\beta)}, \ \dots\dots\dots\dots\dots \ (12),$$

and the relation is obtained.

Multiply all the terms of the transformed equation by a^2, and divide by $\cos^2\beta$; then we have

$$a^2 y^2 + \frac{a^2 \sin\alpha \sin(\alpha + 2\beta)}{\cos^2\beta} x^2 = \frac{a^4 \sin\alpha \sin(\alpha + 2\beta)}{\cos^2\beta}; \ \dots\dots \ (13),$$

or inserting b^2 for its value just found, we get for the ***fundamental equation of the ellipse***

$$a^2 y^2 + b^2 x^2 = a^2 b^2. \ \dots\dots \ \dots\dots\dots\dots\dots (14).$$

CASE III.

The hyperbola.

We shall consider in succession the subordinate cases that can arise from different circumstances of the data.

1. Let $\alpha = \pi$; then we get $\sin \alpha = 0$, and the equation is

$$y^2 \cos^2\beta = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15).$$

This indicates the line of contact of the plane with the surface of the cone, as in the corresponding case of the ellipse. The only difference between the two corresponding cases is, that x was taken positive *towards* V in the ellipse, and towards T in the hyperbola.

2. Let α be so taken that $\cos^2\beta + \sin \alpha \sin (\alpha + 2\beta) = 0$ in the general equation, (*p.* 294); then the equation is reduced to

$$y^2 - x^2 = 2hx \sin \alpha \tan \beta. \ldots\ldots\ldots\ldots\ldots\ldots (16).$$

This species of the curve, the general features of which are the same as those of other hyperbolas, is called the *equilateral hyperbola*.

3. Let $h = 0$; then we have, since $\sin (\alpha + 2\beta)$ is *negative*,

$$y^2 \cos^2\beta - x^2 \sin \alpha \sin \left\{(\alpha + 2\beta) - \pi\right\} = 0; \quad \ldots\ldots \quad \ldots (17),$$

which is of the form

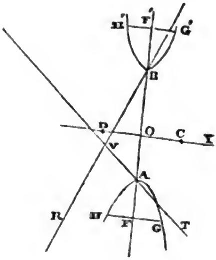
$$(ay + bx)(ay - bx) = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18),$$

and represents two lines through the origin making equal angles with the axis of x. They are the intersections of the cutting-plane with the tangent-planes to the surface of the cone.

4. The general case of the hyperbola subject to no restrictive conditions.

For simplicity of writing, put $\alpha + 2\beta - \pi = \delta$; then the general equation becomes

$$y^2 \cos^2\beta - x^2 \sin \alpha \sin \delta = hx \sin \alpha \sin 2\beta \ldots\ldots\ldots \ldots (19).$$



In this case the plane of section HAG will meet the side VR *produced beyond* V in a point B, and will never meet the side AV of the cone except at the point A; whilst, however far both sheets of the cone be extended, they will always be cut by the plane HFG. There will hence be two indefinite branches, HAG, H′BG′, of the curve, one in each sheet of the cone, having no geometrical termination, and never meeting each other. This will, however, be rendered more apparent by the discussion of the equation.

Again, as in the ellipse, we have two equal and opposite values of y for each value of x, from the equation

$$y = \pm \sec \beta \sqrt{hx \sin \alpha \sin 2\beta + x^2 \sin \alpha \sin \delta}. \quad \ldots\ldots\ldots \quad (20).$$

The branch HAF is therefore similar and equal to the branch GAF, as is also H′BF′ to G′BF′.

Had, as in the ellipse, the preceding investigation been conducted with respect to VB instead of VA, and corresponding changes been made, we should find that in reference to OY, which bisects AB at right angles, the branches FAG, F′BH′, are similar and equal, as are FAH, F′BH′. Wherefore, the entire section is composed of four equal and similar branches, considered with respect to the two axes AB, CD, and centre O.

5. The relation AB to a certain line CD, which is usually estimated on OY, may be obtained precisely as in the case of the ellipse. Denoting this by b, and AB by a, we thus have finally

$$a^2y^2 - b^2x^2 = - a^2b^2. \qquad \ldots\ldots\ldots\ldots (21).$$

PROP. II.

Every equation of the second degree is that of a conic section.

The most general form that such an equation can have is

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0 \qquad \ldots\ldots\ldots\ldots (1).$$

This may be considered in reference to any angle of ordination, and any values, positive, negative, or zero, of any of the coefficients; and by successive transformations, we shall be able to reduce it, under all circumstances, to one or other of the forms found to represent the conic sections in the preceding problem.

1. Transform to axes bisecting the angle of ordination and its supplement; and let the angle of ordination be 2α. Then the formulæ of transformation (*p.* 250) will be found from α and $\frac{1}{2}\pi + \alpha$ being the inclinations of the new axes to the original axis of x. We must then write

$$x = \frac{x' \sin (2\alpha - \alpha) + y' \sin \{2\alpha - (\alpha + \frac{1}{2}\pi)\}}{\sin 2\alpha} = \frac{x' \sin \alpha - y' \cos \alpha}{\sin 2\alpha};$$

$$y = \frac{x' \sin \alpha + y' \sin (\alpha + \frac{1}{2}\pi)}{\sin 2\alpha} = \frac{x' \sin \alpha + y' \cos \alpha}{\sin 2\alpha};$$

and in the result, dropping the accents from $x'y'$, we have

$$\left. \begin{array}{l} (A - B + C)\, y^2 \cos^2\alpha + (A - C)\, xy \sin 2\alpha + (A + B + C)\, x^2 \sin^2\alpha \\ \quad + (D + E)\, x \sin\alpha \sin 2\alpha + (D - E)\, y \cos\alpha \sin 2\alpha + F \sin^2 2\alpha \end{array} \right\} = 0 \;..\; (2);$$

and which we shall designate by

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \qquad \ldots\ldots\ldots\ldots (3).$$

The function $B^2 - 4AC$ will presently be shown to be the test of the particular kind of curve represented by the equation, since $b^2 - 4ac$ will be shown to be so; and hence the dependence of the one function on the other must be established.

Now by substitution,

$$\begin{aligned} b^2 - 4ac &= (A - C)^2 \sin^2 2\alpha - 4(A + B + C)(A - B + C) \sin^2\alpha \cos^2\alpha \\ &= \{(A - C)^2 - (A + C)^2 + B^2\} \sin^2 2\alpha \\ &= \{B^2 - 4AC\} \sin^2 2\alpha; \end{aligned}$$

and hence, whatever *sign* (or zero) belong to $b^2 - 4ac$, the same will belong to $B^2 - 4AC$, and *vice versâ;* that is, as $b^2 - 4ac$ is positive, negative, or zero, $B^2 - 4AC$ will be positive, negative, or zero, respectively, and *vice versâ.*

Next transpose the origin of co-ordinates so as to remove the first powers of x and y; which will be effected if for x and y we write

$$x + \frac{2ae - bd}{b^2 - 4ac}, \text{ and } y + \frac{2cd - be}{b^2 - 4ac}.$$

The effect of this transformation is to give the equation

$$ay^2 + bxy + cx^2 + \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f = 0 \qquad \ldots\ldots\ldots (4).$$

In the next place transpose the axis through an angle β such that

$$\tan 2\beta = - \frac{b}{a - c};$$

in consequence of which the term xy is eliminated, and the equation is changed into

$$\left\{(a+c) \pm \sqrt{(a-c)^2+b^2}\right\} y^2 + \left\{(a+c) \mp \sqrt{(a-c)^2+b^2}\right\} x^2 + \frac{2\{ae^2+cd^2-bde\}}{b^2-4ac} + 2f = 0 \text{ ....} (5).$$

Now this divides itself into cases according to the value of $b^3 - 4ac$, as already premised.

1. Suppose $b^2 - 4ac$ negative. Then, since

$$(a-c)^2 + b^2 = (a+c)^2 + (b^2 - 4ac),$$

the radical part of the coefficients will be less in value than the rational; and hence both the coefficients will be *essentially positive;* or the equation will be that already determined as representing the *ellipse.*

2. Suppose $b^2 - 4ac$ positive; then $\sqrt{(a-c)^2+b^2}$ will be greater than $a + c$, and one or other of the coefficients of (5) will be negative whilst the other is positive; and the form of the equation is that already found as belonging to the *hyperbola.*

3. Suppose $b^2 - 4ac = 0$; then $\sqrt{(a-c)^2+b^2} = (a+c)$, and one of the coefficients, either of x or y, vanishes for this value. At the same time, the term, free from the variables, will become either infinite or indeterminate. In fact, the transformation (4) was to an origin (which had in the discussion of the preceding proposition been shown to be a centre) infinitely distant, the values of the new co-ordinates becoming in such case infinite. Moreover, there is in this case only one variable left, and we cease to have the equation of any definite locus. For these reasons we must adopt a different course of transformation.

The term xy is *first* removed by the transformation through the angle β, as before; in which case the equation is reduced to one of the forms

$$(a+c)\,y^2 + \frac{d\sqrt{a} - e\sqrt{c}}{\sqrt{a+c}}\,y + \frac{d\sqrt{c} + e\sqrt{a}}{\sqrt{a+c}}\,x + f = 0 \text{ ......} (6),$$

$$\text{or } (a+c)\,x^2 + \frac{d\sqrt{c} - e\sqrt{a}}{\sqrt{a+c}}\,y + \frac{d\sqrt{a} + e\sqrt{c}}{\sqrt{a+c}} + f = 0 \text{ ....} (7);$$

the former being applicable to the case where the sign of b is —, and the latter where +.

Lastly, transform the origin of (6) by writing for x and y,

$$x + \frac{d^2a + 2de\sqrt{ac} + e^2c - 4(a+c)^2 f}{4(d\sqrt{c} + e\sqrt{a})\,(a+c)^{\frac{3}{2}}}, \text{ and } y - \frac{d\sqrt{a} - e\sqrt{c}}{2(a+c)^{\frac{3}{2}}}:$$

and the result is

$$(a+c)\,y^2 + \frac{d\sqrt{c} + e\sqrt{a}}{\sqrt{a+c}}\,x = 0 \text{ ....} (8);$$

and a corresponding change of origin will transform (7) to

$$(a+c)\,x^2 + \frac{d\sqrt{a} - e\sqrt{c}}{\sqrt{a+c}}\,y = 0 \text{ ....} (9);$$

either of which is of the general form of the equation of the *parabola* determined in the preceding proposition.

Scholium 1.

It has already been shown under what conditions the co-efficients of the equations furnish immediate information respecting the general character of the one to which they individually relate, when the axes are rectangular; and it has been shown, that when this is the case, the coefficients of the equations referred to oblique angles of ordination have precisely the same relations as to sign or vanishing. Hence, in the *general equation* (1), we are entitled to infer that

$b^2 - 4ac = 0$, designates the *parabola*,
$b^2 - 4ac = -$, the *ellipse*, and
$b^2 - 4ac = +$, the *hyperbola*.

Scholium 2.

It is not, however, to be inferred, that the term xy can *only* be removed when the axes are rectangular.

For let the angle of ordination be 2α, and the inclinations of X′ and Y′ to X be β, γ; then (*p.* 250) we have only to substitute for y and x the functions

$$\frac{x \sin \beta + y \sin \gamma}{\sin 2\alpha}, \text{ and } \frac{x \sin (2\alpha - \beta) + y \sin (2\alpha - \gamma)}{\sin 2\alpha};$$

and the co-efficient of the term involving xy, abating its denominator, becomes

$$\left.\begin{array}{l} 2a \sin \beta \sin \gamma + 2c \sin (2\alpha - \beta) \sin (2\alpha - \gamma) \\ + b \sin \gamma \sin (2\alpha - \beta) + b \sin \beta \sin (2\alpha - \gamma) \end{array}\right\} = 0.$$

As there is only one condition to be fulfilled by this equation, which contains two arbitrary quantities, β, γ, it is clear that whatever value be given to one of them, a value which shall fulfil it can be found for the other; or the elimination of xy can be effected in innumerable different ways, the angles of co-ordination being oblique. It is equally clear, that only one such transformation can be made whilst the angle of ordination is required to be $\frac{1}{2}\pi$. That one, indeed, might have been obtained at once from (1), without passing through (2), by the substitution of the functions of β from

$$\tan 2\beta = \frac{c \sin 4\alpha - b \sin 2\alpha}{a - b \cos 2\alpha + c \cos 4\alpha};$$

but the process would have been much longer, and the reduction more intricate.

Scholium 3.

As the following expressions occur very frequently in researches concerning the conic sections, they ought to be designated by some uniform notation. The following will occasionally be used in this work:

$\mathrm{R}^2 = (a - b \cos \alpha + c)^2 + (b^2 - 4ac) \sin^2\alpha$,
$\mathrm{Q} = a - b \cos \alpha + c$,
$\mathrm{H} = \mathrm{Q} - 2a \sin^2\alpha = a \cos 2\alpha - b \cos \alpha + c$,
$\mathrm{K} = \mathrm{Q} - 2c \sin^2\alpha = c \cos 2\alpha - b \cos \alpha + a$.

PROP. III.

To find the intersections of a given straight line (2) *with a given conic section* (1).

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \ldots\ldots\ldots\ldots (1),$$
$$y = px + q \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Substitute from (2) in (1); and we get

$(ap^2 + bp + c)\, x^2 + (2apq + bq + dp + e)\, x + aq^2 + dq + f = 0 \ldots . (3)$; the roots of which being found, give two values of x, and then from (2) there are obtained the corresponding values of y.

Corollary.

A straight line can intersect a conic section in only two points.

PROP. IV.

To determine the conditions of a line being a tangent to a conic section.

In this case the two points of section must coalesce, and the two roots of equation (3) in the preceding problem must be equal. This will require, as in the case of the circle, that when the first side is rendered a complete square, the second side shall be zero. This will give the condition,

$$(2apq + bq + pd + e)^2 - 4(ap^2 + bp + c)(aq^2 + dq + f) = 0$$

Now this equation involves both p and q; and we may make any further hypothesis which shall fix the value of one of them, and then proceed to the consequent determination of the other. We shall give two examples

1. *The tangent parallel to a given line.*

Whether referred to rectangular or oblique axes, the value of p is in this case given; whence two values of q will be found from (*prop. iv.*), showing that *two such tangents can be drawn,* the values of q being the segments of the axis of y cut off by them, estimated from the origin. These values are readily found, for the equation (4) is reducible to the form

$$\left.\begin{matrix}(b^2 - 4ac)\,q^2 + 2\{(2ae - bd)\,p - (2cd - be)\}q \\ + (d^2 - 4af)\,p^2 + 2\,(ed - 2bf)\,p + (e^2 - 4cf)\end{matrix}\right\} = 0 \;\ldots\ldots\ldots\ldots\; (1).$$

We see also, when $b^2 - 4ac = 0$, or the curve is a *parabola*, there can be but *one such tangent,* the equation (a) being then reduced to the first degree.

2. *The tangent drawn through a given point* (x_1y_1).

In this case the equation of the line is

$$y = px + y_1 - px_1;$$

and writing $y_1 - px_1$ for q in the equation of condition (*prop. iv.*), we shall have, after obvious reductions,

$$\left.\begin{matrix}p^2\{(b^2 - 4ac)\,x_1^2 + 2\,(bd - 2ae)\,x_1 + (d^2 - 4af)\} \\ - 2p\{(b^2 - 4ac)\,x_1y_1 + (bd - 2ae)\,y_1 + (be - 2cd)\,x_1 - (de - 2bf)\} \\ + \{(b^2 - 4ac)\,y_1^2 + 2\,(be - 2cd)\,y_1 + (e^2 - 4cf)\}\end{matrix}\right\} = 0\ (2).$$

Resolving this equation for p, we get the expression

$$p = \frac{\left.\begin{matrix}(b^2 - 4ac)\,x_1y_1 + (bd - 2ae)\,y_1 + (be - 2cd)\,x_1 - (de - 2bf) \\ \pm 2\sqrt{[\lambda\{ay_1^2 + bx_1y_1 + cx_1^2 + dy_1 + ex_1 + f\}]}\end{matrix}\right\}}{(b^2 - 4ac)\,x_1^2 + 2\,(bd - 2ae)\,x_1 + (d^2 - 4af)} \;\ldots\ldots\; (3)[*],$$

where $\lambda = ae^2 + cd^2 + fb^2 - bde - 4acf$.

There can hence from the point x_1y_1 be generally drawn two tangents to the curve of the second degree.

---

[*] The following forms of λ, derived from the operation of completing the square, will often be useful:

$$\begin{aligned}
4a\lambda &= (bd - 2ae)^2 - (b^2 - 4ac) \qquad (d^2 - 4af), \\
4b\lambda &= 2(be - 2cd)\,(bd - 2ae) - 4(bd - 2ae)\,(be - 2cd), \\
4c\lambda &= (be - 2cd)^2 - (b^2 - 4ac)\,(e^2 - 4cf), \\
4d\lambda &= 2(be - 2cd)\,(d^2 - 4af) + 2\,(bd - 2ae)\,(de - 2bf), \\
4e\lambda &= 2(bd - 2ae)\,(e^2 - 4cf) + 2\,(be - 2cd)\,(de - 2bf), \\
4f\lambda &= (de - 2bf)^2 - (d^2 - 4af)\,(e^2 - 4cf).
\end{aligned}$$

These two tangents will be identical, on the vanishing of the radical in (3); and this will be effected either by $\lambda = 0$, or

$$ay_1^2 + bx_1y_1 + cx_1^2 + dy_1 + ex_1 + f = 0 \ldots\ldots\ldots\ldots(4).$$

Now the former of these signifies that the conic section is composed of two straight lines: and the latter, that the point x_1y_1 is situated *in the curve itself*.

In this case of the point x_1y_1 in the curve, or the tangents becoming identical, we have from (2)

$$p^2 = \frac{(b^2 - 4ac)y_1^2 + 2(be - 2cd)y_1 + (e^2 - 4cf)}{(b^2 - 4ac)x_1^2 + 2(bd - 2ae)x_1 + (d^2 - 4af)} \ldots\ldots\ldots(5).$$

Add $4c(ay_1^2 + bx_1y_1 + cx_1^2 + dy_1 + ex_1 + f) = 0$ to the denominator,
and $4a(ay_1^2 + bx_1y_1 + cx_1^2 + dy_1 + ex_1 + f) = 0$ to the numerator;
then we shall have

$$p^2 = \frac{4c^2x_1^2 + 4bcx_1y_1 + 4ecx_1 + b^2y_1^2 + 2bey_1 + e^2}{4a^2y_1^2 + 4abx_1y_1 + 4ady_1 + b^2x_1^2 + 2bdx_1 + d^2} \ldots\ldots\ldots (6);$$

and extracting the roots of both sides, we get

$$p = -\frac{2cx_1 + by_1 + e}{2ay_1 + bx_1 + d} \ldots\ldots\ldots\ldots\ldots\ldots (7);$$

and the equation of the tangent, therefore, becomes

$$y - y_1 = -\frac{2cx_1 + by_1 + e}{2ay_1 + bx_1 + d}(x - x_1) \ldots\ldots\ldots\ldots\ldots(8).$$

If, again, we multiply out the expression for the tangent at the point x_1y_1 in the curve just obtained, we get, after substituting f for $ay_1^2 + bx_1y_1 + cx_1^2 + dy_1 + ex_1$, the following useful form:

$$2ay_1y + b(x_1y + y_1x) + 2cx_1x + d(y + y_1) + e(x + x_1) + 2f = 0 \text{ [*] } \ldots (9),$$

Corollary.

The normal being perpendicular to the tangent, its equation will be (see p. 260), when the ordination is rectangular,

$$y - y_1 = \frac{2ay_1 + bx_1 + d}{2cx_1 + by_1 + e}(x - x_1) \ldots\ldots\ldots\ldots\ldots (10);$$

or when the axes make the angle a,

$$y - y_1 = \frac{(2ay_1 + bx_1 + d) - (2cx_1 + by_1 + e)\cos a}{(2cx_1 + by_1 + e) - (2ay_1 + bx_1 + d)\cos a}(x - x_1) \ldots (11).$$

Scholium.

Expressions for the *sub-normal, sub-tangent, etc.*, are given by some writers in continuation of this proposition; but as the *only* use made of them is in reference to the rectangular axes passing through the centre, it is deemed by us a most inappropriate place to bring them into.

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[*] This formula, whilst quite as well fitted for actual use as that marked (8), has the advantage of being easily recalled to memory. For write the equation in the following form:

$$ayy + bxy + cxx + dy + ex + f = 0.$$

Form a line by writing one factor only with a dash, and then a second line, writing the remaining factors with a dash, leaving in each case the other factors in plain letters; this will give

$$ay_1y + bx_1y + cx_1x + dy_1 + ex_1 + f,$$
$$ayy_1 + bxy_1 + cxx_1 + dy + ex + f.$$

The sum of these two, equated to zero, is the expression (9) just given.

PROP. V.

To find the locus of the middle points of parallel chords from the general equation of the conic section.

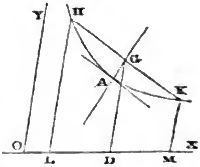
Let the general equation be as before, and denote the system of parallel chords (q being the quantity upon which the particular chord depends), by

$$y = px + q.$$

Insert this in the general equation; then we get (as in *prop. iii.*)

$$x^2 + \frac{2apq + bq + dp + e}{ap^2 + bp + c}\,x + \frac{aq^2 + dq + f}{ap^2 + bp + c} = 0.$$

Let HK be that one of the chords which depends on an assumed value of q, and G be its middle point; and draw the ordinates HL, GD, KM; then OL, OM, are the values of the abscissa x in the above equation, and OD the abscissa of the middle point of the chord HK.



Now OD $= \frac{1}{2}$(OL + OM); and by the theory of roots, the co-efficient of x is the sum of the two roots OL, OM, with their signs changed. That is, denoting now by x and y, the co-ordinates of G, we have

$$x = -\frac{2apq + bq + dp + e}{2(ap^2 + bp + c)}; \text{ or again,}$$

$$q = -\frac{2(ap^2 + bp + c)\,x + (dp + e)}{2ap + b}.$$

Equate this to the value of q in the first equation above; then, after due reduction, we get, as the locus of G,

$$(2ap + b)\,y + (bp + 2c)\,x + (dp + e) = 0.$$

The middles of all the parallel chords are therefore situated in a straight line.

We have next to inquire whether there be any point for which this equation holds good for all values of p. If such be the case, we shall have, by arranging the equation according to p, the form

$$(2ay + bx + d)\,p + 2cx + by + e = 0;$$

And from the indeterminateness of p we get

$$2ay + bx + d = 0, \text{ and } 2cx + by + e = 0;$$

$$\text{or, } x = -\frac{bd - 2ae}{b^2 - 4ac}, \text{ and } y = -\frac{be - 2cd}{b^2 - 4ac}.$$

This result shows that the point to which the origin is transposed to remove the first powers of x and y in the general equation, is that also through which all the diameters pass: in other words, the *centre of the curve* is that in which the co-ordinate origin being placed, the equation of the curve is of the form $a_1y^2 + b_1xy + c_1x^2 + f_1 = 0$.

When $b^2 - 4ac = 0$, the centre is infinitely distant, and the lines converging to it become virtually parallel, agreeing with what has before been determined (*page* 298): but this may be more advantageously viewed under the following aspect.

For c substitute its value $\frac{b^2}{4a}$, or for a its value $\frac{b^2}{4c}$, in the equation of the diameter; then we get

$$2ay + bx + \frac{2a\,(dp + e)}{2ap + b} = 0, \text{ or } by + 2cx + \frac{2c\,(dp + e)}{2c + bp} = 0,$$

and this expresses different lines depending on the value of p; but all of them parallel to the line (both equations representing the same),

$$2ay + bx = 0, \text{ or } by + 2cx = 0.$$

The tangent at the extremity, A, of the diameter, is parallel to HK.

This may be established by finding the co-ordinates of A, and thence forming the equation of the corresponding tangent. It would be found that p has the same value as in the expression of the parallel chords. It, however, follows, from the simple consideration, that if the line drawn through A, parallel to the chords, should again cut the circle, that A is also the middle of the chord so drawn. Whence the tangent is parallei as asserted.

Also, the tangent at the other extremity of the diameter, where the curve is either an ellipse or hyperbola, is parallel to HK.

These accord with properties already deduced geometrically; and those were in fact there made the definition of conjugates.

Corollary.

In the general case, let the equations of two series of lines which vary their positions, as q and q' vary, be

$$y = px + q, \text{ and } y = p'x + q';$$

and let, moreover, the diameter which bisects all the chords of the conic section intercepted upon the first series, be one of the second series; then the intercepted chords of the second series will be bisected by a diameter which is one of the first series.

For, in the case of the first and second hypotheses, we have respectively

$$p' = -\frac{bp + 2c}{2ap + b}, \text{ and } p = -\frac{bp' + 2c}{2ap' + b}.$$

Now these two equations are identical, as will be seen on resolving the former for p, or the latter for p'; but it will readily appear from reducing them, they both becoming

$$2app' + b(p + p') + 2c = 0.$$

This, then, is the criterion of two lines being conjugate in reference to the general conic section,

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

and we shall call it the *equation of ordination.*

From this we learn, that to every line $y = px + q$, there is one system of lines, and only one, conjugate to it. For, to each value of p there is one, and one only, corresponding value of p'.

It also follows, that every diameter has one, and only one diameter, which is conjugate to it; and these are the lines through the centre, parallel to the two lines which are defined as to parallelism by p and p'.

Again, if β, γ, be the angles made with the axis of x by the systems of lines depending on p and p', we have (the angle of ordination being a)

$$p = \frac{\sin\beta}{\sin(a - \beta)}, \text{ and } p' = \frac{\sin\gamma}{\sin(a - \gamma)};$$

and if we insert these in the *equation of ordination*, we get

$$2a\sin\beta\sin\gamma + b\{\sin\beta\sin(a-\gamma) + \sin\gamma\sin(a-\beta)\} + 2c\sin(a-\beta)\sin(a-\gamma) = 0.$$

Now, if we transform the general equation to a system of co-ordinate axes, making the angles β, γ, with the original axis of x (as at p. 250), we shall have the expression on the left side of the equation just obtained, as the co-efficient of xy in the transformed. It follows, therefore, that in an equation of the conic section referred to conjugate lines as axes of co-ordinates, the term involving xy is absent: or, in other words, any equation of the conic sections in which $b = 0$, is referred to conjugate axes of co-ordinates.

There will amongst the systems of conjugate axes of co-ordinates be one through any given point which will be rectangular, and only one.

For let $\gamma = \beta + 90^\circ$; then $\alpha - \gamma = (\alpha - \beta) - 90^\circ$; and the conjugate equation becomes

$$-a \sin 2\beta + b \sin (2\beta - \alpha) + c \sin 2(\alpha - \beta) = 0; \text{ from which}$$

$$\tan 2\beta = \frac{-b \sin \alpha + c \sin 2\alpha}{a - b \cos \alpha + c \cos 2\alpha}.$$

When the original axes are rectangular, this becomes simplified into

$$\tan 2\beta = \frac{-b}{a - c};$$

agreeing with the determination of the same tangent given at *p.* 297.

PROP. VI.

To find when an equation of the second degree represents two straight lines.

Denote the equation of the second degree as (1), and those of two straight lines by (2, 3), as follows:

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

$$py + qx + r = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$p_1 y + q_1 x + r_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Multiply (2, 3) together: then, equating the co-efficients of the resulting equation with those of (1), we get

$$\begin{array}{r|l} pp_1 = a & pq_1 + p_1 q = b \\ qq_1 = c & pr_1 + p_1 r = d \\ rr_1 = f & qr_1 + q_1 r = e \end{array} \left.\vphantom{\begin{array}{l} a \\ b \\ c \end{array}}\right\} \ldots\ldots\ldots\ldots\ldots (4).$$

From the first column of these we get $p_1 = \frac{a}{p}$, $q_1 = \frac{c}{q}$, and $r_1 = \frac{f}{r}$; and these, inserted in the second column, give

$$\left.\begin{array}{r} aq^2 + cp^2 = bpq \\ ar^2 + fp^2 = dpr \\ cr^2 + fq^2 = eqr \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

From the second and third of these equations we get at once

$$\frac{p}{r} = \frac{d \pm \sqrt{d^2 - 4af}}{2f}, \text{ and } \frac{q}{r} = \frac{e \pm \sqrt{e^2 - 4cf}}{2f};$$

which, inserted in (2), gives the equation

$$(d \pm \sqrt{d^2 - 4af})\, y + (e \pm \sqrt{e^2 - 4cf})\, x + 2f = 0 \ldots\ldots\ldots (6);$$

and hence also, from (4) we get

$$(d \mp \sqrt{d^2 - 4af})\, y + (e \mp \sqrt{e^2 - 4cf})\, x + 2f = 0 \ldots\ldots\ldots (7).$$

But we have employed only *two* of the equations (5) for the determination of the co-efficients of (6, 7), whilst all the three conditions are to be fulfilled by them. In fact, from the second and third we can get a value of $\frac{p}{q}$, and likewise from the first, another value of $\frac{p}{q}$. In order, therefore, to fulfil all the conditions, these values must be equal, and we must obtain an equation independent of them by equating these values; or, in other words, we must eliminate p, q, r, from the three equations.

Eliminating r in the usual manner from the second and third equations, we get

$$cdep^2 - (cd^2 + fb^2 + e^2a - 4acf)\,pq + aedq^2 = 0;$$

which combined with the first gives the final equation

$$fb^2 + cd^2 + ae^2 - bde - 4acf = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Now the possibility of the resolution of (1) into the factors (2, 3) depends upon the *co-existence* of the three equations (5), and upon the *reality of the roots* of each. The former condition is expressed by (8); and the latter will require that the three quantities $b^2 - 4ac$, $d^2 - 4af$, and $e^2 - 4cf$, shall be *positive;* as when they are not, the lines must be imaginary, even should equation (8) be fulfilled.

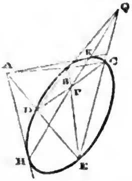
It may, however, be shown, that should (8) and either two of the three equations $b^2 - 4ac$, *etc.* be fulfilled, the remaining one will be fulfilled also; though as each of those expressions requires to be formed for the composition of (8), there is no advantage in attempting to contract the number of tests, especially as they are in some slight degree checks upon each other. Theoretically, however, there are but three conditions essential.

Scholium.

It should be remarked, that $b^2 - 4ac$ being *positive* (one of the above conditions), is that of the curve being the hyperbola: and it has been shown, that one species of hyperbola is composed of merely two straight lines. The consistency of the results is, therefore, apparent.

PROP. VII.

From a fixed point A, within or without a conic section, draw lines to cut the curve in B, C, and D, E; also draw BE, CD, to meet in P; then P is always situated in the polar *of which A is the* pole.



Let the curve be denoted by

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \ldots\ldots (1).$$

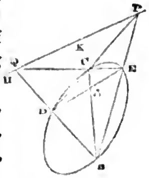
Also, let the conjoint equation of the lines AC, AE, and that of the lines BE, CD, be expressed (*p.* 256) by

$$a(y - y_1)^2 + 2\text{A}(y - y_1)(x - x_1) + \text{B}(x - x_1)^2 = 0 \ldots (2),$$

$$a(y - y_2)^2 + 2\text{A}'(y - y_2)(x - x_2) + \text{B}'(x - x_2)^2 = 0 \ldots (3);$$

x_1y_1 denoting the point A, and x_2y_2 the point P.

Then, since the three equations exist conjointly, all those which result from taking their sums, differences, *etc.*, exist also conjointly with them. Wherefore, expand (2, 3), and equate their sum to the double of (1); the co-efficients of which, again equated term for term, will give the following system of equations:



$$\text{A} + \text{A}' = b \ldots\ldots (4) \quad | \quad \text{B} + \text{B}' = 2c \ldots\ldots (5),$$

$$a(y_1 + y_2) + \text{A}x_1 + \text{A}'x_2 = -d \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$\text{A}y_1 + \text{B}x_1 + \text{A}'y_2 + \text{B}'x_2 = -e \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$a(y_1^2 + y_2^2) + 2\text{A}x_1y_1 + 2\text{A}'x_2y_2 + \text{B}x_1^2 + \text{B}'x_2^2 = 2f \ldots (8).$$

Insert the values of A′, B′, from (4, 5) in the three next, which gives

$$A(x_1 - x_2) = -a(y_1 + y_2) - bx_2 - d \;\ldots\ldots\ldots\ldots (9),$$

$$A(y_1 - y_2) + B(x_1 - x_2) = -by_2 - 2cx_2 - e \;\ldots\ldots\ldots\ldots (10),$$

$$2A(x_1y_1 - x_2y_2) + B(x_1^2 - x_2^2) = -a(y_1^2 + y_2^2) - 2cx_2^2 - 2bx_2y_2 + 2f \;\ldots (11).$$

Multiply (10) by $x_1 + x_2$, and subtract the result from (11); then

$$(x_1 - x_2)(y_1 + y_2)A = (2cx_2 + by_2 + e)x_1 + ex_2 + 2f - ay_1^2 - ay_2^2 - bx_2y_2 \;\ldots (12).$$

Multiply (9) by $y_1 + y_2$, and equate with (12); then, after slight reduction, there results

$$2ay_1y_2 + bx_1y_2 + bx_2y_1 + 2cx_1x_2 + d(y_1 + y_2) + e(x_1 + x_2) + 2f = 0 \;\ldots (13).$$

Now this equation may be written in either of the forms

$$(2ay_1 + bx_1 + d)y_2 + (2cx_1 + by_1 + e)x_2 + dy_1 + ex_1 + 2f = 0 \;\ldots\ldots (14),$$

$$(2ay_2 + bx_2 + d)y_1 + (2cx_2 + by_2 + e)x_1 + dy_2 + ex_2 + 2f = 0 \;\ldots\ldots (15);$$

in which x_1y_1 and x_2y_2 are, obviously, interchangeable.

The former of these equations, x_2y_2 denoting the variable point, is the equation of a straight line, the locus of P, which was the object of our inquiry. Dropping the subscribed numerals, that line is denoted by

$$(2ay_1 + bx_1 + d)y + (2cx_1 + by_1 + e)x + dy_1 + ex_1 + 2f = 0 \;\ldots\ldots (16).$$

This result is easily made the foundation of the *algebraical theory of conjugate polars:* we can, however, here only give a few of the slightest and most elementary consequences, which we shall do in the shape of *corollaries*.

Corollary 1.

As we have not particularized by the formation of (3) which pairs of lines BE, CD, or BD, CE, was to give by their intersection, the point P, the result applies to that of *either pair*, and hence that of *both pairs*. The points P and Q are, therefore, *both* situated in the straight line PQ, designated by (16).

Corollary 2.

If a point x_2y_2, situated in (16), be made the pole, then x_1y_1 will be a point in the polar; for both hypotheses are fulfilled by the same equation (13).

Corollary 3.

If the point A or x_1y_1 be *without* the curve, and tangents AK, AH, be drawn, the points of contact will be situated in PQ.

For in this case BE and CD coalesce, and the equation (3) which represents them is simply reduced to the condition of $A'^2 = aB'$; a circumstance which produces no effect on the subsequent reductions, and which is perfectly accidental in respect of the general system of relations.

Corollary 4.

If through any point A, as *pole*, (either within or without the curve,) lines be drawn to cut the curve, as HK, the tangents at the extremities will always intersect in the *polar* corresponding to A.

For in this case, the only peculiarity attaching to the investigation is, that AC and AD coalesce, and hence that (2) is subject to the additional relation $A^2 = aB$; which, as in the preceding corollary, is without any influence either upon the investigations or the final result.

PROP. VIII.

If parallel to two given lines secants be drawn to cut each other, and a line of the second order, the rectangle under the intercepted segments of the one will have to the rectangle under the segments of the other, a given ratio.

Let $ay^2 + bxy + cx^2 + dy + ex + f = 0$ be the curve, β, γ, the angles made by the given lines with the axis of x, and $x'y'$ the co-ordinates of the point of intersection of the secants.

Refer the curve to new co-ordinate axes, making respectively the angles β, γ, with the axis of x (*prop. ii. p.* 250), and likewise the co-ordinates $x'y'$ to the same new axes. The equation of the curve will then take the form

$$a_1y^2 + b_1xy + c_1x^2 + d_1y + e_1x + f_1 = 0,$$

and the co-ordinates of the point will become x_1y_1. Then again, remove the origin of co-ordinates to x_1y_1, and the equation of the curve will be transformed to

$$a_2y^2 + b_2xy + c_2x^2 + d_2y + e_2x + f_2 = 0.$$

The curve is now referred to the two secants as co-ordinate axes, and the intercepted segments of these axes are the segments to which the theorem relates. To find these segments, put successively x and y equal to zero; then

$$a_2y^2 + d_2y + f_2 = 0,$$
$$c_2x^2 + e_2x + f_2 = 0,$$

are the equations whose roots are the segments of the secants.

Now by the theory of equations, $\frac{f_2}{a_2}$ and $\frac{f_2}{c_2}$ are the products of the respective pairs of roots, and it is to be shown that these have a constant ratio; or that $c_2 : a_2$ is constant for all positions of the two secants as long as they are parallel to the given lines.

In respect of the previous transformations, it becomes only necessary to attend to the composition of the co-efficients a_2 and c_2. Let, then, α be the angle of ordination to which the given equation is constructed. Then the first transformation requires us to write for x and y the values (*p.* 250)

$$\frac{x \sin(\alpha - \beta) + y \sin(\alpha - \gamma)}{\sin \alpha} \text{ and } \frac{x \sin \beta + y \sin \gamma}{\sin \alpha}.$$

Again, a_1 and c_1 are the co-efficients of y^2 and x^2 in the first transformation; in other words, to the first and third terms of the equation, which involve terms of the second degree. These give

$$a_1 = \frac{a \sin^2\gamma + b \sin\gamma \sin(\alpha - \gamma) + c \sin^2(\alpha - \gamma)}{\sin^2\alpha},$$

$$c_1 = \frac{a \sin^2\beta + b \sin\beta \sin(\alpha - \beta) + c \sin^2(\alpha - \beta)}{\sin^2\alpha}.$$

Also, it is clear that $a_2 = a_1$, and $c_2 = c_1$, since the transformation of the origin does not affect the co-efficients of the second powers of the variables. Whence the alleged ratio is constant; viz.

$$\frac{c_2}{a_2} = \frac{a \sin^2\beta + b \sin\beta \sin(\alpha - \beta) + c \sin^2(\alpha - \beta)}{a \sin^2\gamma + b \sin\gamma \sin(\alpha - \gamma) + c \sin^2(\alpha - \gamma)}.$$

Scholium.

It may be well for the student to bear in mind that, universally, the co-efficients of the second powers y^2, xy, x^2, are affected only by the transformation of co-ordinates, as regards *direction;* and the co-efficients of the first powers, d, e, as well as f, are affected by the transformation of the *origin* of co-ordinates.

PROP. IX.

In the plane of every curve represented by an equation of the second degree, and referred to any co-ordinate angle, there can always be found a point and a right line, such that the distances of any point in the curve, one from the point to be found, and the other from the line to be found, shall have a constant ratio, which ratio can also be found.

Let the equations of the given curve and of the line to be found be

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \ldots\ldots\ldots\ldots (1).$$

$$py + qx + r = 0 \ldots\ldots\ldots\ldots (2).$$

Also, let hk denote the point to be found, m the ratio to be found, and α the angle of ordination.

Retaining the parallelism of the co-ordinate axes, transfer the origin to the centre of the curve (*p.* 297); then we have the preceding equations reduced to

$$ay^2 + bxy + cx^2 + f' = 0 \ldots\ldots\ldots\ldots (3),$$

$$py + qx + r' = 0 \ldots\ldots\ldots\ldots (4);$$

$$\text{where } r' = \frac{2cd - be}{b^2 - 4ac} \cdot p + \frac{2ae - bd}{b^2 - 4ac} \cdot q + r \ldots\ldots\ldots\ldots (a),$$

$$\text{and } f' = \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f \ldots\ldots\ldots\ldots (b).$$

Also, if D and P denote the distance and the length of the perpendicular, we have $D^2 = m^2P^2$; and the values of these being inserted (*pages* 253, *eq.* 1, and 263, *eq.* 34), we get

$$(x-h')^2 + 2(x-h')(y-k')\cos\alpha + (y-k')^2 = \frac{m^2(py + qx + r')\sin^2\alpha}{p^2 - 2pq\cos\alpha + q^2} \ldots\ldots (5);$$

where $h'k'$ denotes the point in question referred to the new axes.

Expand this, and equate the co-efficients of the result to those of (3). We thus obtain the following six equations for the determination of the six quantities, p, q, r', h', k', and m; premising that u^2 is put for $p^2 - 2pq\cos\alpha + q^2$.

$$a = u^2 - m^2p^2\sin^2\alpha \ldots\ldots\ldots\ldots (6),$$

$$b = 2u^2\cos\alpha - 2m^2pq\sin^2\alpha \ldots\ldots\ldots\ldots (7),$$

$$c = u^2 - m^2q^2\sin^2\alpha \ldots\ldots\ldots\ldots (8),$$

$$0 = u^2(k' + h'\cos\alpha) + m^2pr'\sin^2\alpha \ldots\ldots\ldots\ldots (9),$$

$$0 = u^2(h' + k'\cos\alpha) + m^2qr'\sin^2\alpha \ldots\ldots\ldots\ldots (10),$$

$$f' = u^2(h'^2 + 2h'k'\cos\alpha + k'^2) - m^2r'^2\sin^2\alpha \ldots\ldots\ldots\ldots (11).$$

Multiply (7) by $\cos\alpha$, and subtract from the sum of (6) and (8); then

$$a - b\cos\alpha + c = 2u^2\sin^2\alpha - m^2\sin^2\alpha\,(p^2 - 2pq\cos\alpha + q^2)$$

$$= u^2(2 - m^2)\sin^2\alpha \ldots\ldots\ldots\ldots (12).$$

Again, from the same three equations we have

$$m^2p^2\sin^2\alpha = u^2 - a \ldots\ldots\ldots\ldots (13),$$

$$2m^2pq\sin^2\alpha = 2u^2\cos\alpha - b \ldots\ldots\ldots\ldots (14),$$

$$m^2q^2\sin^2\alpha = u^2 - c \ldots\ldots\ldots\ldots (15).$$

In this, $(2m^2pq\sin^2\alpha)^2 = 4m^2p^2\sin^2\alpha \,.\, m^2q^2\sin^2\alpha$, or

$(2u^2\cos\alpha - b)^2 = 4(u^2 - a)(u^2 - c)$; or again,

$4u^4\sin^2\alpha - 4(a - b\cos\alpha + c)u^2 = b^2 - 4ac$, and

$$u^2 = \frac{a - b\cos\alpha + c \pm \sqrt{(a - b\cos\alpha + c)^2 + (b^2 - 4ac)\sin^2\alpha}}{2\sin^2\alpha}$$

$$= \frac{Q + R}{2\sin^2\alpha} \text{ (see } p.\text{ 299, } Schol. \text{ 3)} \ldots\ldots\ldots\ldots (16).$$

Again, this equation becomes $2u^2 \sin^2 a - Q = R$, which, inserted in (12), gives

$$m^2 = \frac{R}{u^2 \sin^2 a} = \frac{2R}{Q + R} \dots\dots\dots\dots\dots\dots (17);$$

the constant ratio is therefore determined.

We next proceed to p and q; which are obtained at once from (13) and (15) respectively by means of (16, 17).

For (*p*. 299, *Schol*. 3),

$$p^2 = \frac{u^2 - a}{m^2 \sin^2 a} = \frac{(Q + R - 2a \sin^2 a)(Q + R)}{4R \sin^4 a} = \frac{(H + R)(Q + R)}{4R \sin^4 a} \dots (18),$$

$$q^2 = \frac{u^2 - c}{m^2 \sin^2 a} = \frac{(Q + R - 2c \sin^2 a)(Q + R)}{4R \sin^4 a} = \frac{(K + R)(Q + R)}{4R \sin^4 a} \dots (19).$$

The signs of p and q will be the same or different according as $2u^2 \cos a - b$, or $(Q + R) \cos a - b \sin^2 a$, is positive or negative; as at once appears from (14).

Next, resolving (9, 10) for h' and k', we get

$$h' = -\frac{m^2 r' (q - p \cos a)}{u^2} \dots\dots\dots\dots\dots\dots (20),$$

$$k' = -\frac{m^2 r' (p - q \cos a)}{u^2} \dots\dots\dots\dots\dots\dots (21).$$

Also, multiply (9) by k' and (10) by h', and add the results: then,

$$u^2 (h'^2 + 2h'k' \cos a + k'^2) = - m^2 r' (pk + qh) \sin^2 a \dots\dots (22).$$

Insert (20, 21) in (22), and reduce: then,

$$u^2 (h'^2 + 2h'k' \cos a + k'^2) = m^4 r'^2 \sin^2 a \dots\dots\dots \quad (23).$$

Substitute for the left side of (23) its value in (11), and we get

$$r'^2 = \frac{f'}{m^2 (m^2 - 1) \sin^2 a} = \frac{(R + Q)^2 f'}{2R (R - Q) \sin^2 a} \dots\dots\dots (24).$$

Lastly, substitute the values of r', p, q, u^2, in (20, 21); and there results

$$h' = -\frac{\sqrt{K + R} - \sqrt{H + K} \,.\, \cos a}{\sqrt{b^2 - 4ac}} \cdot \frac{\sqrt{\frac{1}{2} f'}}{\sin^2 a} \dots\dots\dots (25),$$

$$k' = -\frac{\sqrt{H + R} - \sqrt{K + R} \,.\, \cos a}{\sqrt{b^2 - 4ac}} \cdot \frac{\sqrt{\frac{1}{2} f'}}{\sin^2 a} \dots\dots\dots (26).$$

To these, adding the co-ordinates of the centre, we have h, k, the co-ordinates of the point to be found; and f and r are found by means of (a) and (b).

Scholium.

The case of the parabola is not *directly* demonstrable by this process, since the centre is at an infinite distance. However, the equations in reference to the original origin, admit of comparatively easy solution when $b^2 - 4ac = 0$; and the student is recommended to exercise himself in the discussion of this case, as the expressions reduce to tolerably simple forms.

The problem in its general form appears to have been first discussed by M. Bret, *Ann. des Math. tom. viii.*; and it has subsequently been discussed by Sir John Lubbock, in the *Phil. Mag.* Aug. 1831. The method adopted by both these distinguished geometers is, to find a point in the plane of the conic section such, that all lines drawn from it to the curve shall be rational functions of the data of the curve. This method did not lead them to consider the focus in connexion with the directrix—a view which it will be evident from the preceding solution, is far simpler in all its details than that which they employed. Bret's method is, however, generalised in the following problem.

A process very analogous to the one here employed, the origin being transferred to a point in the curve itself, is applicable to all the three curves together. For the case of the parabola the reader may consult a paper by the editor of this work, in the *Phil. Mag.* for January 1843, *pp.* 15—22.

PROP. X.

The distance of the focus (hk), *from any point* (x_1y_1) *in the general curve of the second degree, referred to any system of axes*, $ay^2+bxy+cx^2+dy+ex+f=0$, *is a rational linear function of the co-ordinates* x_1, y_1, *of that point.*

Or, in other words:—

$(x_1-h)^2+2(x_1-h)(y_1-k)\cos a+(y_1-k)^2$ *is of the form* $(p_1y_1+q_1x_1+r_1)^2$, p_1, q_1, r_1 *being determinable functions of the co-efficients* a, b, c, d, e, f.

For, multiplying the general equation of the curve in reference to x_1y_1 by the arbitrary quantity λ, we shall have the two following expressions identical:

$$\lambda\,(ay_1^2+bx_1y_1+cx_1^2+dy_1+ex_1+f), \text{ and}$$
$$(x_1-h)^2+2(x_1-h)(y_1-k)\cos a+(y_1-k)^2-(p_1y+q_1x+r_1)^2.$$

Equating the co-efficients of the homologous terms of x_1, y_1, we get

$$\begin{array}{ll|ll}
\lambda a = 1 - p_1^2 \;\ldots\ldots\ldots\ldots & (1) & \lambda d = -2\,(k + h\cos a + p_1r_1) \;. & (4)\\
\lambda b = 2\cos a - p_1q_1 \;\ldots\ldots\ldots & (2) & \lambda e = -2\,(h + k\cos a + q_1r_1) \;.. & (5)\\
\lambda c = 1 - q_1^2 \;\ldots\ldots\ldots\ldots & (3) & \lambda f = h^2 + 2hk\cos a + k^2 - r_1^2 \;.. & (6)
\end{array}$$

We have to show that p_1, q_1, r_1, λ, h, k are determinable and real. The work only sketched out, leaving the details for the student's exercise.

From (1, 2, 3,) we have successively

$$p_1^2 = 1-\lambda a, \text{ and } q_1^2 = 1-\lambda c; \text{ and hence}$$
$$4\,(1-\lambda a)\,(1-\lambda c) = 4p_1^2q_1^2 = (2p_1q_1)^2 = (2\cos a - \lambda b)^2;$$

or, collecting and arranging the terms,

$$\left.\begin{array}{l}
(b^2-4ac)\,\lambda^2 + 4\lambda\,(a - b\cos a + c) = 4\sin^2 a, \text{ or}\\
\dfrac{4\sin^2 a}{\lambda^2} - \dfrac{4\,(a-b\cos a+c)}{\lambda} = b^2-4ac
\end{array}\right\}\;\ldots\ldots\ldots\;(7).$$

This is identical with (19) of the preceding solution, where $u^2=\dfrac{1}{\lambda}$; and hence we have, as in that place, and by resolving the above for λ, the two forms

$$\frac{1}{\lambda}=\frac{Q+R}{2\sin^2 a}, \text{ or } \lambda = \frac{2\,(R-Q)}{b^2-4ac} \;\ldots\ldots\ldots\ldots\ldots\;(8),$$

or λ is the reciprocal of the determining ratio.

Again, from (1, 3) we have

$$p_1^2 = 1-\lambda a = 1-\frac{2a\sin^2 a}{Q+R}=\frac{H+R}{Q+R}\;\ldots\ldots\ldots\ldots\;(9),$$
$$q_1^2 = 1-\lambda c = 1-\frac{2c\sin^2 a}{Q+R}=\frac{K+R}{Q+K}\;\ldots\ldots\ldots\ldots\;(10),$$

and p_1, q_1 must have the same or different signs according $2\cos a - \lambda b$ is positive or negative, from (2).

Again, h, k are determined from (4, 5, 6), as in the former problem, and their values are hence known.

Lastly, the value of r_1 may now be found from *either* of the equations (4, 5, 6), and as it is of the first degree in (4, 5), it is evidently real, and all the required magnitudes are now found. It will, however, to obtain symmetrical results, be necessary either to obtain it from (6) or from some symmetrical combinations, as the sum or difference, of (4) and (5).

Corollary.

The distance of the point x_1y_1 from the directrix is also a rational linear function of x_1 and y_1. For it has a fixed ratio to the focal distance.

PROP. XI.

To find the equation of a conic section through five given points, four of which are in the axes of co-ordinates.

Denote the points by $(x_1 0)$, $(x_2 0)$, $(x_3 y_3)$, $(0 y_4)$, $(0 y_5)$; then the curve being $ay^2 + bxy + cx^2 + dy + ex + f = 0$, we have the five equations of condition.

$$ay_3^2 + bx_3y_3 + cx_3^2 + dy_3 + ex_3 + f = 0 \;\ldots\ldots\ldots\ldots\; (1),$$

$$cx_1^2 + ex_1 + f = 0 \;\ldots\ldots\; (2), \qquad ay_4^2 + dy_4 + f = 0 \;\ldots\ldots\; (5),$$

$$cx_2^2 + ex_2 + f = 0 \;\ldots\ldots\; (3). \qquad ay_5^2 + dy_5 + f = 0 \;\ldots\ldots\; (6).$$

From (2, 3) we have

$$c = \frac{f}{x_1x_2}, \text{ and } e = -\frac{f(x_1 + x_2)}{x_1x_2}$$

From (5, 6) we have

$$a = \frac{f}{y_4y_5}, \text{ and } d = -\frac{f(y_4 + y_5)}{y_4y_5}$$

Insert these values of a, c, d, e, in (1); and then we get

$$b = -\frac{f}{x_3y_3}\left\{\frac{(y_3 - y_5 - y_4)\,y_3}{y_4y_5} + \frac{(x_3 - x_2 - x_1)\,x_3}{x_1x_2} + 1\right\}$$

$$= f\left\{-\frac{y_3 - y_5}{x_3y_4y_5} - \frac{x_3 - x_1}{x_1x_2y_3} + \frac{x_3 - x_1}{x_1x_2y_3} + \frac{1}{x_3y_5}\right\}.$$

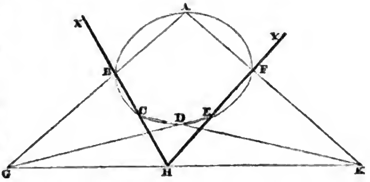
Scholium.

The general problem, when all the points are *any how given*, leads to reductions so laborious, that the formulæ of solution have never, I believe, been given.

PROP. XII.

If the three pairs of opposite sides of a hexagon inscribed in a conic section be produced to meet, the three points of intersection will range in the same straight line. (Pascal's hexagram.)

First solution, by Mr. Rutherford.



Let A, B, C, D, E, F, be six points in the curve of a conic section, and let the opposite sides as enunciated meet in G, H, K; then G, H, K, are in one straight line. For take HCB and HEF for the axes of co-ordinates of x and y respectively, and designate by a, a_1, β, β_1, the distances of the points E, F, C, B, from the origin H; and by x_1y_1 and x_2y_2 the co-ordinates of the points D and A. Consequently, we have

(1) .... $\frac{\alpha - x_1}{\alpha y_1} . y + \frac{x}{\alpha} = 1$, for the equation of DE.

(2) .... $\frac{\beta_1 - y_2}{\beta_1 x_2} . x + \frac{y}{\beta_1} = 1$, for the equation of AB.

(3) .... $\frac{\beta - y_1}{\beta x_1} . x + \frac{y}{\beta} = 1$, for the equation of CD.

(4) .... $\frac{\alpha_1 - x_2}{\alpha_1 y_2} . y + \frac{x}{\alpha_1} = 1$, for the equation of AF.

Subtracting equation (2) from equation (1), and also (3) from (4), we obtain

(5) . .. $\left\{\frac{1}{\alpha} - \frac{\beta_1 - y_2}{\beta_1 x_2}\right\} x + \left\{\frac{\alpha - x_1}{\alpha y_1} - \frac{1}{\beta_1}\right\} y = 0$, for the equation of HG.

(6) .... $\left\{\frac{1}{\alpha_1} - \frac{\beta - y_1}{\beta x_1}\right\} x + \left\{\frac{\alpha_1 - x_2}{\alpha_1 y_2} - \frac{1}{\beta}\right\} y = 0$, for the equation of HK.

Let XY be the co-ordinates of H, and X_1Y_1 those of the point K; then from (5) and (6) we find

$$\frac{X}{Y} = \frac{\alpha y_1 + \beta_1 x_1 - \alpha\beta_1}{\alpha y_2 + \beta_1 x_2 - \alpha\beta_1} . \frac{x_2}{y_1} \text{ and } \frac{X_1}{Y_1} = \frac{\alpha_1 y_2 + \beta x_2 - \alpha_1\beta}{\alpha_1 y_1 + \beta x_1 - \alpha_1\beta} . \frac{x_1}{y_2}.$$

Now if the equation of the curve be represented by

(7) .... $y^2 + bxy + cx^2 + dy + ex + f = 0$,

it must be satisfied by substituting the co-ordinates of each of the six points, A, B, C, D, E, F, in the curve: hence we have

$c\alpha^2 + e\alpha + f = 0$ .... (8)

$c\alpha_1^2 + e\alpha_1 + f = 0$ .... (9)

$\beta^2 + d\beta + f = 0$ .... (10)

$\beta_1^2 + d\beta_1 + f = 0$ .... (11)

$y_1^2 + bx_1y_1 + cx_1^2 + dy_1 + ex_1 + f = 0$ .... (12)

$y_2^2 + bx_2y_2 + cx_2^2 + dy_2 + ex_2 + f = 0$ .... (13)

From (8), (9), (10), (11), we obtain $e = -c(\alpha_1 + \alpha)$; $f = c\alpha\alpha_1$; $d = -(\beta_1 + \beta)$, and $f = \beta\beta_1$. Now equating these values of f, we get

$$c = \frac{\beta\beta_1}{\alpha\alpha_1}, \text{ and } e = -\frac{\beta\beta_1}{\alpha\alpha_1}(\alpha_1 + \alpha).$$

Multiply equation (12) by x_2y_2, and (13) by x_1y_1, and subtract; then we have

$$x_2y_2(y_1^2 + cx_1^2 + dy_1 + ex_1 + f) = x_1y_1(y_2^2 + cx_2^2 + dy_2 + ex_2 + f).$$

In this equation substitute the preceding values of c, d, e, f, and we get

$$\frac{x_1y_1}{x_2y_2} = \frac{\alpha\alpha_1 y_1^2 + \beta\beta_1 x_1^2 - \alpha\alpha_1(\beta + \beta_1) y_1 - \beta\beta_1(\alpha + \alpha_1) x_1 + \alpha\alpha_1\beta\beta_1}{\alpha\alpha_1 y_2^2 + \beta\beta_1 x_2^2 - \alpha\alpha_1(\beta + \beta_1) y_2 - \beta\beta_1(\alpha + \alpha_1) x_2 + \alpha\alpha_1\beta\beta_1}.$$

But since $\frac{x_1y_1}{x_2y_2} = \frac{(\alpha\beta + \alpha_1\beta_1) x_1y_1}{(\alpha\beta + \alpha_1\beta_1) x_2y_2}$; consequently, (*vol. i. theor.* 71, *p.* 325,)

$$\frac{x_1y_1}{x_2y_2} = \frac{\alpha\alpha_1 y_1^2 + (\alpha\beta + \alpha_1\beta_1) x_1y_1 + \beta\beta_1 x_1^2 - \alpha\alpha_1(\beta + \beta_1) y_1 - \beta\beta_1(\alpha + \alpha_1) x_1 + \alpha\alpha_1\beta\beta_1}{\alpha\alpha_1 y_2^2 + (\alpha\beta + \alpha_1\beta_1) x_2y_2 + \beta\beta_1 x_2^2 - \alpha\alpha_1(\beta + \beta_1) y_2 - \beta\beta_1(\alpha + \alpha_1) x_2 + \alpha\alpha_1\beta\beta_1}$$

$$= \frac{(\alpha y_1 + \beta_1 x_1 - \alpha\beta_1)(\alpha_1 y_1 + \beta x_1 - \alpha_1\beta)}{(\alpha y_2 + \beta_1 x_2 - \alpha\beta_1)(\alpha_1 y_2 + \beta x_2 - \alpha_1\beta)}; \text{ therefore}$$

$$\frac{\alpha y_1 + \beta_1 x_1 - \alpha\beta_1}{\alpha y_2 + \beta_1 x_2 - \alpha\beta_1} . \frac{x_2}{y_1} = \frac{\alpha_1 y_2 + \beta x_2 - \alpha_1\beta}{\alpha_1 y_1 + \beta x_1 - \alpha_1\beta} . \frac{x_1}{y_2}.$$

Consequently, from what has been done above, we have $XY_1 - X_1Y = 0$; and therefore the three points G, H, K, range in the same straight line GHK.

Scholium.

Another solution different from this may be seen in the *Phil. Mag.* July, 1842; and others in the *Ladies' Diary*, 1843. See also *pages* 190, 233, of this work.

Second solution, by Mr. Fenwick.

Take the same axes of x and y as before, and denote the six points, in the same order as C, B, A, *etc.*; thus,

$$x_1 0,\ x_2 0,\ x_3 y_3,\ 0 y_4,\ 0 y_5,\ x_6 y_6.$$

Then we may represent the conic section by the general equation (each term being divided by f),

$$ay^2 + bxy + cx^2 - dy - ex + 1 = 0 \ \ldots\ldots\ldots\ldots\ldots \ (1).$$

The equations of the lines which meet in M and N will obviously be

$$(\text{AB}) \ \ldots\ldots\ldots\ldots\ y(x_2 - x_3) + xy_3 = x_2 y_3 \ \ldots\ldots\ldots\ldots\ldots \ (2),$$
$$(\text{DE}) \ \ldots\ldots\ldots\ldots\ yx_6 + x(y_5 - y_6) = y_5 x_6 \ \ldots\ldots\ldots\ldots\ldots \ (3),$$
$$(\text{AF}) \ \ldots\ldots\ldots\ldots\ yx_3 + x(y_4 - y_3) = y_4 x_3 \ \ldots\ldots\ldots\ldots\ldots \ (4),$$
$$(\text{CD}) \ \ldots\ldots\ldots\ldots\ y(x_1 - x_6) + xy_6 = y_6 x_1 \ \ldots\ldots\ldots\ldots\ldots \ (5).$$

Combining (2, 3), and also (4, 5), so that the absolute terms in the resulting equations may be eliminated, we get the equations of GH, KH, viz.,

$$(\text{GH}) \ \ldots\ldots\ y\left\{\frac{x_2 - x_3}{x_2 y_3} - \frac{1}{y_5}\right\} + x\left\{\frac{1}{x_2} - \frac{y_5 - y_6}{y_5 x_6}\right\} = 0 \ \ldots\ (6),$$

$$(\text{KH}) \ \ldots\ldots\ y\left\{\frac{x_1 - x_6}{x_1 y_6} - \frac{1}{y_4}\right\} + x\left\{\frac{1}{x_1} - \frac{y_4 - y_3}{y_4 x_3}\right\} = 0 \ \ldots\ (7).$$

Our object will now be attained, provided we can prove that (6) and (7) make equal angles with the axis of x; and this will be the case, when

$$\alpha\beta' = \alpha'\beta \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (8),$$

α, β being the co-efficients of y and x in (6), α', β', those of y and x in (7).

$$\text{Now, } \alpha\beta' = \left\{\frac{x_2 - x_3}{x_2 y_3} - \frac{1}{y_5}\right\}\left\{\frac{1}{x_1} - \frac{y_4 - y_3}{y_4 x_3}\right\}$$
$$= \left\{-\frac{x_3}{x_2 y_3} + \frac{1}{y_3} - \frac{1}{y_5}\right\}\left\{\frac{y_3}{y_4 x_3} - \frac{1}{x_3} + \frac{1}{x_1}\right\}.$$
$$= -\frac{1}{x_2 y_4} - \frac{y_3 - y_5}{x_3 y_4 y_5} - \frac{x_3 - x_1}{x_1 x_2 y_3} + \frac{x_3 - x_1}{x_1 x_3 y_3} + \frac{1}{x_3 y_5} - \frac{1}{x_1 y_5} \ \ldots\ (9).$$

In a similar way we get

$$\alpha'\beta = -\frac{1}{x_1 y_5} - \frac{y_6 - y_4}{x_6 y_4 y_5} - \frac{x_6 - x_2}{x_1 x_2 y_6} + \frac{x_6 - x_2}{x_2 x_6 y_6} + \frac{1}{x_6 y_4} - \frac{1}{x_2 y_4} \ \ldots\ (10).$$

Again, substituting successively in (1), the co-ordinates of the four points in the axes, and of the point $x_3 y_3$ (or A), we get five equations, from which we readily find (*prop. xi.*), making due allowance for changes of sign,

$$b = -\frac{y_3 - y_5}{x_3 y_4 y_5} - \frac{x_3 - x_1}{x_1 x_2 y_3} + \frac{x_3 - x_1}{x_1 x_3 y_3} + \frac{1}{x_3 y_5}.$$

In a precisely similar way, using the points in the axes, and the point $x_6 y_6$, or D, we get

$$b = -\frac{y_6 - y_4}{x_6 y_4 y_2} - \frac{x_6 - x_2}{x_1 x_2 y_6} + \frac{x_6 - x_2}{x_2 x_6 y_6} + \frac{1}{x_6 y_4}.$$

And since (9) and (10) are equal, G, H, K, are in a straight line by (8).

PROP. XIII.

If a hexagon be circumscribed to a conic section, the three diagonals which join the three pairs of opposite summits, will pass through the same point.

Let $p_1 p_2 p_3$, &c. be the points of contact, which we shall denote by

$$x_1 y_1,\ x_2 y_2,\ x_3 y_3,\ x_4 y_4,\ x_5 y_5,\ x_6 y_6;$$

and AB, BC, *etc.*, the tangents at these points.

Having joined CF, EB, intersecting in O, and also DO, AO; refer the system to CF, BE, as axes of co-ordinates, and denote the conic section by the general equation:—

$$ay^2 + bxy + cx^2 + dy + ex + 1 = 0 \ldots (\text{A}).$$

The several tangents will then be denoted by the equations :—

(AB) .... $y\,(2ay_1+bx_1+d)+x(2cx_1+by_1+e)+dy_1+ex_1+2=0$ .... (1),
(BC) .... $y\,(2ay_2+bx_2+d)+x(2cx_2+by_2+e)+dy_2+ex_2+2=0$.... (2),
(CD) .... $y\,(2ay_3+bx_3+d)+x(2cx_3+by_3+e)+dy_3+ex_3+2=0$.... (3),
(DE) .... $y\,(2ay_4+bx_4+d)+x(2cx_4+by_4+e)+dy_4+ex_4+2=0$.... (4),
(EF) .... $y\,(2ay_5+bx_5+d)+x(2cx_5+by_5+e)+dy_5+ex_5+2=0$.... (5),
(FA) .... $y\,(2ay_6+bx_6+d)+x(2cx_6+by_6+e)+dy_6+ex_6+2=0$.... (6).

Combining (3, 4), and also (1, 6), so that the absolute terms in the resulting equations may be eliminated, we get the equations of DO, AO :—

$$(\mathrm{DO})\,..\,y\left\{\frac{2ay_3+bx_3+d}{dy_3+ex_3+2}-\frac{2ay_4+bx_4+d}{dy_4+ex_4+2}\right\}+x\left\{\frac{2cx_3+by_3+e}{dy_3+ex_3+2}-\frac{2cx_4+by_4+e}{dy_4+ex_4+2}\right\}=0\;..\,(7),$$

$$(\mathrm{AO})\,..\,y\left\{\frac{2ay_1+bx_1+d}{dy_1+ex_1+2}-\frac{2ay_6+bx_6+d}{dy_6+ex_6+2}\right\}+x\left\{\frac{2cx_1+by_1+e}{dy_1+ex_1+2}-\frac{2cx_6+by_6+e}{dy_6+ex_6+2}\right\}=0\;..(8).$$

It will suffice now to show that the products of the co-efficients of y, x, in (7) and (8), taken in a *cross order*, are equal, or, rather, the products of the numerators, since the products of the denominators are evidently equal.

Performing the subtractions indicated by the signs, with each of these co-efficients, we get

$\alpha=(2ae-bd)\,(y_3x_4-x_3y_4)+(4a-d^2)\,(y_3-y_4)+(2b-de)\,(x_3-x_4)$.. (9),
$\beta'=(2cd-be)\,(x_1y_6-x_6y_1)+(4c-e^2)\,(x_1-x_6)+(2b-de)\,(y_1-y_6)$..(10),
$\alpha'=(2ae-bd)\,(y_1x_6-x_1y_6)+(4a-d^2)\,(y_1-y_6)+(2b-de)\,(x_1-x_6)$..(11),
$\beta=(2cd-be)\,(x_3y_4-y_3x_4)+(4c-e^2)\,(x_3-x_4)+(2b-de)\,(y_3-y_4)$..(12),

α, β being the co-efficients of y and x in (7), without the denominators, α', β' those of y and x in (8).

Again, since the lines (2, 3) meet in the axis of x, put $y=0$ in each of these, and equate the resulting values of x, then we get the following relation :

$(2cd-be)\,(y_2x_3-y_3x_2)+(4c-e^2)\,(x_3-x_2)+(2b-ed)\,(y_3-y_2)=0$ .... (13).

Reasoning in a similar way with the lines (5, 6), (1, 2), and (4, 5), we get the following :—

$(2cd-be)\,(x_6y_5-x_5y_6)+(4c-e^2)\,(x_6-x_5)+(2b-ed)\,(y_6-y_5)=0$ .. (14),
$(2ae-bd)\,(y_1x_2-x_1y_2)+(4a-d^2)\,(y_1-y_2)+(2b-ed)\,(x_1-x_2)=0$ .. (15),
$(2ae-bd)\,(y_4x_5-x_4y_5)+(4a-d^2)\,(y_4-y_5)+(2b-de)\,(x_4-x_5)=0$ .. (16).

If, now, we substitute the values of $(2cd-be)$, $(2ae-bd)$, $(4c-e^2)$, and $(4a-d^2)$ derived from (13, 14, 15, 16), in (9, 10, 11, 12), and multiply out, we shall get the relation

$$\alpha\beta'=\alpha'\beta,$$

hence A, O, D are in one straight line.

For this very elegant investigation of Brianchon's theorem, which has not, I believe, before been completely effected by the method of co-ordinates, I am indebted to my colleague, Mr. Fenwick. A geometrical demonstration has been given at *p.* 191. The idea of *eliminating the co-efficients* of the equation of the curve, so as to *produce an identical equation in terms of the co-ordinates of the six points of contact*, is, I think, both new, and likely to be effective in other cases.

It should, however, be remarked, that a co-ordinate investigation of the theorem adapted to the case of the parabola, was given by Sir John Lubbock, in the *Philosophical Magazine*, for August 1838; and he points out the mode of extending it to the other cases.

RECAPITULATORY SYNOPSIS OF THE EQUATIONS.

For any angle of ordination, α; and when $\alpha = \frac{1}{2}\pi$ *.

General equations.

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \quad\ldots\ldots\ldots\ldots (1)$$

$$y = -\frac{bx + d \mp \sqrt{(b^2 - 4ac)\,x^2 + 2\,(bd - 2ae)\,x + (d^2 - 4af)}}{2a} \quad\ldots\ldots (2).$$

$$x = -\frac{by + e \mp \sqrt{(b^2 - 4ac)\,y^2 + 2\,(be - 2cd)\,y + (e^2 - 4cf)}}{2c} \quad\ldots\ldots (3).$$

Co-ordinates of the centre of the curve.

$$x' = \frac{2ae - bd}{b^2 - 4ac};\ y' = \frac{2cd - be}{b^2 - 4ac} \quad\ldots\ldots\ldots\ldots (4);$$

which are infinite when $b^2 - 4ac = 0$. These will render $d' = 0$ and $e' = 0$ in the transformed equation.

Equation of conjugation of two lines.

$$2app_1 + b\,(p + p_1) + 2c = 0 \quad\ldots\ldots\ldots\ldots (5);$$

the lines being denoted by the equations

$$y = px + q, \text{ and } y = p_1x + q_1.$$

Intersections of the curve with the axes of co-ordinates.

$$y = -\frac{d \pm \sqrt{d^2 - 4af}}{2a};\ x = -\frac{e \pm \sqrt{e^2 - 4cf}}{2c} \quad\ldots\ldots (6).$$

The curve touches the axes of y or x when $d^2 - 4af = 0$, or $e^2 - 4cf = 0$; and both when both are fulfilled. When these quantities are negative, the curve does not meet the corresponding axes of co-ordinates.

Origin and inclination of co-ordinate axes.

The co-ordinate axes are *conjugate* whether the system be oblique or rectangular, when $b = 0$.

The transformation is effected by (5), where

$$p = \frac{\sin\beta}{\sin(\alpha - \beta)},\ \text{and } p_1 = \frac{\sin\beta_1}{\sin(\alpha - \beta_1)} \quad\ldots\ldots\ldots\ldots (a);$$

and β, β_1, are the angles made by the new axes of x and y with the old axis of x.

The corresponding values of the coefficients of y^2 and x^2, are

$$\frac{a - b\cos\alpha + c \pm \mathrm{R}}{2\sin^2\alpha},\ \text{and } \frac{a - b\cos\alpha + c \mp \mathrm{R}}{2\sin^2\alpha} \quad\ldots\ldots\ldots (b);$$

where $\mathrm{R}^2 = (a - b\cos\alpha + c)^2 + (b^2 - 4ac)\sin^2\alpha$, and the upper sign of the value of R being that which belongs to the value of $c\sin 2\alpha - b\sin\alpha$.

If, moreover, the origin be transferred to the centre of the curve, the two conjugate semi-axes will be expressed by

$$\frac{-2\sin^2\alpha}{a - b\cos\alpha + c \pm \mathrm{R}.}\left\{\frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f\right\} \quad\ldots\ldots\ldots (c).$$

---

* The reader will remark, that in the conic sections, as in the straight line, a considerable number of the equations are alike when the axes of co-ordinates are oblique and when right-angled. The cases, where they become simplified by $\alpha = \frac{1}{2}\pi$, are put down immediately following the general case and the equations marked †.

When the original and the new axes are rectangular, these expressions become simplified, for $\cos \alpha = 0$, and $\sin \alpha = 1$. The student will do well to write out these particular results for his future use.

The new axes of co-ordinates will be rectangular and conjugate when the new axis of x makes an angle β with the old, such that

$$\tan 2\beta = \frac{c \sin 2\alpha - b \sin \alpha}{a - b \cos \alpha + c \cos 2\alpha}, \text{ and } \beta_1 - \beta = \tfrac{1}{2}\pi \ldots\ldots\ldots (7\,\dagger).$$

Special curves under certain conditions amongst the co-efficients.

PARABOLA. $b^2 - 4ac = 0$.	ELLIPSE. $b^2 - 4ac = -$	HYPERBOLA. $b^2 - 4ac = +$
CASES.	CASES.	CASE.
$bd - 2ae = 0$ *two parallel lines.*	$c = a,\ b = 2a \cos \alpha$ *a circle.*	$(bd - 2ae)^2 = (b^2 - 4ac)(d^2 - 4af)$ *two straight lines.*
$bd - 2ae = 0$ $d^2 - 4af = 0$ *one straight line.*	$(bd - 2ae)^2 = (b^2 - 4ac)(d^2 - 4af)$ *a point.*	$b = 0,\ a + c = 0$ *the equilateral hyperbola.*
$bd - 2ae = 0$ $d^2 - 4af = -$ *imaginary locus.*	$(bd - 2ae)^2 < (b^2 - 4ac)(d^2 - 4af)$ *imaginary locus.*	

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

The tangent to the curve drawn through a point x_1y_1 in the curve is expressed by

$$y - y_1 = -\frac{2cx_1 + by_1 + e}{2ay_1 + bx_1 + d}(x - x_1) \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

or by $(2ay_1 + bx_1 + d)\, y + (2cx_1 + by_1 + e) + dy_1 + ex_1 + 2f = 0 \ldots. (10)$

or by $2ay_1y + b\,(x_1y + y_1x) + 2cx_1x + d(y + y_1) + e(x + x_1) + 2f = 0 \ .. \ (11)$.

The normal to the curve at the point x_1y_1 in it is

$$y - y_1 = \frac{(2ay_1 + bx_1 + d) - 2(cx_1 + by_1 + e) \cos \alpha}{(2cx_1 + by_1 + e) - 2(ay_1 + bx_1 + d) \cos \alpha}(x - x_1) \ \ldots. (12),$$

$$y - y_1 = \frac{2ay_1 + bx_1 + d}{2cx_1 + by_1 + e} x - x_1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13\,\dagger).$$

Equation of the tangent to the curve from a point x_1y_1 without it, see p. 300.

Equation of the corresponding chord of contact *.

$$(2ay_1 + bx_1 + d)\, y + (2cx_1 + by_1 + e)\, x + dy_1 + ex_1 + 2f = 0 \ldots. (14).$$

Equation of a diameter of the curve through any point x_1y_1.

$$\left\{(b^2 - 4ac)x_1 + (bd - 2ae)\right\}(y - y_1) - \left\{(b^2 - 4ac)y_1 + (be - 2cd)\right\}(x - x_1) = 0;$$

and the lines conjugate to this diameter are .... (15),

$$(2ay_1 + bx_1 + d)\, y + (2cx_1 + by_1 + e)\, x + k = 0 \ldots\ldots\ldots (16),$$

where the values of k individualise the particular lines.

In the parabola, the diameter is expressed either by

$$2a(y - y_1) + b(x - x_1) = 0, \text{ or } 2c(x - x_1) + b(y - y_1) = 0 \ldots. (17, 18).$$

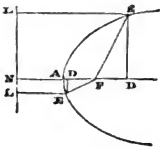
---

* This is, more generally, the equation of the *conjugate polar* to the *pole* x_1y_1, in reference to the conic section (*prop. vii.*).

PARTICULAR CASES AND PROPERTIES OF LINES OF THE SECOND ORDER.

PROP. I.

A point and a line being given in position, and a variable point being taken, so that its distances from the given point and line be in a given ratio: then the locus of this variable point is a line of the second order.



Let F be the given point, NL the given line, and E a position of the variable point; also let e be the given ratio, such that $\mathrm{FE} = e \,.\, \mathrm{EL}$. Denote the parts of the line FN by $\mathrm{NA} = h$, $\mathrm{AF} = k$, it being so divided in A that $\mathrm{AF} = e \,.\, \mathrm{AN}$, or $k = eh$. Draw ED perpendicular to NF, and make $\mathrm{AD} = x$, $\mathrm{DE} = y$. Then

$$\mathrm{FD}^2 + \mathrm{DE}^2 = \mathrm{FE}^2 = e^2\mathrm{EL}^2 = e^2\mathrm{ND}^2, \text{ or}$$

$$(x - k)^2 + y^2 = e^2 (x + h)^2, \text{ or by reduction}$$

$$y^2 = 2k (1 + e) x - (1 - e^2) x^2 \ldots\ldots\ldots\ldots \quad (1).$$

This being of the second degree, the locus of E is a line of the second order.

Applying to this the criterion (*page* 299), we have

$$\mathrm{B}^2 - 4\mathrm{AC} = -4 (1 - e^2) \ldots\ldots\ldots\ldots \quad (2),$$

we learn, therefore, that when $e = 1$, the curve is the *parabola;* that when e is less than 1, it is the *ellipse;* and that when e is greater than 1, it is the *hyperbola*.

Corollary 1.

When $e = 1$, then $y^2 = 4kx$, the equation of the *parabola* referred to its principal axis, agreeing with the determination at *p.* 298.

When $e < 1$, we have, by putting $(1 - e) a$ for k, the form $y^2 = (1 - e^2) (2ax - x^2)$, adapted to the *ellipse*, see *p.* 299. When $e > 1$, we have $1 - e^2$ negative; and hence, if we put $k = -(1 - e)a$, we have for the equation of the hyperbola, $y^2 = (e^2 - 1) (2ax + x^2)$. See *p.* 299.

Corollary 2.

The rectangle under the abscisses in the ellipse and hyperbola varies as the square of the ordinate. For $2ax - x^2$ in the ellipse, and $2ax + x^2$ in the hyperbola are the said rectangles, and y^2 in both cases varies as they do.

Corollary 3.

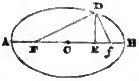
In the same way the square of the ordinate of the parabola varies as the abscissa x.

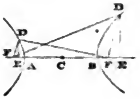
Scholium.

This is the converse of a general property established in the preceding chapter, *p.* 308. We have used the letter e in this place instead of the m in that one: the reason of this will appear in a subsequent proposition, in which we shall have occasion to recur to this subject. See also *pages* 109, 132, and 152, of the geometrical treatise in this volume. The point and line are the *focus* and *direction.—Defs* 18, 24, *p.* 106, 107.

PROP. II.

If lines be drawn from two given points to meet in a third point, such that their sum or difference be given, the locus of the third point will respectively be the ellipse and hyperbola.





Let F, f be the given points from which lines drawn to any point D have their sum or difference equal to a given line $2a$; also denote Ff by $2ae$. Draw DE perpendicular to Ff, and bisect Ff, in C. Then, if CE $= x$, and ED $= y$, we shall have the conditions expressed by

$$\sqrt{(ae + x)^2 + y^2} + \sqrt{(ae - x)^2 + y^2} = 2a,$$
$$\sqrt{(ae + x)^2 + y^2} - \sqrt{(ae - x)^2 + y^2} = 2a.$$

Reducing these in the ordinary manner we shall have the equations

$$y^2 = (1 - e^2)(a^2 - x^2),$$
$$\text{and, } y^2 = (e^2 - 1)(x^2 - a^2).$$

When e is less than 1, the criterion gives the *ellipse*, and the first form is adapted; and when e is greater than 1, the second form applies, and the criterion shows that the curve is the *hyperbola*.

PROP. III.

To investigate the forms of the equations of the ellipse and hyperbola referred to conjugate diameters as co-ordinate axes, and assign the constants in terms of those diameters [*].

THE general equation of the curve being written

$$Ay^2 + Bxy + Cx^2 + Dy + Ex = F,$$

we have D $= 0$ and E $= 0$, because it is referred to the centre; and we likewise have B $= 0$, because it is referred to conjugate axes (*p.* 303): wherefore the equation is in this case reduced to

$$Ay^2 + Cx^2 = F \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now, according to the *signs* of these co-efficients, we shall have three cases to consider.

(1.) Let them all be + as in (1); then applying the criterion $B^2 - 4AC$, we find the curve so written to be the *ellipse*.

---

[*] As it is of the utmost consequence to employ one uniform system of estimating the *ordinate directions on the axes*, we shall here always adopt the following method: it being, in fact, that which we have uniformly employed in discussing the straight line.

We suppose ourselves on one side of the *horizontal* axis of x, (however that axis may lie in our actual figure with respect to the horizontal and vertical edges of the paper,) and that the positive values of x are on the *right*, whilst the negative are on the *left* of the horizon. The positive values of y are estimated on the other axes above or beyond the axis of x, and the negative below the same axis. When, also, a transfer of origin, in the following pages, is made from the centre, it is invariably done by going to the left for x and downwards for y; that is, by passing negatively. This causes the substitution for x and y to become $x - h$ and $y - k$ where $-h$ and $-k$ are the respective values of the new origin in reference to the original one.

Angular magnitude is also estimated *upwards* from x, and entirely round the origin, as in the discussion at *p.* 414. *vol. i.*

(2.) We may have C *minus*, and the equation will then be

$$Ay^2 - Cx^2 = F \ldots\ldots (2).$$

The criterion shows this to be the *hyperbola*.

(3.) We may take A *minus*, giving

$$-Ay^2 + Cx^2 = F, \text{ or } Ay^2 - Cx^2 = -F \ldots\ldots (3).$$

This is also in the same way the equation of the *hyperbola*.

The only variety, so far as sign is concerned, yet left is

$-Ay^2 - Bx^2 = -F$, which is equivalent to (1).

Let the curve cut the axes of co-ordinates; then y and x are at those points respectively equal to zero. Denote the portion of the intercepted axis of x by a, and that of y by b; then we have

$$a = \pm \sqrt{\frac{F}{C}}, \text{ and } b = \pm \sqrt{\frac{F}{A}}.$$

The double signs showing that the intercepted portions of the axes are bisected at the origin of co-ordinates.

These also give

$$A = \frac{F}{b^2} \text{ and } C = \frac{F}{a^2}; \text{ which convert (1) into}$$

$$a^2y^2 + b^2x^2 = a^2b^2 \ldots\ldots (4).$$

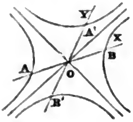
In the same way it will be found that (3) and (2) become convertible into

$$a^2y^2 - b^2x^2 = -a^2b^2 \ldots\ldots (5),$$

$$a^2y^2 - b^2x^2 = +a^2b^2 \ldots\ldots (6).$$

The values of a and b are both real or both imaginary in the ellipse (4): for, by hypothesis, A and C have the same sign. Hence both diameters meet the curve defined by the equation, or neither of them does; that is, in the latter case, the curve is imaginary, since the origin is *within* the curve by its nature.

In each of the equations (5, 6), however, there is one value real and one imaginary. In (5) the semidiameters are $\pm a$ and $\pm b\sqrt{-1}$, and in (6) they are $\pm a\sqrt{-1}$ and b. The *absolute values* are the same in reference to both curves; but one curve is met by one axis and the other by the other. The two equations (5, 6) therefore represent the two pairs of hyperbolas, which, in the geometrical treatise (*p.* 105), have been called *conjugate hyperbolas*.



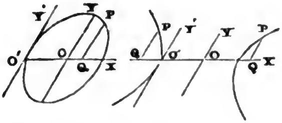
It is this circumstance to which reference is made at (*p.* 162, *note*).

PROP. IV.

To find the equations of the three curves referred to a point in the curve as origin, and the diameter and tangent through that point as co ordinate axes.

In all the curves, when the axes of co-ordinates are conjugate, we have B=0; and the axes of y or x, as the case may be, will be co-incident with, or parallel to, the corresponding axis in the preceding proposition.

The ellipse and *the primary hyperbola:* these will only require us to write $x - a$ for x: in which case, if OY be the conjugate axis to OX, O'Y' will be parallel to it; and the equations are at once transformed to



$$a^2y^2 + b^2(x - a)^2 = +a^2b^2, \text{ or } a^2y^2 + b^2x^2 - 2ab^2x = 0 \ldots\ldots (7),$$

$$a^2y^2 - b^2(x - a)^2 = -a^2b^2, \text{ or } a^2y^2 - b^2x^2 + 2ab^2x = 0 \ldots\ldots (8).$$

Again, for equation (6) of *the conjugate hyperbola*, to refer it to a point in the curve itself, we must make our transformation upon the axis of y, by writing $y - b$ for y; which transforms it into

$$a^2 (y - b)^2 - b^2x^2 = a^2b^2, \text{ or } a^2y^2 - b^2x^2 - 2a^2by = 0 \; \dots\dots\dots \; (9).$$

The parabola. Since $B^2 - 4AC = 0$, and that $B = 0$, the system being conjugate, either A or C must be zero; whence the equation is either

$$Ay^2 + Dy + Ex = F,$$
$$\text{or } Cx^2 + Dy + Ex = F;$$

which are of the same general form, the co-ordinate axes only being interchanged. It will, therefore, be sufficient to discuss the first of them.

For x and y write $x + h$ and $y + k$; then the equation becomes

$$Ay^2 + 2Aky + Ak^2 + Dy + Dk + Ex + Eh = F.$$

In this, as we have two indeterminate quantities h and k, we may make any convenient hypothesis respecting them. Let it be that y shall be absent, and the absolute term be zero. These will be fulfilled by

$$2Ak + D = 0, \text{ and } Ak^2 + Dk + Eh = F; \text{ or}$$
$$k = -\frac{D}{2A}, \text{ and } h = \frac{D^2 + 4AF}{4AE};$$

and the equation has the form

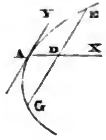
$$y^2 = 4ax \; \dots\dots\dots\dots\dots\dots \; (10).$$

Now the original axes are conjugate by the hypothesis $B = 0$; and since the absolute term is absent, the origin is in the curve itself. Also, since a tangent and diameter through the same point in the curves are always conjugate; and by hypothesis, the axis of x is a diameter, the above equation (10) expresses the relation of the absciss and ordinate in the sense required by the problem.

PROP. V.

To trace the general figures of the curves from their equations deduced in proposition iv.

1. *The parabola,* $y^2 = 4hx$ [*].



The equation takes the form $y = \pm 2 \sqrt{hx}$. For all positive values of x there are two real and equal values of y with contrary signs. Take AX, AY, the co-ordinate axes making the conjugate angle, to which the equation is adapted.

Then, if $x = 0$, we have also $y = 0$, and the curve passes through A.

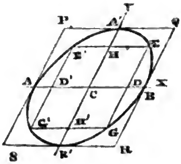
Next, take any positive value AD of x; then DE, DG, being set off equal to the corresponding values of y, give two points, E, G, in the curve, symmetrically situated with respect to the axis AX. The same will occur whenever to the right of A, and however great the distance from it, the point D be taken; whilst all negative values of x give y imaginary. The curve, therefore, is of unlimited extension to the right of AY, and lies wholly on that side of it.

Moreover, the curve passing through A, and having all its other points to the right of that line, the line AY is a tangent to the curve at A.

---

[*] If the equation of the parabola referred to the vertex and principal diameter be transformed to a point $x_1 y_1$ in the curve, taking $\beta = 0$, $\gamma = \alpha =$ the conjugate angle: then it will be found that $h' = h \operatorname{cosec}^2 \gamma$. See also *prop. ix. p.* 115.

2. *The ellipse*, $a^2y^2 + b^2x^2 = a^2b^2$.



Draw AB, A'B', equal to $2a$, $2b$, and bisecting each other in C, and inclined to each other under the conjugate angle to which the equation is referred; and draw the parallelogram PQRS, having its sides parallel to the axes AB, A'B'.

Take $x = 0$; then $y = \pm b$; whence the curve passes through A' and B'.

Take $y = 0$; then $x = \pm a$; whence the curve also passes through A and B.

Put the equation in the form $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$; then it is clear that $\pm a$ are the greatest possible positive or negative values of x that will render y real. The curve does not, thence, extend beyond A and B in the direction of the axis of x. In a similar manner, it is shown that the curve does not extend beyond A' and B'.

Again, take any value of x, as CD; then the equation gives two equal values of y, one on the positive and the other on the negative side of AB, which will be represented by DE and DG. Both these are less than AC' or B'C, since $\pm\frac{b}{a}\sqrt{a^2 - x^2}$ is less than b.

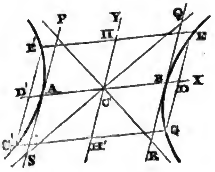
The same holds if CD' be taken on the negative side of the origin, and the positive and negative values of x being taken of the same absolute value, the values of y corresponding to them will also be equal.

In the same manner, if the equation were put in the form $x = \pm\frac{a}{b}\sqrt{b^2 - y^2}$, there would result similar conclusions. The figure indicates the lines upon which the student is to conduct the reasoning.

It follows, therefore, that the curve is composed of four symmetrical branches, viewed in reference to the axes of co-ordinates, viz. AA', A'B, BB', B'A.

Again, since HE and ED are respectively less than CA' and CB, the point E lies within the parallelogram BQ A'C: and the same applies to every point in the curve except A, A', B, B'. The four sides, PQ, QR, RS, SP, of the parallelogram are therefore tangents to the curve at A', B, B', A. They are also conjugate, being by construction parallel to conjugate diameters, and forming a parallelogram, it is the conjugate parallelogram (*def.* 21, *p.* 106).

3. *The hyperbola*, $a^2y^2 - b^2x^2 = -a^2b^2$.



Here $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$. Hence for all positive and negative values of x, absolutely less than a, the corresponding values of y are imaginary; and for all absolutely greater, it is real, whether taken positively or negatively.

Let CX, CY, produced both ways, be taken as axes of x and y. Make AC and CB equal to a, on both sides of the origin; and draw the indefinite lines QR, PS, through B and A, parallel to CY.

Then, when $x = \pm a$, we have $y = 0$, and the curve passes through A and B; and no part of the curve can lie between the parallels QR, PS, since for all the values of x so taken, those of y are imaginary.

All positive and all negative values of x greater than a, give two corresponding values of y: as in the figure, if $\mathrm{CD} = x$, then y is represented by DE and DG; and if CD′ be represented by $-x$, then y will be represented by D′E′ and D′G′. The curve, therefore, is composed of two branches capable of indefinite extension in contrary directions, (since all such values of x render y real,) commencing at A and B, and each of them symmetrically situated with respect to the axis CX, and lying respectively to the left and right of the two parallels QR and PS.

They are also symmetrically situated with respect to the axis CY, as the resolution of the equation for x will show, it giving $x = \pm \frac{a}{b} \sqrt{y^2 + b^2}$; and as is illustrated in the figure by the lines EE′ and GG′.

But there are also two other lines between which the curves are confined. For putting the equation under the form $y = \pm \frac{bx}{a} \sqrt{1 - \frac{a^2}{x^2}}$; and expanding the radical,

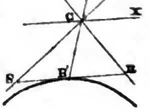
$$y = \pm \frac{bx}{a} \left\{ 1 - \frac{a^2}{2x^2} - \frac{a^4}{8x^4} - \frac{a^6}{16x^6} - \dots ad\ inf. \right\}.$$

Now if we suppose the two lines $y = \pm \frac{bx}{a}$ to be drawn, we see at once that by increasing the absolute magnitude of x, we shall have the values of y in the curve *always less* than the corresponding ones in the straight line; and whilst as x is increased, the corresponding values become nearer and nearer to equality. They can only become actually equal when x is infinite. The curve is therefore situated wholly in the angles QCR, PCS.

This conclusion agrees with the property shown in the geometrical treatise (*pp.* 156—160) to belong to the *asymptotes* of the hyperbola.

4. *The conjugate hyperbola,* $a^2y^2 - b^2x^2 = a^2b^2$.

This gives $y = \pm \frac{b}{a} \sqrt{x^2 + a^2}$; and hence for *every* value of x, positive and negative, there are two real values of y. The curve is in this case capable of indefinite extension, as far as regards the axis of x; and it is, as in the preceding case, limited between two lines $y = \pm \frac{bx}{a}$; but with this difference, that the value of y in the curve is now *always greater* than that in the line. It is, therefore, altogether situated in the other pair of opposite angles QCP, RCS; and it may be shown, as in the preceding case, that it is composed of branches symmetrically situated with respect to the axes of co-ordinates CX, CY.



The resolution for x, however, shows that the two lines PQ, RS, pass through A′, B′, and are parallel to CX; and hence the curve is in all respects analogous to the preceding one, and has the same asymptotes. This is characteristic of the *conjugate hyperbolas.* See *pages* 156—160 of the geometrical treatise in this volume.

Indeed, as the equation is the same with $b^2x^2 - a^2y^2 = -a^2b^2$, we might have seen that the curve would be in this case of the same general form and character with the preceding one, only interchanging the axes of x and y, and their corresponding diameters $2a$ and $2b$.

PROP. VI.

To investigate the parameters, foci, and directrices of lines of the second order.

Here we shall employ the rectangular system, from the greater simplicity conferred upon the inquiry by means of it.

1. *The parameters* (*def.* 17, *p.* 106).

Denote by p the parameter and a, b, the semi-axes; then we have,

(1). $\frac{1}{2}p = \frac{b^2}{a}$ in the *ellipse;*

(2). $\frac{1}{2}p = \frac{b^2}{a}$ in the *primary hyperbola;*

(3). $\frac{1}{2}p = \frac{a^2}{b}$ in the *conjugate hyperbola;*

(4). $\frac{1}{2}p = 2h$ in the *parabola.*

2. *The foci* (*def.* 18, *p.* 106).

In the *ellipse* put $y = \frac{1}{2}p$ and $x = u$; then,

$$\tfrac{1}{4}a^2p^2 + b^2u^2 = a^2b^2;$$

or resolving and substituting for p^2,

$$u = \pm\sqrt{\frac{a^2(b^2 - \frac{1}{4}p^2)}{b^2}} = \pm\sqrt{a^2 - b^2};$$

or the square of the distance of the focus from the centre is equal to $a^2 - b^2$.

In both the systems of *hyperbolas* we have similarly,

$$u = \pm\sqrt{a^2 + b^2}.$$

In the *parabola*, estimated from the vertex, we have, putting u for the distance of the focus,

$$\tfrac{1}{4}p^2 = 4hu; \text{ or } u = \frac{\frac{1}{4}p^2}{4h} = \frac{4h^2}{4h} = h.$$

3. *The directrices* (*def.* 24, *p.* 107).

The equations are those of straight lines perpendicular to the transverse axes. Their equations follow at once to be

(1). In the ellipse .... $x \mp \frac{a^2}{\sqrt{a^2 - b^2}} = 0.$

(2). The primary hyperbola .... $x \mp \frac{a^2}{\sqrt{a^2 + b^2}} = 0.$

(3). The conjugate hyperbola ... $x \mp \frac{b^2}{\sqrt{a^2 + b^2}} = 0.$

(4). The parabola .... $x + h = 0.$

Corollary 1.

If we write a^2e^2 for $a^2 - b^2$ in the ellipse, referred to the centre, we shall have $u = ae$, and the curve will then be expressed by

$$y^2 = (1 - e^2)(2ax - x^2).$$

In a similar manner, the two hyperbolas give $u = ae$, and $u = be$, and the curves themselves respectively become

$$y^2 = (e^2 - 1)(2ax + x^2), \text{ and}$$
$$x^2 = (e^2 - 1)(2by + y^2).$$

Corollary 2.

The radius vector drawn from the focus being written r, we shall have

(1). In the ellipse $r = a \pm ex$, the upper sign being employed when the focus and the point in the curve are on different sides of the conjugate axes.

$$\begin{aligned} \text{For } r^2 &= y^2 + (x \pm ae)^2 \\ &= (1 - e^2)(a^2 - x^2) + (x \pm ae)^2 \\ &= (a \pm ex)^2. \end{aligned}$$

(2). Similarly, in the *primary hyperbola*,

$$r^2 = (ex \pm a)^2.$$

(3). In the *conjugate hyperbola*,

$$r^2 = (ey \pm b)^2.$$

(4). In the *parabola*,

$$r^2 = (h + x)^2.$$

For, $y^2 = 4hx$, and $r^2 = y^2 + (h - x)^2 = (h + x)^2$.

Corollary 3.

The quantity e is the *determining ratio*, as appears by comparing this with *prop. i. p.* 317. The determining ratio is therefore that which the transverse axis bears to the focal distance from the centre in the ellipse and hyperbolas.

Corollary 4.

From *cor. i.* are also easily inferred the remarkable relations: viz. that

$r = x + h$, in the *parabola*,
$r = ex + a$, in the *ellipse*,
$r = ex - a$, in the *primary hyperbola*.
$r = ey - b$, in the *conjugate hyperbola*.

The property in the hyperbolas is to be understood as referred in each case to the *real foci* of the respective pairs. The expression for a point in one pair of them, in reference to the focus of the other pair, takes a much more complex form; which, as it is of little utility in investigation, need not be here specified.

PROP. VII.

To find the equation of the hyperbolas when the co-ordinate axes are parallel to the asymptotes.

The equation of conjugation is $a^2 \tan\beta \tan\gamma + b^2 = 0$, where β and γ are the angles made with the axis of x. Let us then take one of the asymptotes as the new axis of x, and the centre as origin. Then,

$$\tan\beta = \frac{b}{a}; \text{ and hence } \tan\gamma = -\frac{b^2}{a^2}\cot\beta = -\frac{b}{a};$$

or the other asymptote is the conjugate axis of y. Whence also $\alpha = \pi - 2\beta$; and

$$\sin\beta = \frac{b}{\sqrt{a^2 + b^2}} \qquad \cos\beta = \frac{a}{\sqrt{a^2 + b^2}}$$
$$\sin\gamma = \frac{b}{\sqrt{a^2 + b^2}} \qquad \cos\gamma = \frac{-a}{\sqrt{a^2 + b^2}}$$

For y write $x \sin \beta + y \sin \gamma$; and for x write $x \cos \beta + y \cos \gamma$: then,

$$\left.\begin{array}{r} a^2x^2 \sin^2\beta + 2a^2xy \sin \beta \sin \gamma + a^2y^2 \sin^2\gamma \\ - b^2x^2 \cos^2\beta - 2b^2xy \cos \beta \cos \gamma - b^2y^2 \cos^2\gamma \end{array}\right\} = a^2b^2;$$

in which the first and third vertical pairs cancel, and,

$$2\,(a^2 \sin \beta \sin \gamma - b^2 \cos \beta \cos \gamma)\,xy = a^2b^2.$$

But by the values of $\sin \beta$, *etc.* above, this reduces to

$$xy = \tfrac{1}{4}\,(a^2 + b^2).$$

This is the equation referred to the asymptotes; and that referred to axes parallel to them is found by writing $x + x_1$ for x and $y + y_1$ for y, giving

$$xy + x_1y + y_1x + x_1y_1 = \tfrac{1}{4}\,(a^2 + b^2);$$

x_1y_1 being the co-ordinates of the origin.

The converse process obviously transforms the last equation to the preceding; and this, again, to the original form adapted to rectangular conjugate axes.

Corollary.

All the conjugate parallelograms have their angular points situated on the asymptotes.

The points x_1y_1 and x_2y_2, at which the tangents are drawn, will be one (x_1y_1) situated in the primary hyperbola, and the other (x_2y_2) in its conjugate. The equations of the tangents at these points will, therefore, be

$$a^2y_1y - b^2x_1x = -\,a^2b^2 \qquad \Big| \qquad a^2y_2y - b^2x_2x = +\,a^2b^2;$$

and by addition we get the equation of the line passing through the origin and intersection of the tangents; viz.

$$a^2\,(y_1 + y_2)\,y = b^2\,(x_1 + x_2)\,x.$$

But the diameters being conjugate in reference to the primary hyperbola, we have the conjugate equation

$$\tan \beta \tan \gamma = \frac{b^2}{a^2}\,; \text{ where } \tan \beta = \frac{y_1}{x_1}, \text{ and } \tan \gamma = \frac{y_2}{x_2}.$$

Also, since x_1y_1 is in the primary, and x_2y_2 in the conjugate hyperbola, we have

$$a^2y_1^2 - b^2x_1^2 = -\,a^2b^2 \qquad \Big| \qquad a^2y_2 - b^2x_2^2 = +\,a^2b^2.$$

Eliminate x_1y_1, x_2y_2, by means of the last four equations, and there results the equation of the asymptotes; viz.

$$y^2 = \frac{b^2}{a^2}\,x^2, \text{ or } y = \pm\,\frac{b}{a}\,x.$$

PROP. VIII.

To find the equations of the tangents and normals passing through given points.

This general problem has been already solved at *pp.* 300—1; and the student may either obtain them by giving to the co-efficients of the general equation the special forms which they have in our present chapter; or he may deduce them, *ab initio*, by corresponding processes. The following tables will show the results in a form adapted for convenient reference. The tables themselves need but little remark, as their structure is obvious.

The numbered parts are the same in both tables, and in the second the numbers only without the names of the parts are repeated, on account of the room; but being placed on opposite pages, are both visible at the same time.

TABLE I.

The functions of the ellipse and hyperbola, referred to any conjugate system of diameters; the lower sign in the hyperbola referring to the conjugate of that expressed by the upper sign.

		FUNCTION.	ELLIPSE.	HYPERBOLAS.
1	Any angle of ordination a.	Curve	$a^2y^2 + b^2x^2 = a^2b^2$	$a^2y^2 - b^2x^2 = \mp a^2b^2$
2		Tan (1)*	$\frac{y-y_1}{x-x_1} = \frac{x_1y_1 + F'}{x_1^2 - a^2}$	$\frac{y-y_1}{x-x_1} = \frac{-x_1y_1 + F'}{x_1^2 \mp a^2}$
3		Tan (2)	$a^2y_1y + b^2x_1x = a^2b^2$	$a^2y_1y - b^2x_1x = \mp a^2b^2$
4		Subt. on x	$\frac{a^2y_1^2}{b^2x_1}$ or $\frac{a^2 - x_1^2}{x_1}$	$-\frac{a^2y_1^2}{b^2x_1}$ or $-\frac{x_1^2 \mp a^2}{x_1}$
5		Subt. on y	$\frac{b^2x_1^2}{a^2y_1}$ or $\frac{b^2 - y_1^2}{y_1}$	$-\frac{b^2x_1^2}{a^2y_1}$ or $-\frac{y_1^2 \pm b^2}{y_1}$
6		Normal	$\frac{y-y_1}{x-x_1} = \frac{a^2y_1 - b^2x_1 \cos a}{b^2x_1 - a^2y_1 \cos a}$	$\frac{y-y_1}{x-x_1} = -\frac{a^2y_1 + b^2x_1 \cos a}{b^2x_1 + a^2y_1 \cos a}$
7		Subn. on x	$-\frac{(b^2x_1 - a^2y_1 \cos a)y_1}{a^2y_1 - b^2x_1 \cos a}$	$\frac{(b^2x_1 + a^2y_1 \cos a)\, y_1}{a^2y_1 + b^2x_1 \cos a}$
8		Subn. on y	$-\frac{(a^2y_1 - b^2x_1 \cos a)\, x_1}{b^2x_1 - a^2y_1 \cos a}$	$\frac{(a^2y_1 + b^2x_1 \cos a)\, x_1}{b^2x_1 + a^2y_1 \cos a}$
9	Rectangular equation.	Normal	$a^2y_1x - b^2x_1y = (a^2 - b^2)\, x_1y_1$	$a^2y_1x + b^2x_1y = (a^2 + b^2)\, x_1y_1$
10		Subn. on x	$-\frac{b^2x_1}{a^2}$	$\frac{b^2x_1}{a^2}$
11		Subn. on y	$-\frac{a^2y_1}{b^2}$	$\frac{a^2y_1}{b^2}$

* By F′ is meant $\pm \sqrt{a^2 y_1^2 + b^2 x_1^2 - a^2 b^2}$ in the ellipse, and $\pm \sqrt{\pm (a^2 y_1^2 - b^2 x_1^2) + a^2b^2}$ in the hyperbolas. In this latter case, the sign ∓ refers to the two cases in the equation of the hyperbola, the order of the signs to be taken as in the equation of the first line.

The tangent (1) refers to the point x_1y_1 being without the curve, and (2) to its being in the curve.

Moreover, they are the values of $x - x_1$ and $y - y_1$ in the *sub-tangent* and *sub-normal* that are invariably put down in these tables.

The following is the process for the case of the *tangent* being drawn to the curve from a point x_1y_1 situated without it.

The equation of any line through x_1y_1 is $y = y_1 + p\,(x - x_1)$ where p is indeterminate. Insert this in the equation of the curve (1): then we get

$$(a^2p^2 + b^2)\, x^2 + 2a^2p\,(y_1 - px_1)\, x + a^2\left\{(y_1 - px_1)^2 - b^2\right\} = 0.$$

The values of x in this are the abscisses of the points of intersection of the line with the curve; and that the line may be a tangent, these points must coalesce, and the values of x be equal. This will require the equation above to be a complete square; and hence,

$$4a^2\,(a^2p^2 + b^2)\left\{(y_1 - px_1)^2 - b^2\right\} = \left\{2a^2p\,(y_1 - px_1)\right\}^2.$$

Expanding and resolving this for p, we have the second side of (2) as its value. Whence the equation of the cutting (or now touching) line is as above stated.

TABLE II.

The functions of the three curves referred to any point in the curve as origin and the diameter and tangent to the curve through that point.

	PARABOLA.	ELLIPSE.	HYPERBOLA.
1	$y^2 = 4ax$	$a^2y^2 + b^2x^2 - 2ab^2x = 0$	$a^2y^2 - b^2x^2 + 2ab^2x = 0$
2*	$\frac{y-y_1}{x-x_1} = y_1 \pm \frac{\sqrt{y_1^2 - 4ax_1}}{2x_1}$	$\frac{y-y_1}{x-x_1} = \frac{y_1(x_1-a) + F'}{x_1^2 - 2ax_1}$	$\frac{y-y_1}{x-x_1} = \frac{y_1(x_1-a) + F'}{x_1^2 - 2ax_1}$
3	$y_1y = 2a(x + x_1)$	$a^2y_1y + b^2x_1x - ab^2(x+x_1) = 0$	$a^2y_1y - b^2x_1x + ab^2(x+x_1) = 0$
4	$-2x_1$	$\frac{2ax_1 - x_1^2}{x_1 - a}$	$-\frac{2ax_1 - x_1^2}{x_1 - a}$
5	$-\frac{1}{2}y_1$	$\frac{b^2x_1 - ay_1^2}{ay_1}$	$-\frac{b^2x_1 - ay_1^2}{ay_1}$
6	$\frac{y-y_1}{x-x_1} = -\frac{y_1 + 2a\cos a}{2a + y_1\cos a}$	$\frac{y-y_1}{x-x_1} = \frac{a^2y_1 - b^2(x_1-a)\cos a}{b^2(x_1-a) - a^2y_1\cos a}$	$\frac{y-y_1}{x-x_1} = -\frac{a^2y_1 + b^2(x_1-a)\cos a}{b^2(x_1-a) + a^2y_1\cos a}$
7	$\frac{(2a + y_1\cos\alpha)y_1}{y_1 + 2a\cos\alpha}$	$-\frac{\{b_2(x_1-a) - a^2y_1\cos\alpha\}y_1}{a^2y_1 - b^2(x_1-a)\cos\alpha}$	$\frac{\{b^2(x_1-a) + a^2y_1\cos\alpha\}y_1}{a^2y_1 + b^2(x_1-a)\cos\alpha}$
8	$\frac{(y_1 + 2a\cos\alpha)x_1}{2a + y_1\cos\alpha}$	$-\frac{\{a^2y_1 - b^2(x_1-a)\cos\alpha\}x_1}{b^2(x_1-a) - a^2y_1\cos\alpha}$	$\frac{\{a^2y_1 + b^2(x_1-a)\cos\alpha\}x_1}{b^2(x_1-a) + a^2y_1\cos\alpha}$
9	$\frac{y-y_1}{x-x_1} = -\frac{y_1}{2a}.$	$\frac{y-y_1}{x-x_1} = \frac{a^2y_1}{b^2x_1}$	$\frac{y-y_1}{x-x_1} = -\frac{a^2y_1}{b^2x_1}$
10	$2a$	$-\frac{b^2x_1}{a_2}$	$\frac{b^2x_1}{a^2}$
11	$\frac{x_1y_1}{2a}$	$-\frac{a^2y_1}{b^2}$	$\frac{a^2y_1}{b^2}$

* In these, the values of F′ are $\pm\sqrt{a^2y_1^2 + b^2x_1^2 - 2ab^2x_1}$ in the ellipse, and $\pm\sqrt{a^2y_1^2 - b^2x_1^2 + 2ab^2x_1}$ in the hyperbola. The origin is the extremity of the diameter to the *left* of the centre, in both curves, the positive value of x being reckoned to the right of the origin.

For illustration with respect to the *normal*, we shall take the *parabola*.

The point x_1y_1 being in the curve, (the only case for which in all the curves in both tables the normal is given,) we have its equation

$$yy_1 = 2a(x + x_1), \text{ or } y = \frac{2ax}{y_1} + \frac{2ax_1}{y_1}.$$

The perpendicular to this through x_1y_1 to the angle of ordination α is (*p.* 260, *eq.* 28)

$$\frac{y - y_1}{x - x_1} = -\frac{1 + \frac{2a}{y_1}\cos\alpha}{\frac{2a}{y_1} + \cos\alpha} = -\frac{y_1 + 2a\cos\alpha}{2a + y_1\cos\alpha};$$

which is the equation (6) in the table.

Except that the expressions are a little more complex for the normals in all the cases of all the curves above tabulated, the process of finding them is precisely similar to the one here put down.

PROP. IX.

Lines drawn from any point in the ellipse and hyperbola to the extremity of any diameter are conjugate chords.

Take, for instance, the ellipse, and refer it to the proposed diameter and its conjugate, as axes of y and x: put b for the proposed semi-diameter, and a for the semi-diameter conjugate to it. Then, if x_1y_1 be the assumed point, we have $a^2y_1^2 + b^2x_1^2 = a^2b^2$, or

$$-\frac{b^2}{a^2} = \frac{y_1^2 - b^2}{x_1^2} = \frac{y_1 + b}{x_1}\cdot\frac{y_1 - b}{x_1}.$$

But the equations of the two chords themselves are

$$\frac{y_1 + b}{x_1} = \frac{y + b}{x}, \text{ and } \frac{y_1 - b}{x_1} = \frac{y - b}{x};$$

and, if $y = p_1x + q_1$ and $y = p_2x + q_2$ be the lines, we have

$$p_1 = \frac{y_1 + b}{x_1}, \text{ and } p_2 = \frac{y_1 - b}{x_1},$$

and hence $$p_1p_2 = \frac{y_1 + b}{x_1}\cdot\frac{y_1 - b}{x_1} = -\frac{b^2}{a^2}.$$

Thus, the criterion of the lines being conjugate (*p.* 315) is fulfilled; and the conclusion follows.

In precisely the same way the property is proved in the hyperbola.

Corollary.

If the assumed diameter be the transverse or the conjugate, the angle of ordination is $\frac{1}{2}\pi$; and the factors p_1, p_2 are the tangents of the angles made by the chords with the axis of x. We thus get, adapted to this case

$$\tan\beta_1 \tan\beta_2 = \mp\frac{b^2}{a^2}$$

And further, when $b = a$ in the ellipse, we have

$$\tan\beta_1 \tan\beta_2 = -1;$$

or the chords are at right angles; thus showing that the angle in a semi-circle is a right angle.

PROP. X.

Given the principal parameter of a parabola, to find that of any given diameter.

Let the principal parameter $4h = p$, and that belonging to the diameter through the point x_1y_1 of the curve be $4h_1 = p_1$: then the equations of the parabola referred to the diameter through the vertex and the point x_1y_1 are

$y^2 = px$ .... (1) | $y^2 = p_1x$ .... (2).

Also, the equations for transformation to conjugate co-ordinate axes through x_1y_1, the axes of x being parallel in both cases, and the angle of conjugation $= \alpha$, are found upon insertion of $x_1 + x + y\cos\alpha$ for x, and $y_1 + y\sin\alpha$ for y, (these being the forms to which the equations of transformation reduce under these particular circumstances) to be

$$2y_1\sin\alpha = p\cot\alpha, \text{ or } \tan\alpha = \frac{p}{2y_1}, \text{ and } y_1^2 = 4hx_1.$$

Whence the equation referred to the new axes and origin, is

$$y^2 = p\cos^2\alpha . x = 4(h + x_1)\,x = p_1x \ldots\ldots\ldots\ldots \quad (3),$$

which is the equation of the parabola referred to the diameter and tangent through x_1y_1.

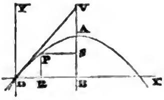
Hence $p_1 = 4(h + x_1)$ the parameter required, is found. See also *prop. xxi. cor.* 2.

Corollary.

The parameters of the different diameters are to one another inversely, as the squares of the lines of the angles which the tangents at their extremities make with the principal axis.

PROP. XI.

A parabola is referred to rectangular co-ordinates, of which one is perpendicular to the principal axis, and the other parallel to it, the origin being on the curve: to find its equation.



Let D be the origin; and let any two fixing conditions be assumed. Suppose the angle VDB made by the tangent at D, with the line DX perpendicular to the axis AB, to be one; and the length DV of the tangent at D intercepted by the axis to be the other. Let P be a point in the curve.

Put VDB $= \alpha$, DV $= a$, DR $= x$, RP $= y$: then

$$\text{DB} = a \cos \alpha, \ \text{AB} = \tfrac{1}{2}a \sin \alpha, \text{ and } p = \frac{\text{DB}^2}{\text{AB}} = \frac{2a \cos^2\alpha}{\sin \alpha}.$$

But we have $p.\text{AS} = \text{PS}^2$ by the curve; or substituting

$$p(\text{AB} - \text{BS}) = (\text{DB} - \text{DR})^2, \text{ or}$$

$$\frac{2a \cos^2\alpha}{\sin \alpha}\{\tfrac{1}{2}a \sin \alpha - y\} = \{a \cos \alpha - x\}^2, \text{ or again}$$

$$y = x \tan \alpha - \frac{x^2}{2a} . \tan \alpha \sec \alpha.$$

which is the form required.

Scholium.

This proposition virtually contains the whole practice of calculating the circumstances connected with the flight of a projectile *in vacuo*. We shall have frequent occasion to recur to it under that head.

PROP. XII.

To find the polar equations of the curves and of their tangents and normals; both when referred to the centre and to the focus, as origin of polar co-ordinates.

It is obvious that by using the proper substitution, we may refer the origin of polar co-ordinates to any point whatever; but the two mentioned in the enunciation of the proposition are all that commonly offer any advantages in our inquiries respecting these curves. It is equally obvious, too, that we might investigate these equations directly and without the intervention of rectilinear co-ordinates; but such methods, though very elegant, are somewhat operose, from the reductions to which they lead. We, therefore, prefer in this place to adopt the method of transformation. This merely consists in writing for x and y their values $r \cos \theta$, and $r \sin \theta$. The processes involved being of the most elementary and simple kind, it has been deemed sufficient to tabulate the results as in *prop. vii.*

It must, in these equations, be kept in mind in respect to the ellipse and hyperbola referred to the focus, that the focus employed is invariably that *to the left of the centre of the curve*, and that the positive branch of the axis is that to the right of the focus, whilst in the parabola, the same rule applies, the centre being at an infinite distance to the right of the accessible focus of the curve.

The upper sign throughout applies to θ measured from the positive branch as the initial position of the radius vector, and the lower signs to θ measured from the negative branch. Should it be desirable to refer to the other foci as the respective origins of polar co-ordinates, or to reverse the position of the parabola; then either the positive and negative branches of the axis must be reversed, or else the upper and lower signs in the equations of the curves respectively interchanged.

Moreover, in reference to the best equations to be employed for this transformation, it may be remarked, that in all cases the curves referred to the same origin as the polar co-ordinates are, and rectangular conjugate axes, will be found preferable to any other, for the curve, its tangent, and its normal. For the curve alone, the property in *prop. vi. cor.* 4, is well adapted; but it furnishes no assistance for the cases of the tangent and normal.

TABLE III.

The polar equations of the conic sections, with their tangents and normals.

EQUATION OF		ORIGIN AT THE FOCUS.	ORIGIN AT THE CENTRE.
PARABOLA.	Curve	$r = \frac{2h}{1 \mp \cos\theta}$	....
	Tangent	$r\{\cos(\theta - \theta_1) \mp \cos\theta\} = 2r_1$	....
	Normal	$r\{\sin(\theta - \theta_1) \mp \sin\theta_1\} = \mp r_1 \sin\theta_1$	....
ELLIPSE.	Curve	$r = \frac{a(1 - e^2)}{1 \mp e\cos\theta}$	$r^2 = \frac{a^2(1 - e^2)}{1 - e^2\cos^2\theta}$
	Tangent	$r\{\cos(\theta - \theta_1) \mp e\cos\theta\} = r_1(1 \mp e\cos\theta_1)$	$r\{\cos(\theta - \theta_1) - e^2\cos\theta_1\cos\theta\} = r_1(1 - e^2\cos^2\theta_1)$
	Normal	$r\{\sin(\theta - \theta_1) \mp e\sin\theta\} = \mp er_1\sin\theta_1$	$r\{\sin(\theta - \theta_1) - e^2\cos\theta_1\sin\theta\} = -r_1 e\sin\theta_1\cos\theta_1$
PRIMARY HYP.	Curve	$r^{*} = \frac{a(e^2 - 1)}{1 \mp e\cos\theta}$	$r^2 = \frac{a^2(e^2 - 1)}{e^2\cos^2\theta - 1}$
	Tangent	$r\{\cos(\theta - \theta_1) \mp e\cos\theta\} = r_1(1 \pm e\cos\theta_1)$	$r\{\cos(\theta - \theta_1) - e^2\cos\theta_1\cos\theta\} = r_1(1 - e^2\cos^2\theta_1)$
	Normal	$r\{\sin(\theta - \theta_1) \pm e\sin\theta\} = \pm er_1\sin\theta_1$	$r\{\sin(\theta - \theta_1) - e^2\cos\theta_1\sin\theta\} = -r_1 e\sin\theta_1\cos\theta_1$
CONJUGATE HYP.	Curve *	$r^{*} = \frac{b(e'^2 - 1)}{1 \pm e'\cos\theta}$	$r^2 = \frac{-a^2(e^2 - 1)}{e^2\cos^2\theta - 1}$
	Tangent	$r\{\cos(\theta - \theta_1) \pm e'\cos\theta\} = r_1(1 \pm e'\cos\theta_1)$	$r\{\cos(\theta - \theta_1) - e^2\cos\theta_1\cos\theta\} = r_1(1 - e^2\cos^2\theta_1)$
	Normal	$r\{\sin(\theta - \theta_1) \pm e'\sin\theta_1\} = \pm e'r_1\sin\theta_1$	$r\{\sin(\theta - \theta_1) - e^2\cos\theta_1\sin\theta\} = -r_1 e\sin\theta_1\cos\theta_1$
		Note. $e' = \frac{ae}{b}$, or $be' = ae$.	

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* In these cases each hyperbola is referred to *one of its own foci*, the formulæ for each referred to the others' foci being omitted as inelegant and useless. The equations of the tangent

PROP. XIII.

To find, in the ellipse and hyperbola, the position and magnitude of the diameter conjugate to a given diameter.

REFER the system to polar co-ordinates originating at the centre and transverse axis. Let the given diameter make an angle θ_1 with the positive branch of the transverse, and its conjugate, an angle θ_2 with the same branch. Then the equation of conjugation is (*p.* 315).

$$\tan\theta_1 \tan\theta_2 = \mp \frac{b^2}{a^2} = -(1-e^2), \text{ or}$$

$$\tan\theta_2 = -(1-e^2)\cot\theta_1 \ldots\ldots\ldots\ldots \quad (1),$$

which gives the *position* of the conjugate diameter.

Again, $r_1\theta_1$ and $r_2\theta_2$ being points in the curve, we have

$$r_1{}^2 = \frac{a^2(1-e^2)}{1-e^2\cos^2\theta_1} \ldots\ldots \quad (2), \qquad \Big| \qquad r_2{}^2 = \frac{a^2(1-e^2)}{1-e^2\cos^2\theta_2} \ldots\ldots \quad (3).$$

From (1, 2) we get

$$\cos^2\theta_2 = \frac{1}{1+(1-e^2)^2\cot^2\theta_1} = \frac{\sin^2\theta_1}{(e^2-1)^2\cos^2\theta_1+\sin^2\theta_1} = \frac{a^2-r_1{}^2}{e^2\{a^2-r_1{}^2e^2\cos^2\theta_1\}}.$$

Insert this in (3) and reduce: then we have the value

$$r_2{}^2 = a^2 - r_1{}^2e^2\cos^2\theta_1 \ldots\ldots\ldots\ldots \quad (4),$$

and r_1, r_2 are the *magnitudes* of the given semi-diameter and its conjugate respectively.

Scholium.

The expression $(1-e^2)^2\cos^2\theta_1 + \sin^2\theta_1$ occurs so often in this class of researches, that it will be convenient to put down a few of the forms which it is capable of taking.

From the equation of the curve we have

$r_1{}^2e^2\cos^2\theta_1 = r_1{}^2 - a^2(1-e^2)$; and hence $r_1{}^2e^2\sin^2\theta_1 = (a^2-r_1{}^2)(1-e^2)$.

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and normal of the parabola also admit of further simplification; but they are left in their present state to keep the table uniform. Their reduced forms are

	TANGENT.	NORMAL.
For upper sign	$r\cos(\theta-\frac{1}{2}\theta_1)\cos\theta_1 = h$	$r\cos(\theta-\frac{1}{2}\theta_1) = r_1\cos\theta_1$
... lower ...	$r\sin(\theta-\frac{1}{2}\theta_1)\sin\frac{1}{2}\theta_1 = h$	$r\sin(\theta-\frac{1}{2}\theta_1) = r_1\sin\theta_1$

It will also be remarked, that the equations of the tangent and normal of the curves referred to the *centre* are precisely alike; but the second sides may be transformed by eliminating r_1 by means of the equation of the curve. Little advantage, however, attends this change in the tangent, and none in the normal.

Though it anticipates the student's advance, it may not be irrelevant to state, that the expressions for the tangent and normal in the table have been checked by the use of the two following formulæ, first given by the editor in the *Edinb. Trans., vol. xii.*; and which will be investigated in the Differential Calculus in the present volume.

$$\textit{Tangent.} \quad r\left\{\cos(\theta-\theta_1) - \sin(\theta-\theta_1)\frac{dr_1}{r_1 d\theta_1}\right\} = r_1$$

$$\textit{Normal.} \quad r\left\{\sin(\theta-\theta_1) + \cos(\theta-\theta_1)\frac{dr_1}{r_1 d\theta_1}\right\} = r_1 \cdot \frac{dr_1}{r_1 d\theta_1}.$$

Substitute these; then,

$$r_1^2e^2\{(1-e^2)^2\cos^2\theta_1+\sin^2\theta_1\}=(1-e^2)\{r_1^2(1-e^2)-a^2(1-e^2)^2+a^2-r_1^2\}\ldots(1),$$
$$=(1-e^2)\{(1-e^2)a^2+a^2-r_1^2\}\ \ldots\ldots\ (2),$$
$$r^2\{(1-e^2)^2\cos^2\theta_1+\sin^2\theta_1\}=(1-e^2)\{(2-e^2)a^2-r_1^2\}\ldots\ldots\ldots\ldots(3),$$
$$=(1-e^2)\{r_1^2-r_1^2e^2\cos^2\theta_1+a^2-r_1^2\}\ldots(4),$$
$$=(1-e^2)\{a^2-r_1^2e^2\cos^2\theta_1\}\ldots\ldots\ldots\ldots(5);$$

Or again,

$$(1-e^2)^2\cos^2\theta_1+\sin^2\theta_1=1-2e^2\cos^2\theta_1+e^4\cos^2\theta_1\ldots\ldots\ \ldots\ldots(6),$$
$$=(1-e^2)(1-e^2\cos^2\theta_1)+e^2\sin^2\theta_1\ \ldots\ldots\ (7),$$

etc., etc., etc.

PROP. XIV.

The sum of the squares of any pair of conjugate diameters in the ellipse, and their difference in the hyperbola, is equal to the respective sum and difference of the squares of the principal diameters.

For in the ellipse,

$$r_1^2+r_2^2=a^2+r_1^2-r_1^2e^2\cos^2\theta_1=a^2+a^2(1-e^2)=a^2+b^2:$$

and in the hyperbola, r_1^2 and b^2 are essentially negative in the algebraical expressions. Let their absolute values be r'^2 and b'^2: then $+r_1^2=-r'^2$ and $+b^2=-b'^2$; and the expression becomes

$$r_1^2-r'^2=a^2-b'^2.$$

PROP. XV.

The area of any conjugate parallelogram is equal to the rectangle of the principal diameters.

For, from *prop. xiii. eq* (1) we have

$$\sin\theta_1\sin\theta_2=-(1-e^2)\cos\theta_1\cos\theta_2,\text{ or }\cos(\theta_2-\theta_1)=e^2\cos\theta_1\cos\theta_2;$$

hence

$$r_1^2r_2^2\sin^2(\theta_2-\theta_1)=\frac{r_1^2\{a^2-r_1^2e^2\cos^2\theta_1\}.\,a^2\{1-e^2\cos^2\theta_1\}}{a^2-r_1^2e^2\cos^2\theta_1}=a^2.a^2(1-e^2)=a^2b^2,$$

$$\text{or } 4r_1r_2\sin(\theta_2-\theta_1)=4ab.$$

which establishes the conclusion enunciated in the proposition.

PROP. XVI.

The rectangle under the focal radii-vectores of any point in the ellipse or hyperbola, is equal to the square of the axis conjugate to that point.

Denote them by ρ_1 and ρ_2, and let the point be $r_1\theta_1$, the reference being, as before, to the centre and transverse: then,

$$\rho_1^2=r_1^2-2r_1ea\cos\theta_1+a^2e^2,\text{ and }\rho_2^2=r_1^2+2r_1ea\cos\theta_1+a^2e^2,$$
$$\rho_1^2\rho_2^2=(r_1^2+a^2e^2)^2-4r_1^2a^2e^2\cos^2\theta_1\ =(a^2-r_1^2e^2\cos^2\theta_1)^2;$$

hence $\rho_1\rho_2=a^2-r_1^2e^2\cos^2\theta_1$; which by (*xiii.*) proves the theorem.

PROP. XVII.

To find the lengths of focal radii-vectores of a point in the ellipse and hyperbola.

For, from the preceding proposition, employing also its notation,

$$\rho_1^2+\rho_2^2=2r_1^2+2a^2e^2,\text{ and }2\rho_1\rho_2=2a^2-2r_1^2e^2\cos^2\theta_1.$$

Hence,

$$\rho_1{}^2+2\rho_1\rho_2+\rho_2{}^2=2r_1{}^2(1-e^2\cos^2\theta_1)+2a^2(1+e^2)=2a^2\{(1-e^2)+(1+e^2)\}=4a^2,$$

$$\rho_1{}^2-2\rho_1\rho_2+\rho_2{}^2=2r_1{}^2(1+e^2\cos^2\theta_1)-2r_1{}^2(1-e^2\cos^2\theta_1)=4r_1{}^2e^2\cos^2\theta_1:$$

Wherefore, $\rho_1+\rho_2=2a$, and $\rho_1-\rho_2=+2r_1e\cos\theta_1$; and hence,

$$\rho_1=a+r_1e\cos\theta_1,\text{ and }\rho_2=a-r_1e\cos\theta_1.$$

PROP. XVIII.

To find the locus of the intersections of the tangent to a conic section, with the perpendiculars drawn from them to the foci.

1. *The parabola.* Take the focus as origin of polar co-ordinates: then the tangent at $r_1\theta_1$ is expressed by (*p.* 331, *note*).

$$r\cos(\theta-\tfrac{1}{2}\theta_1)=h\sec\tfrac{1}{2}\theta_1\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also, the equation of the perpendicular upon this from the origin is

$$=\beta=\tfrac{1}{2}\theta_1,\text{ or }r_1\sin(\theta-\tfrac{1}{2}\theta_1)=0\ldots\ldots\ldots\ldots\ldots(2).$$

Hence, $r^2\{\cos^2(\theta-\tfrac{1}{2}\theta_1)+\sin^2(\theta-\tfrac{1}{2}\theta_1)\}=h^2\sec^2\tfrac{1}{2}\theta_1$, and

$$r=-h\sec\tfrac{1}{2}\theta_1=-h\sec\theta,\text{ or }h=r\cos(\theta-\pi)\ldots\ldots(3).$$

This is the equation of a line perpendicular to the axis of co-ordinates, having $p=h$; in other words the *tangent at the vertex of the parabola.*

2. *The ellipse and hyperbola.* Refer the system to polar co-ordinates originating at the centre. Then the equation of the tangent given in the table may be written

$$r\,.\,\frac{(1-e^2)\cos\theta_1\cos\theta+\sin\theta_1\sin\theta}{\sqrt{(1-e^2)^2\cos^2\theta_1+\sin^2\theta_1}}=\frac{r_1(1-e^2\cos^2\theta_1)}{\sqrt{(1-e^2)^2\cos^2\theta_1+\sin^2\theta_1}}\ldots\ldots(4).$$

$$\text{Put }\tan\beta=\frac{\tan\theta_1}{1-e^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Then (4) is converted into

$$r\cos(\theta-\beta)=\frac{r_1(1-e^2\cos^2\theta_1)}{\sqrt{(1-e^2)^2\cos^2\theta_1+\sin^2\theta_1}}\ldots\ldots\ldots\ldots\ldots(6).$$

Again, by (5) and *eq* (18) *page* 267, the perpendicular from $r_2\theta_2$ (where $r_2=ae$ and $\theta_2=0$) upon the line (6) is

$$r\sin(\theta-\beta)\,[=ae\sin\beta]=\frac{-ae\sin\theta_1}{\sqrt{(1-e^2)^2\cos^2\theta_1+\sin^2\theta_1}}\ldots\ldots\ldots\ldots(7).$$

To find r eliminate θ from (6, 7); which will be best effected by adding the squares of (6, 7): thus giving

$$r^2=\frac{r_1{}^2(1-e^2\cos^2\theta_1)^2+a^2e^2\sin^2\theta_1}{(1-e^2)^2\cos^2\theta_1+\sin^2\theta_1}=\frac{a^2(1-e^2)(1-e^2\cos^2\theta_1)+a^2e^2\sin^2\theta_1}{(1-e^2)^2\cos^2\theta_1+\sin^2\theta_1}=a^2;$$

whence the locus sought is, in this case, *the circle described on the transverse diameter.*

PROP. XIX.

To find the lengths of the perpendiculars from the foci upon a tangent to the ellipse or hyperbola.

Denote them by λ_1 and λ_2, keeping the same notation for the other parts, as before.

By *eq* (18) *page* 268, we have, since $r_2=ae$ and $\theta_2=0$,

$$\lambda_1=p-r_2\cos(\theta_2-\beta)=p-ae\cos\beta.$$

Also, as in the preceding solution we readily obtain

$$p = \frac{r_1(1-e^2\cos^2\theta_1)}{\sqrt{(1-e^2)^2\cos^2\theta_1+\sin^2\theta_1}}\text{; and } \cos\beta = \frac{(1-e^2)\cos\theta_1}{\sqrt{(1-e^2)^2\cos^2\theta_1+\sin^2\theta_1}}.$$

Hence

$$\lambda_1 = \frac{r_1(1-e^2\cos^2\theta_1) - ae(1-e^2)\cos\theta_1}{\sqrt{(1-e^2)^2\cos^2\theta_1+\sin^2\theta_1}},$$

$$= \frac{r_1^2(1-e^2\cos^2\theta_1) - ar_1e(1-e^2)\cos\theta_1}{r_1\sqrt{(1-e^2)^2\cos^2\theta_1+\sin^2\theta_1}},$$

$$= \frac{a^2(1-e^2) - ar_1e(1-e^2)\cos\theta_1}{\sqrt{(1-e^2)\{a^2-r_1^2e^2\cos^2\theta_1\}}}\text{; that is,}$$

$$\lambda_1 = a\sqrt{1-e^2}.\sqrt{\frac{a-r_1e\cos\theta_1}{a+r_1e\cos\theta_1}} = b\sqrt{\frac{a-r_1e\cos\theta_1}{a+r_1e\cos\theta_1}}$$

Similarly

$$\lambda_2 = a\sqrt{1-e^2}.\sqrt{\frac{a+r_1e\cos\theta_1}{a-r_1e\cos\theta_1}} = b\sqrt{\frac{a+r_1e\cos\theta_1}{a-r_1e\cos\theta_1}}$$

Corollary 1.

The rectangle under the perpendiculars from the foci of the ellipse or hyperbola is equal to the square of the semi-conjugate diameter.

For,

$$\lambda_1\lambda_2 = b^2\sqrt{\frac{a-r_1e\cos\theta_1}{a+r_1e\cos\theta_1}\cdot\frac{a+r_1e\cos\theta_1}{a-r_1e\cos\theta_1}} = b^2.$$

Corollary 2.

The focal radii-vectores make equal angles with the tangent.

For, let the angles so made be χ_1 and χ_2: then

$$\sin^2\chi_1 = \frac{\lambda_1^2}{\rho_1^2} = \frac{b^2}{a^2-r_1^2e^2\cos^2\theta_1}\text{, and } \sin^2\chi_2 = \frac{\lambda_2^2}{\rho_2^2} = \frac{b^2}{a^2-r_1^2e^2\cos^2\theta_1};$$

whence χ_1, χ_2 estimated either way are equal or supplementary.

Corollary 3.

The radii vectores from the foci to the point of contact, have the same ratio as the perpendiculars upon the tangent.

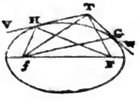
For by the preceding corollary,

$$\frac{\lambda_1}{\rho_1} = \frac{\lambda_2}{\rho_2}.$$

which establishes the corollary.

PROP. XX.

Lines drawn from the foci of a conic section to the intersection of the tangents, make equal angles with them.



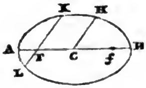
Let G, H, be the points of contact, T the intersection of the tangents, and F, f, the foci. Denote by χ_1, χ_2, the equal angles made at G and H by the radii-vectores from F and f; by $2\omega_1$ and $2\omega_2$, the angles HFG, HfG; by δ_1 and δ_2, the angles FTG and fTH, and by γ the angle GTH. Then,

$$FTG + TGF + GFT = \pi = FTH + THF + HFT; \text{ and}$$
$$fTG + TGf + GfT = \pi = fTH + THf + HfT; \text{ that is,}$$
$$\delta_1 + (\pi - \chi_1) + \omega_1 = (\gamma - \delta_1) + \chi_2 + \omega_1, \text{ or } \gamma - 2\delta_1 = \pi - (\chi_1 + \chi_2)$$
$$(\gamma - \delta_2) + \chi_1 + \omega_2 = \delta_2 + (\pi - \chi_2) + \omega_2, \text{ or } \gamma - 2\delta_2 = \pi - (\chi_1 + \chi_2)$$

Whence, $\gamma - 2\delta_1 = \gamma - 2\delta_2$, or $\delta_1 = \delta_2$, as enunciated.

PROP. XXI.

To find the relations amongst the segments of the chords drawn through the foci of a conic section.



Let F be the focus, and the system referred to this as origin; also put KFB $= \theta_1$ (and hence the salient angle LFB $= \pi + \theta_1$), and KF $= r_1$, FL $= r_2$. Then we have

$$r_1 = \frac{a(1-e^2)}{1 - e\cos\theta_1}, \text{and } r_2 = \frac{a(1-e^2)}{1 - e\cos(\pi - \theta_1)} = \frac{a(1-e^2)}{1 + e\cos\theta_1},$$

$$\text{or } r_1 = \frac{\frac{1}{2}p}{1 - e\cos\theta_1}, \text{ and } r_2 = \frac{\frac{1}{2}p}{1 + e\cos\theta_1};$$

p being the parameter of the principal diameter $= \dfrac{b^2}{\frac{1}{2}a} = \dfrac{2b^2}{a}$ in all the curves.

Then we have,

$$r_1 + r_2 = \frac{p}{1 - e^2\cos^2\theta_1},$$

$$r_1 r_2 = \frac{\frac{1}{4}p^2}{1 - e^2\cos^2\theta_1},$$

$$\text{and } \frac{r_1 r_2}{r_1 + r_2} = \tfrac{1}{4}p.$$

Corollary 1.

If a diameter CH be drawn parallel to KL, and denoted by ρ, we have

$$\rho^2 = \frac{a^2(1-e^2)}{1 - e^2\cos^2\theta_1} = \frac{r_1 r_2}{1 - e^2}, \text{ or } a^2 r_1 r_2 = b^2\rho^2.$$

Corollary 2.

In the *parabola*, the chord through the focus conjugate to any diameter is the parameter of that diameter.

For in this case, $e = 1$, and

$$r_1 + r_2 = \frac{p}{1 - \cos^2\theta_1} = \frac{p}{\sin^2\theta_1} = p_1.$$ See *prop. x. p.* 328, and *note, p.* 320.

PROP. XXII.

The lines drawn from the focus of a conic section to the points of contact of two tangents, form an angle which is bisected by the line drawn from the focus to the intersection of the same tangents.

Let $r_1\theta_1$ and $r_2\theta_2$ be the points of contact; then the equations of the tangents referred to the focus are for the ellipse and hyperbola,

$$r\{\cos(\theta - \theta_1) \pm e \cos\theta\} = a(1 - e^2) \quad \ldots\ldots (1),$$
$$r\{\cos(\theta - \theta_2) \pm e \cos\theta\} = a(1 - e^2) \quad \ldots\ldots (2).$$

Subtract the second of these from the first and simplify; then

$$2 \sin \tfrac{1}{2}\{2\theta - (\theta_1 + \theta_2)\} \sin(\theta_2 - \theta_1) = 0 \quad \ldots\ldots (3);$$

that is,

$$2\theta = \theta_1 + \theta_2; \text{ or } \theta_2 - \theta = \theta - \theta_1 \quad \ldots\ldots (4);$$

which is the algebraical expression of the proposition.

The process and the result for the parabola are precisely the same.

Corollary 1.

Substitute (4) in (1) or (2); then we have the value of r at the point of intersection; viz. in the *ellipse* and *hyperbola*,

$$r = \frac{a(1 - e^2)}{\cos\frac{1}{2}(\theta_2 - \theta_1) \pm e \cos\frac{1}{2}(\theta_2 + \theta_1)} \quad \ldots\ldots (5);$$

and in the *parabola* we obtain, in precisely the same manner,

$$r = \frac{h}{\cos\frac{1}{2}\theta_1 \cos\frac{1}{2}\theta_2} \quad \ldots\ldots (6).$$

Corollary 2.

The semi-angles at the focus are each $= \frac{1}{2}(\theta_2 - \theta_1)$.

PROP. XXIII.

If a circle be described about the triangle formed by any three tangents to a parabola, it will pass through the focus.

Let $r_1\theta_1$, $r_2\theta_2$, $r_3\theta_3$, be the three points referred to polar co-ordinates originating at the focus; then the equations of the three tangents are

$$r\cos(\theta - \tfrac{1}{2}\theta_1) = h \sec\tfrac{1}{2}\theta_1 \quad \ldots\ldots (1),$$
$$r\cos(\theta - \tfrac{1}{2}\theta_2) = h \sec\tfrac{1}{2}\theta_2 \quad \ldots\ldots (2).$$
$$r\cos(\theta - \tfrac{1}{2}\theta_3) = h \sec\tfrac{1}{2}\theta_3 \quad \ldots\ldots (3).$$

Denote by $R_1\Theta_1$, $R_2\Theta_2$, $R_3\Theta_3$, the intersections of the pairs of tangents, (2, 3), (3, 1), (1, 2), of these equations. Then, proceeding exactly as in *prop. xxii.* we have

$$\begin{array}{l|l}
\Theta_1 = \frac{1}{2}(\theta_2 + \theta_3) & R_1 = h \sec\frac{1}{2}\theta_2 \sec\frac{1}{2}\theta_3 \\
\Theta_2 = \frac{1}{2}(\theta_3 + \theta_1) & R_2 = h \sec\frac{1}{2}\theta_3 \sec\frac{1}{2}\theta_1 \\
\Theta_3 = \frac{1}{2}(\theta_1 + \theta_2) & R_3 = h \sec\frac{1}{2}\theta_1 \sec\frac{1}{2}\theta_2.
\end{array}$$

From these we very simply obtain the following ones

$$R_1 \sin(\Theta_2 - \Theta_3) = h\{\tan\tfrac{1}{2}\theta_3 - \tan\tfrac{1}{2}\theta_1\}$$
$$R_2 \sin(\Theta_3 - \Theta_1) = h\{\tan\tfrac{1}{2}\theta_1 - \tan\tfrac{1}{2}\theta_3\}$$
$$R_3 \sin(\Theta_1 - \Theta_2) = h\{\tan\tfrac{1}{2}\theta_2 - \tan\tfrac{1}{2}\theta_3\}$$

and by addition,

$$R_1 \sin(\Theta_2 - \Theta_3) + R_2 \sin(\Theta_3 - \Theta_1) + R_3 \sin(\Theta_1 - \Theta_2) = 0 \quad \ldots\ldots (4).$$

Whence the circle passes through the origin; (4) being the criterion (*p.* 286, *eq.* 20).

Scholium.

This remarkable proposition has in this country been generally, but erroneously, ascribed to Dr. Wallace, that gentleman having made a distinct and positive claim to it, in his *Conic Sections, page* 167. It was, however, published by the celebrated Lambert in his *Insigniores orbitæ Cometarum proprietates* (*section i.*), nearly forty years before Dr. Wallace republished it in the *Mathematical Repository.* See also *p.* 118 of this work; and Poncelet, *Traité des Propriétés Projectives*, p. 269.

This property has been usually considered an unfavourable one for the application of the co-ordinate methods, without deriving aid from the geometrical properties of the figure. I think that, except the one here given (which I first published in the *Phil. Mag.* July 1842,) there has been only one solution given, founded *purely* on the principles of co-ordinates; which was by Sir John Lubbock, *Phil. Mag.* August, 1838. The method here employed, is well calculated to illustrate the utility of the *polar equation of the tangent.*

EXERCISES ON LINES OF THE SECOND ORDER.

1. A LINE passing through $(-1, 2)$ and $(-2, 1)$ cuts both an ellipse and hyperbola whose axes are 6 and 8, and are parallel to the rectangular co-ordinate axes; and the co-ordinates of the common centre of the two curves are $(-1{\cdot}5, 2{\cdot}5)$: what portion of it is intercepted by the curves respectively?

2. The co-ordinates of the vertex of a parabola are $(-2, 1)$, its axis makes an angle of $-45°$ with the axis of x, and the curve passes through the origin of co-ordinates: find its equation.

3. Two parabolas, whose parameters are p and p_1, having a common vertex at the origin of co-ordinates, and their axes inclined at an angle of 60° to each other, and 50° and $-40°$ to the axis of x, intersect in a second point, the co-ordinates of which it is required to find.

4. Trace the curves represented by the equations:

$$(1).\ 3xy + 6x - 9y - 12 = 0.$$

$$(2).\ 4x^2 - 6xy \pm 8y^2 - 10x + 10y = 120.$$

5. Given the base and either the sum or difference of the angles at the base, to find the loci of the vertices, and of the centres of the circumscribed, inscribed, and escribed circles.

6. Given the base and sum or difference of the sides, to find the loci of the same parts as in the preceding example.

7. Given two focal radii-vectores, r_1 and r_2, and the angle a comprised between them, to find the polar equation of the parabola.

8. Given three focal radii of an ellipse, r_1, r_2, r_3, and the angles a_1, a_2, comprised between them, to find the polar equation of the ellipse.

9. A point P so moves that its distance from a given point has always the same ratio to its distance from a given line: assign the polar equation of the locus of P, and specify the diversity of cases that can arise.

10. A given polygon is so moved that two of its angular points always rest upon given lines: what curves are traced out by its other angles?

11. If parabolas have the same given vertex, and touch the same given line, what curve is traced out by the focus: and if they have the same focus and touch the same given line, determine the path of the vertex.

12. Tangents to a conic section intersect under a right angle; and others are parallel to conjugate diameters: determine the paths of the points of intersection in both cases.

13. The base of a triangle is given, and either the sum, difference, product, or

ratio of the tangents of the angles at the base, is also given, to find the locus of the vertex.

14. The equations of loci which intersect, are referred to an angle of co-ordination a, and denoted by

$$\frac{1}{ax}+\frac{1}{by}=\frac{1}{xy}+\frac{1}{ab}, \text{ and } \frac{1}{a_1x}+\frac{1}{b_1y}=\frac{1}{xy}+\frac{1}{a_1b_1};$$

determine their nature, and their points of intersection.

15. The sides containing the vertical angle and a point through which the base passes are given in position: prove that the locus of the centre of the circumscribing circle is an hyperbola.

16. If the rectangular ordinates of a conic section be produced till they are respectively equal to the normals of the same points: show that their extremities trace out *portions* of other conic sections, and assign the limits of these loci.

17. Find the co-ordinates of the foci of the conic section denoted by

$$r^2(a\sin^2\theta+b\sin\theta\cos\theta+c\cos^2\theta)+r(d\sin\theta+e\cos\theta)+f=0.$$

18. The triangle formed by three tangents to a parabola is circumscribed by a circle: it is required to prove that this circle always passes through the focus of the parabola, employing *rectangular co-ordinate axes*. (See *page* 336.)

19. The points of contact of a quadrilateral circumscribing a conic section are made the angular points of an inscribed one: then the diagonals of both quadrilaterals intersect in one point; and the intersections of the opposite sides of the two quadrilaterals intersect in points which are all four in one straight line.

20. From the extremities and the point of contact of one of the sides of a pentagon with a conic section, draw lines to the opposite angles: these will intersect in one point.

21. A line moves always parallel to itself, and cuts the three sides of a given triangle; and a point is taken in it so that the three segments into which it is divided by the lines and that point shall be either in arithmetical, geometrical, or harmonical proportion: it is required to prove, that in each case the locus of the point taken in the line is a conic section.

22. From a point lines are drawn making given angles with given lines; and they are so related that the sums or differences of the squares, or of the rectangles, of any number of them, taken in a given order, shall have a given ratio, sum, or difference: the locus of the point from which the lines are drawn is a conic section.

23. If an ellipse and hyperbola have the same diameters, then, if from any point in either curve lines be drawn from the extremities of either axis to cut the other curve: the lines joining the other points of intersection of the lines with the curve will be parallel to the other axis.

24. Of points, lines, and circles, any two being given, to show that the locus of the centres of the circles which touch them will be conic sections.

25. In the ellipse, if CP, CQ, be drawn at right angles to each other: prove that $\frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{CA^2}+\frac{1}{CB^2}$, where CA, CB are the semi-axes.

26. Let two parabolas have the same vertex, and their axes be perpendicular to each other; also let their parameters be a and $2a$: then the co-ordinates of their second point of intersection are two mean proportionals between a and $2a$; and the former is the side of a cube which is double of a^3.

27. Let AC, BC, be any two given lines meeting in C, and let AV, BV, be inflected to a point V so that $\tan VAC = m \tan VBC$: show that V is always in a conic section.

28. Let ABCD be a given quadrilateral, and P such a point that perpendiculars PF, PG, PH, PK, being drawn to AB, BC, CD, DA, there shall exist the relation $PF^2 + PG^2 : PH \,.\, PK :: m : n$; then P is in a conic section.

29. If two equal parabolas have axes coincident in position, then any line a tangent to the interior one, and a chord to the other, will be bisected at the point of contact.

30. In any two ellipses, or any two hyperbolas, situated on the same plane, there can always be found one pair of conjugate diameters in one parallel to one pair in the other: and if the two curves cut in four points, there can be a third curve described through them which shall have one pair of conjugate diameters also parallel to the former two pairs.

31. The equation $r = \dfrac{\frac{1}{2}h}{1 + e \cos \theta}$ represents an ellipse, parabola, or hyperbola, according as e is less than, equal to, or greater than unity; where h is the parameter of the curve.

32. Given the equation of the polar and of the conic section to which it is referred; viz.

$$my + nx = 1, \text{ and } ay^2 + bxy + cx^2 + dy + ex + f = 0:$$

to show that the co-ordinates of the pole are

$$y_1 = -\frac{(2cd - be) - m(e^2 - 2cf) - n(de - bf)}{(b^2 - 4ac) + m(be - 2cd) + n(bd - 2ae)},$$

$$x_1 = +\frac{(2ae - bd) - m(de - bf) - n(d^2 - 2af)}{(b^2 - 4ac) + m(be - 2cd) + n(bd - 2ae)}.$$

33. Assign the equation of a conic section which shall touch two given lines, each in a given point, and also pass through a third given point

34. Point out the method and assign the equations of condition from which we can find the equation of a conic section which passes through five given points, any how given, with respect to a system of rectilinear co-ordinates. What is the reason that these co-efficients have never been assigned?

35. Transform the general equation of the lines of the second order from a rectangular to a polar form, the origin of co-ordinates being the same in both systems; and prove from this that five independent conditions are necessary and sufficient for the determination of the curve.

36. Let the focal radius-vector make an angle χ with the tangent at the point of contact $r\theta$; and let ω be the angle made by the supplemental chords drawn from $r\theta$ to the extremities of the transverse diameter: then it is required to prove that,

$$2 \tan \chi = e \tan \omega.$$

37. Find the position and parameter of the parabola $\sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{y}{b}} = 1$.

38. A chord and its conjugate diameter make an angle α, and the tangent at the extremity of the chord makes with it the angles β_1, β_2: show that

$$\cot \beta_1 \sim \cot \beta_2 = 2 \cot \alpha.$$

39. Prove the following relations between the radius-vector and perpendiculars from the foci and centre upon the corresponding tangent:—

Origin.	*Parabola.*	*Ellipse.*	*Hyperbola.*
Focus	$p^2 = hr$	$p^2 = \dfrac{b^2 r}{2a - r}$	$p^2 = \dfrac{b^2 r}{2a + r}$
Centre	(no case.)	$p^2 = \dfrac{a^2 b^2}{a^2 + b^2 - r^2}$	$p^2 = \dfrac{-a^2 b^2}{a^2 - b^2 - r^2}$

40. If tangents to a conic section be drawn continually at right angles to each other, their intersections will trace a circle whose radius is $\sqrt{a^2 + b^2}$, in the ellipse and hyperbola; and a straight line tangent at the vertex, in the parabola.

41. If from the focus of a conic section a perpendicular be drawn to any focal chord, it will meet the directrix in the same point with the two tangents at the extremity of that chord.

42. Conjugate tangents in the ellipse and hyperbola intersect in points of another ellipse and hyperbola, which are similar and similarly situated with the original curves.

43. A variable tangent to the ellipse or hyperbola cuts from two fixed and parallel tangents, segments, whose rectangle is constant, and equal to the square of the semi-diameter conjugate to that which joins the fixed points of contact.

44. Find the lengths of the perpendiculars from the foci of each of the conic sections upon tangents, at the point x_1y_1, the curves being referred to rectangular co-ordinates.

45. The ellipse and hyperbola may be represented by the equations $y^2 = (1-e^2)(a^2 - x^2)$ and $y^2 = (e^2 - 1)(x^2 - a^2)$, where e is the determining ratio of the curve.

46. The lengths of two conjugate diameters $2a'$, $2b'$ of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, which make a given angle α with each other are

$$2a' = \sqrt{a^2 + b^2 + 2ab \operatorname{cosec} \alpha} \pm \sqrt{a^2 + b^2 - 2ab \operatorname{cosec} \alpha},$$
$$2b' = \sqrt{a^2 + b^2 + 2ab \operatorname{cosec} \alpha} \mp \sqrt{a^2 + b^2 - 2ab \operatorname{cosec} \alpha}.$$

47. Let three parabolas be escribed to the three sides of a triangle, and have the lines bisecting the interior angles of the triangle for their principal axes; let also a, b, c, be the sides of triangle, p_1, p_2, p_3, the parameters of the three parabolas, and r the radius of the circle inscribed in the triangle: then,

$$p_1, p_2\, p_3 : (a + b)(b + c)(c + a) :: r^3 : abc.$$

48. Let there be any line of the second order, referred to axes having an angle of ordination α, denoted by

$$ay^2 + bxy + cx^2 + dy + ex + f = 0:$$

then the two following hyperbolas will intersect in its foci: viz.

$$(b^2-4ac)(y^2-x^2) + 2(be-2cd)y-2(bd-ae)x + (d^2-4af)-(e^2-4af) = 0,$$

and,

$$(b^2-4ac)(xy-x^2\cos\alpha)+(bd-2ae)(y-2x\cos\alpha)+2bf-ed+(d^2-4af)\cos\alpha = 0.$$

49. A right line of given length a is erected perpendicular to a horizontal plane: to show that the locus of the extremity of its shadow is expressed by

$$y^2\cos^2\varpi + x^2(\cos^2\varpi - \cos^2\lambda) - ax\sin 2\lambda + a^2(\cos^2\varpi - \sin^2\lambda) = 0,$$

where ϖ = polar dist. of sun, and λ the latitude of the place.

50. Suppose the earth and sun to be spherical, the sun being in the focus of the elliptic orbit of the earth: find the locus of the vertex of the earth's conical shadow.

Ans. An ellipse similar to the earth's orbit, whose parameter is $\frac{pr}{r - r'}$, p being the parameter of the orbit, and r, r' the radii of the sun and earth.

51. A normal at any point of the equilateral hyperbola is equal to the distance of that point from the centre.

52. Three straight lines, A, B, C, are given on a plane; and circles are described to touch B, having their centres in A; and C for a common polar: show that the locus of the poles is an hyperbola.

THE GENESIS AND EQUATIONS OF VARIOUS CURVES.

It has been already explained, and sufficient examples have been given in illustration, that when one datum only is absent from the conditions which are requisite for the solution of a determinate problem, there are innumerable points in the same plane from which lines being drawn to fulfil all the other conditions, this remaining one will also be fulfilled. Thus, the base and altitude of a triangle being given, to find the triangle, then lines from *any point* in a certain line parallel to the base, fulfil the conditions: or, if it be required to find a point at a given distance from a given point, the condition is fulfilled by *any point* which is in the circumference of a circle whose centre is the given point, and radius the given distance: or, again, if it be required to find a point such that the sum or difference of its distance from two given points be given, any point in a corresponding ellipse or hyperbola satisfies the conditions. In all these cases one datum more would have fixed the point, or, at least, have reduced them to a definite number, all of which, with that datum annexed, could have been found with more or less *algebraical* difficulty.

The process for other curves is of exactly the same character as for those just quoted, or for other cases analogous to them, already discussed with a greater or less degree of detail. The method consists in assuming the unknown point (like the unknowns in common algebra) as represented by the co-ordinates x and y in rectilinear co-ordinates, or by $r\theta$ in polar; and then expressing all the conditions of the problem in terms of the data, and of xy, or $r\theta$, as the case may require. The result is the *equation of the curve;* and is that which the student is, in this section of his study, required to find.

It will be at once obvious that curves may be imagined without limit: but in the examples which follow we shall only select those which are interesting, either in connexion with the progress of mathematical science, or else for their application to practical purposes in mechanics or the arts. Even of these we shall only be able to put down the mere results. However, there is no case that is likely to create any insuperable obstacle to the student; any details of the investigation are unnecessary to bring it fully within his reach. Several of them, also, will be again noticed under the *differential and integral calculus.*

The chief of the curves which have received distinct names in consequence of some peculiar interest or value attached to them, are given by means of their respective geneses: then follow a few others, which are only varied methods of generating the same curves, or else of curves which have not received separate appellations.

No figures are given, as the *verbal description* will be a sufficient guide to their construction; and the student is earnestly urged to construct them *seriatim*, and, as far as possible, to trace the course of the several curves, and assign their intersection, with the axes. It will be a really useful practice for him.

I. The lemniscate. Its general equation is of the form

$$(x^2 + y^2)^2 + d^2y^2 + e^2x^2 = f^4.$$

No geometrical genesis is known to give the form above with three arbitrary magnitudes, d, e, f: but one or two varieties have much mathematical interest.

1. *The Cassinian ellipse.* Given the base, $2a$, of a triangle and the rectangle of its sides, ab, to find the locus of the vertex.

Take the base as axis of x, and the middle of the base as origin of rectangular co-ordinates; then the equation is readily found to be

$$\{y^2 + (a + x)^2\}\{y^2 + (a - x)^2\} = a^2b^2; \text{ or by reduction}$$

$$(y^2 + x^2)^2 + 2a^2(y^2 - x^2) + a^2 = a^2b^2.$$

The polar equation, by the usual transformation, becomes

$$r^4 - 2a^2r^2\cos 2\theta + a^2 = a^2b^2.$$

This curve was conjectured by Cassini to be the form of the planetary orbits, from which its name is derived. More recently it has been, with much probability, conceived by Sir John Herschel, to be the form of the lines of equal tint in polarised light. It will be advantageous to the student to construct a series of them to scale.

2. *The intersection of the perpendiculars from the centres of the ellipse and hyperbola upon the tangents, with those tangents.*

This problem gives the three equations,

$$a^2y_1^2 \pm b^2x_1^2 = \pm a^2b^2, \text{ the curve,}$$
$$a^2y_1y \pm b^2x_1x = \pm a^2b^2, \text{ the tangent,}$$
$$a^2y_1x \mp b^2x_1y = 0, \text{ the perpendicular.}$$

Eliminate x_1y_1 from these; and there results the final equation,

$$(x^2 + y^2)^2 = a^2x^2 \pm b^2y^2$$

(3). When $a = b$, or the equilateral hyperbola is taken, we get the simplified form

$$(x^2 + y^2)^2 + a^2(y^2 - x^2) = 0.$$

This curve is the *lemniscate of James Bernoulli*, which possesses many very curious properties.

II. Agnesi's Witch. At the extremity B of the diameter AB ($= a$) of a circle, draw a tangent BH to meet any other line AH drawn from A to cut the circle in E; also from H draw HP parallel to AB, and from E draw DEP parallel to BH, meeting HP in P: then P traces the curve.

Ans. Its equation is $xy^2 = a^2(a - x)$.

This curve is more remarkable as being proposed by a lady, the Donna Agnesi, professor of mathematics at Bologna, than for any properties which it possesses.

III. The cissoïd of Diocles. Take two points P, S, variable in position, but always equidistant from A, B, the extremities of the diameter of a circle, and draw ST, PM perpendicular to AB; and through T where ST meets the circle draw AT to meet PM in M, the point which traces the cissoïd.

Let $a =$ the diameter of the circle; then $y^2 = \dfrac{x^3}{a - x}$ is the equation required.

This curve was devised for the trisection of an angle and the duplication of the cube.

IV. The conchoïd of Nicomedes. A line PB revolves about a given point P, and cuts a given line AX in B; and in the line PB a point M is taken such that BM is always of the same given length. The point M traces the conchoïd.

Draw PA perpendicular to AX, and take AX for axis of x and A for origin. Let PA $= a$, and BM $= b$. Then the equation is

$$y^2x^2 - (y^2 - b^2)(y + a)^2 = 0.$$

Also, if P be taken as polar origin, and PA as origin of θ, the polar equation of the curve is

$$(r \mp b) = a \sec \theta.$$

according as M is on the same side of AX or not, with P.

This curve was proposed for the same purpose as the preceding one.

V. THE TRISECTRIX. From a point A in the circumference of a circle, draw chords, and from the other extremity C of each chord, set off CP, CP′, each equal to the radius: the points P, P′ will describe the branches of the curve.

The rectangular equation referred to the point A as origin, and having the diameter through A for axis of x, is

$$(x^2 + y^2)^2 - a\,(a + 2x)\,(x^2 + y^2) + 4a^2x^2 = 0.$$

The polar equation is

$$r = (2 \cos \theta \pm 1)\, a,$$

the upper sign applying to the exterior and the lower one to the interior branch of the curve.

The object of this curve, as its name implies, is to trisect an angle. It was first proposed by Captain Burton, R. N., for that purpose, in the *United Service Journal*, twelve or fourteen years ago.

VI. THE QUADRATRIX OF DINOSTRATUS. About the centre C of a circle whose radius is a, a line CP revolves with an uniform angular velocity, originating at the radius CB, whilst parallel to a tangent at B, and originating with it, a line DP moves also with an uniform angular velocity, such that D moves through the radius BC, whilst the revolving line CP describes a right angle: these two lines meet in P, which describes the curve.

If BC be the axis of rectangular co-ordinates, and B the origin, the curve will be denoted by

$$y = (a - x) \tan\left(\frac{x}{a}\cdot\frac{\pi}{2}\right).$$

If the curve be referred to polar co-ordinates, the centre being the pole, and CB the origin of θ, the polar equation will be

$$r = a \sec \frac{\pi - 2\theta}{\pi}.$$

This curve was invented by Dinostratus for the multisection of an angle, and the multiplication of a cube. It is, however, remarkable chiefly from its furnishing an explanation of the curious optical phenomenon exhibited by viewing the wheel of a carriage through the vertical bars of a Venetian blind. See Dr. Roget, *Phil. Trans.*

VII. THE QUADRATRIX OF TSCHIRNHAUSEN. If whilst the uniformly revolving line move from the radius drawn to one extremity, B, of a quadrant, the other line move uniformly from the other extremity, A, of the quadrant, parallel to the tangent at A, the intersection of these lines trace the curve.

Employing the same notation as in the preceding case, the rectangular and polar equations are respectively,

$$y = r \sin\left(\frac{x}{a}\cdot\frac{\pi}{2}\right), \text{ and } r = a \sec \frac{2\theta}{\pi}$$

This curve, like the preceding, was devised for the multisection of an angle, and the multiplication of the cube.

VIII. WATT'S PARALLEL MOTION. The line ABC $= 2b$ has its extremities moving on the circumference of two equal circles, whose radius is c, and the distance of their centres CD $= 2a$: the middle point of AB describes the curve.

Take M, the middle of CD, for origin of co-ordinates, CD being the axis of x; then x_1y_1, x_2y_2, xy denoting A, B, P, we have the following five equations from which to eliminate x_1, y_1, x_2, y_2.

$$(a - x_1)^2 + y_1{}^2 = c^2 \qquad\Big|\qquad 2x = x_2 + x_1$$

$$(a + x_2)^2 + y_2{}^2 = c^2 \qquad\Big|\qquad 2y = y_2 + y_1$$

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = 4b^2.$$

Put $d^2 = b^2 - c^2$: then the result is,

$$(a^2 - d^2 + x^2 + y^2)^2 x^2 + (a^2 + d^2 + x^2 + y^2)\, y^2 = 4a^2b^2y^2.$$

Or, again, if referred to polar co-ordinates, the origin of θ being the previous axis of y, the result takes the form

$$\sin^2\theta = \frac{4a^2b^2 - (r^2 + a^2 + b^2 - c^2)^2}{4a^2\,(c^2 - r^2)}.$$

The object of Watt in this construction was, to give to P by means of the motions of the three bars CA, AB, BD about the joints at A, B, C, D, a *rectilinear vertical motion.* Within the limits of the actual motion in the circumstances of the constructions employed, it becomes *very near* to a fulfilment of that purpose, though not accurately. For a lengthened discussion of this subject, see Prony, *Arch. Hydr. tom ii. pp.* 133—145. Also *Gent. Diary*, 1817 and 1838.

IX. THE MAGNETIC CURVE. The base $(= 2a)$ and sum of the cosines of the angles θ_1, θ_2 at the base is given $(\cos \Theta_1, + \cos \theta_2 = 2 \cos \beta)$: the vertex traces the curve.

Refer to the middle of the base as origin, the base itself being the axis of x: then the equation is

$$\frac{a + x}{\sqrt{(a + x)^2 + y^2}} + \frac{a - x}{\sqrt{(a - x)^2 + y^2}} = 2 \cos \beta,$$

which, when deprived of its radicals and denominators, is of the eighth degree.

Many of the properties of this curve are given by the editor of this work in two papers in the *Philosophical Transactions* for 1835 and 1836.

X. THE INVOLUTE OF THE CIRCLE. A string being unwound from the circumference of a circle; its extremity traces the curve.

Let the line from which θ is measured be the radius through the position of the extremity of the unwrapped string, and the centre of the circle be the origin; also let a be the radius of the circle: then the polar equation is

$$\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\frac{a}{r}.$$

XI. THE LOGARITHMIC CURVE. If abscisses be taken on a given line in arithmetical progression, and their ordinates be taken in geometrical progression, the extremities of these ordinates trace out the curve.

Let a be a constant; then the curve is denoted by

$$y = ae^x.$$

Scholium.

The curve represented by the equation

$$x + a = \tfrac{1}{2}\, a \left\{ e^{\frac{y}{a}} + e^{-\frac{y}{a}} \right\},$$

being that of a flexible chain suspended at its extremities, is called the *Catenary.*

XII. THE TRIGONOMETRICAL CURVES. These are formed by opening out the circumference of a circle into a straight line, and at each point of this line setting off an ordinate at right angles to it, equal to the sine, cosine, tangent, *etc.*, of the portion of the curve from the point which we took as origin. Their equations, it is obvious, will be

$$y = \sin x, \ y = \cos x, \ y = \tan x, \ etc.$$

They are often called *the figure of sines, the figure of cosines, etc.*

XIII. THE HARMONIC CURVE. If a right cylinder be cut by a plane, and the surface be opened out upon a plane, the cut-edge of the surface becomes this curve. Its equation is

$$y = m \sin x;$$

where m depends upon the inclination of the cutting-plane to the axis of the cylinder. When it cuts the axis under an angle of 45°, $m = 1$, and the curve becomes the figure of sines.

XIV. THE ALGEBRAIC SPIRAL. A line revolving about a pole, whilst a point in it is so taken, that the radius-vector is a given multiple of a given power of the arc of a given circle concentric with the pole; then this point traces the algebraic spiral.

Let a be the radius of the given circle; then the equation is readily seen to be

$$r = ma\theta^n.$$

Of these there are several cases which have been much discussed. The following are a few of the number that have received names.

$r = a\theta$, the *spiral of Archimedes*,

$r = a\theta^{-1}$, the *reciprocal* or *hyperbolic spiral*,

$r = a\theta^{-\frac{1}{2}}$, the *lituus*, one of *Cotes's spirals*.

XV. THE EXPONENTIAL SPIRAL. The point in the revolving line is expressed by some multiple power of the angle described.

Its equation, therefore, is $r = ae^{m\theta}$.

When $m = 1$, the curve is the *logarithmic spiral*, and possesses many remarkable properties, mainly discovered by James Bernouilli.

XVI. THE PARABOLIC SPIRAL. If the axis of a parabola be wound round a circle, the curve itself becomes the parabolic spiral.

Let p be the parameter of the parabola, a the radius, and c the circumference of the circle; also measure θ from the radius drawn to the vertex of the parabola; then the polar equation is

$$r = a \pm \sqrt{pc\frac{\theta}{\pi}};$$

where the upper and lower signs refer respectively to the spirals formed by the two semi-parabolas.

XVII. THE CYCLOÏD OR ROULETTE. A given circle rolls upon a given line, then any point in the plane of the circle traces the cycloïd.

Let the axis of x be the given line or directrix, and the origin that point where the radius through the tracing point is perpendicular to the axis of x. Denote the distance of the tracing point from the centre of the circle by r, and let mr be the radius. Let also the angle made by the radius in contact with the directrix with the radius through the describing point, at any period of the generating motion be ω: then

$$x = r\,(m\omega - \sin\omega),\ \text{and}\ y = r\,(1 - \cos\omega).$$

For many of the purposes of investigation, these equations are very convenient, and they are easily obtained; but to have the definite equation of the curve, the angle ω must be eliminated. This gives

$$y - r + r\cos\frac{x + \sqrt{2ry - y^2}}{mr} = 0,$$

$$\text{or}\ y = 2r\sin^2\frac{x + \sqrt{2ry - y^2}}{2mr}.$$

The cycloïd is the *prolate, common,* or *curtate,* according as m is less than 1, equal to 1, or greater than 1.

The cycloïd has many and remarkable properties, the greater part of which were discovered by Pascal.

XVIII. THE COMPANION TO THE CYCLOÏD. The ordinate to a circle being produced till it is equal to the arc, estimated from the diameter to which the ordinate is drawn, traces with its extremity this curve.

If r be the radius of the circle, and ω the angle at the centre subtended by the arc: then

$$y = r\,(1 + \cos\omega),\ \text{and}\ x = r\omega.$$

Or if ω be eliminated, as in the cycloïd, we have

$$y - r - r\cos\frac{x}{r} = 0;\ \text{or again,}$$

$$y = 2r\cos^2\frac{x}{2r}.$$

This curve receives its name from the circumstance, that if the ordinates of the common cycloïd be diminished by the corresponding ordinates of the circle, their extremities will describe the companion to the cycloïd. It does not differ in reality from *the figure of sines.*

XIX. THE EPICYCLOÏD. If a circle roll upon a fixed circle, then any point invariably connected with the rolling circle, traces this curve.

The curve is properly called the *epicycloïd* when the tracing point is in the circumference of the rolling circle: it is called the *epitrochoïd* when the point is not in the circumference, and the rolling circle is *without* the base or directrix; and the *hypotrochoïd,* when the rolling circle is *within* the directrix, or touches its concavity.

Take the straight line in which the two centres and the tracing point situated in the course of their motion, as axis of y, and a line perpendicular to it through the centre of the fixed circle, as axis of y. Let r be the radius of the fixed circle, r' that of the rolling one, and b the distance of the tracing point from the centre of the latter; also, let ω be the angle made by the line joining the centres of the two circles in any position with the axis of y: then the values of x and y in terms of ω will be

$$y = (r \pm r')\cos\omega + b\cos\left(\frac{r \pm r'}{r'}\,\omega\right),$$

$$x = (r \pm r')\ \sin\omega \pm b\sin\left(\frac{r \pm r'}{r'}\,\omega\right);$$

the upper sign referring to the *epitrochoïd,* and the lower one to the *hypocycloïd.*

If the tracing point be in the circumference of the rolling circle, we have $b = r'$, and the values for the *epicycloïd* and *hypocycloïd*, as the upper or lower signs are taken, are

$$y = (r \pm r) \cos \omega + r' \cos \left(\frac{r \pm r'}{r'} \omega \right),$$

$$x = (r \pm r') \sin \omega \pm r' \sin \left(\frac{r \pm r'}{r'} \omega \right).$$

Some varieties of these curves are not without interest. For instance:

(1). When the radius of the rolling circle which generates the epicycloïd is half that of the fixed circle, or $r' = \frac{1}{2}r$, the curve is an *ellipse.*

(2). When the radii have the same relation in the hypocycloïd, the point traces a diameter of the fixed circle.

(3). When the radii of the circles are equal, the epicycloïd becomes *Castillon's cardioïd;* a figure resembling a heart, and which possesses some curious properties. Its rectangular and polar equations are

$$(y^2 + x^2 - a^2)^2 = 4a^2 \left\{ y^2 + (x + a)^2 \right\}$$

$$r = 2a\,(1 + \cos \theta) = 4a \cos^2 \tfrac{1}{2}\theta.$$

(4.) If hypocycloïds be described by generating circles whose radii are $\frac{1}{2}(r + r')$ and $\frac{1}{2}(r - r')$, they will be identical: r being the radius of the fixed circle, and r' any line less than r.

XX. THE EPICYCLE. The centre of the moving circle moving uniformly in the circumference of another, revolves with a uniform angular velocity about that centre: then a point in the circumference of the moving one traces the curve.

Let the initial position be taken when the tracing point C is in line with O, B, the centres of the fixed and moving circles; and let B be another position of the centre of the moving circle, corresponding to the position P of the tracing point. Denote the angle AOB by ω, and the angle DBP (D being in continuation of OB) by $m\omega$; also let r and r' be the radii of the fixed and moving circles: then,

$$x = r \sin \omega + r' \sin (m + 1)\,\omega,$$

$$y = r \cos \omega - r' \cos (m + 1)\,\omega,$$

are the values of the rectangular co-ordinates, y having the position OA, and x perpendicular to it.

The epicycle has an important place in the history of astronomy, it being the curve in which Ptolemy conceived the celestial objects to move round the earth.

XXI THE COROLLA. These curves are so called from their resemblance to the corolla of an open flower. They are comprised in the general polar equation,

$$r = a \sin m\theta; \text{ or } r = a \cos m\theta.$$

When m is an integer, the corolla will be composed of an even number of petals; when $2m$ is an integer and m not so, it will be composed of an odd number. When m is irrational, it will be composed of an infinite number overlaying and crossing one another.

The student will be interested in the construction of a few of the simpler ones.

In the *Philosophical Transactions*, there is a paper by Guido Grandi, in which several of them are figured. It was presented under the quaint designation of "a bouquet of flowers for the Royal Society."

FURTHER EXERCISES ON CO-ORDINATES.

1. One parabola rolls upon another in the manner of the epitrochoïd, their vertices being in contact at the commencement of the motion: show that the focus of the rolling one describes a straight line, and that its vertex describes the cissoïd.

2. Let BC, CI, be indefinite lines at right angles, and a point E be taken in BC; let two rulers HG, GF, meet in G, and be fixed at right angles to one another; also let the ruler FG be of a given length, and GH be unlimited: then if this rectangular ruler be so moved that HG always passes through E, and F always rests upon CI, the middle of FG will trace the cissoïd.

This is Newton's instrument for the construction of the curve.

3. If a circle be described on the principal diameter of the ellipse or hyperbola, (or a tangent drawn at the vertex of the parabola,) and a rectangular ruler, as in (2), have one leg passing through the focus of the curve, and the angular point moving on the circle, (or tangent, in the case of the parabola,) then the other leg of the ruler will be a tangent to the conic section.

4. A semicircle being described on a given line AB ($= 2a$) as diameter, let an indefinite straight line EF be drawn parallel to that diameter at the distance b ($< a$) from it; also let the portion DD′ of any radius CD′ intercepted by this line and the circle be divided in P in the ratio of m to n: then the locus of P is the conchoïd, whose pole is C the centre of the circle; the distance of its pole from the directrix is $\frac{nb}{m+n}$; and the constant distance of the tracing point from the directrix is $\frac{ma}{m+n}$.

5. The intersection of the perpendicular from the centre of an ellipse upon the tangent with the tangent is denoted by the polar equation

$$r^2 = a^2 \cos^2\theta + b^2 \sin^2\theta.$$

6. Given the base $2a$, and altitude b of a triangle, to find the locus of the centre of the inscribed circle.

$$2(x^2 - a^2)y - b(y^2 + x^2) + a^2b = 0.$$

7. The corner of a leaf of paper is doubled down; find the locus of the angular point, θ being measured from the line bisecting the original corner of the leaf.

(1). When the length of the crease is given $= 2a$;

(2). When the folded area is given $= \Delta$.

Ans. $r = 2a \cos 2\theta$, and $r^2 = 2\Delta \cos 2\theta$ respectively.

8. The three angles of a variable triangle DEF continually move upon the three sides of a given triangle ABC; also two of the sides DE, EF, are always divided by two given lines in given ratios: show that the locus of the point which divides the third side FD in a given ratio is a conic section.

9. DG, FG, are two lines given in position, and E a given point: show that when the two lines form the side of a variable triangle whose base passes through E, the locus of the centre of the circle circumscribing the triangle is an hyperbola.

10. The equation of the cycloïd referred to the middle of the base, and to the vertex, becomes respectively

$$y - r \cos \frac{x - \sqrt{2ry - y^2}}{mr} \mp r = 0$$

DIFFERENTIAL CALCULUS.

INTRODUCTION.

1. In the practice of elementary algebra, quantities which from the conditions of their introduction into an investigation were *given*, are conveniently called in the higher departments of the science, *constants*. The two words have, however, precisely the same meaning; viz. a fixed, though arbitrary, value throughout the investigation.

The constants here, as the *data* there, are always designated by the earlier letters of the alphabet, *a*, *b*, *c*, etc.

2. Every determinate problem in elementary algebra terminates in a single equation, involving only the *data* and a single *quæsitum* of the problem. The symbol of that quæsitum constitutes the *unknown* of the equation; and the resolution of the equation furnishes either one, or a defined number, of the values of that unknown, which are all the values which that symbol can take in such an equation, consistently with the conditions of the problem which gave rise to it.

All the terms of the equation being brought to one side, the result is commonly called a function of that unknown; and if x be that unknown, it is written $f(x)$, $F(x)$, $\phi(x)$, X, *etc.*, according to convenience. (See *vol. i. pp.* 207, 231.)

In the equation $f(x) = 0$, *etc.*, there is nothing indeterminate: since (*theoretically* at least), there is only a definite number of values of x; or at least its values are separated by *finite intervals*, instead of being capable of taking continuous successive values without limitation.

3. In elementary algebra, there is a class of equations comprising only given quantities, and a single quantity whose value has not been laid down or rendered constant. Such are

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots \quad (\textit{vol. i. p.}\ 163),$$

$$(1+x)^n = 1 + \frac{n}{1}.x + \frac{n(n-1)}{1.2}x^2 + \ldots\ldots\ldots \quad (\text{——}\ \ 242),$$

$$\log_e x = 1 + (x-1) + \tfrac{1}{2}(x-1)^2 + \ldots\ldots\ldots \quad (\text{——}\ \ 250),$$

$$\sin x = 1 - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots\ldots \quad (\text{——}\ \ 435).$$

Such equations only express the *identity* of the second side with the first, when the indicated or implied operations for expansion are supposed to be performed. In these, however, x is capable (and in the investigations which gave rise to the equations *assumed* to be) of all *consecutive* and *continuous values*. The symbol x is in these cases called *indeterminate;* and the reader will perceive in this circumstance the perfect propriety of the term applied to the method by which the successive coefficients were obtained, so as to fulfil the several equations—*the method of indeterminate coefficients.*

4. In the preceding case, the coefficients, though determinable by the methods employed in their investigation, and therefore *given*, were not arbitrary. When, however, an equation terminated so as to involve *more than one quantity* whose values were not given or constant (see *vol. i. pp.* 174, 175), then it is clear that either of them may admit of indefinite and continuous variation consistent with

the conditions which gave rise to the equation in which they occurred. A case familiar to the student may be quoted in illustration: the *doctrine of loci*. Take, for instance, the equation of the straight line

$$y = ax + b.$$

In this it has been shown, that for *every* value of x there is one corresponding value of y; and *vice versâ*, that for every value of y there is one corresponding value of x: the variations of the fundamental symbol, whether x or y, being *continuous*, or *without finite interval*, and the values of the dependent one, whether y or x, being also *continuous*.

In the equation of the second degree, also,

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

the same remarks apply; and though the possible values of x and y may be contained within *finite limits*, their variations are *continuous*, or *without finite intervals*.

5. It is not, however, to be understood that there can only enter *one* such independent quantity, whether x or y, into an equation; for an equation may be conceived to have any specified number of its component symbols of variable value. The uses to be made of such an assumption in subsequent inquiries, is evidently the only limit to the number of assumed variables. The *continuity of their variations according to a uniform law*, is the only condition to which such symbols are to be considered subjected in these investigations.

6. A *function of a variable* (or *of several variables*) is any algebraic expression which contains it (or them), either with or without the intermixture of given or constant quantities. The notation for a function of a single variable x, is $f(x)$, $\phi(x)$, u_x, X, *etc.*; and that for a function of several variables, x, y, z, is $f(x, y, z, \dots)$, $\phi(x, y, z, \dots)$, $u_{x, y, z,}$ *etc.* Nothing beyond the expressions which we have under consideration being composed of $x, y, z, \dots$ is to be understood by these notations; and either of them may be used according to circumstances. (*vol. i. Finite Differences.*)

7. Where a series of functions of any number of variables are derived each from the preceding one according to any given law, it is usual to express the circumstance by the following notation:—

Given function.	1st derived funct.	2nd derived funct.	3rd derived funct.	
$f(x, y, \dots)$	$f_1(x, y, \dots.)$	$f_2(x, y, \dots)$	$f_3(x, y, \dots)$	......
$\phi(x, y, \dots)$	$\phi'(x, y, \dots)$	$\phi''(x, y, \dots)$	$\phi'''(x, y, \dots)$	......
$u_{x, y, \dots}$	$u'_{x, y, \dots}$	$u''_{x, y, \dots}$	$u'''_{x, y, \dots}$	......
U	U_1	U_2	U_3	......

In the last case U is put (as also u often is) to represent the function, without the subscribed letters of $u_{x, y, \dots}$

8. Whatever functions enter into an expression, they always appear as component parts of an equation. When the entire function is brought to one side, the other will be zero; or the equation will be written in one of the forms $f(x, y, \dots) = 0$, $u_{x, y, \dots} = 0$, $U = 0$, *etc.* Such functions are called *implicit functions* of the variables. When, on the other hand, the equation is resolved, or assumed to be, for one of the variables (for instance z) in terms of the other or others, still entering into the general implicit form; then the equation is written $z = f(x, y, \dots.)$, and z is called an *explicit function* of the other variables, $x, y, \dots$

9. As the variation which any one variable undergoes depends upon all those which the remaining ones undergo; and as all the others but this one admit of variation independently of each other, these others are called *independent variables*, and the remaining one the dependent variable. The expression $f(x, y, \ldots) = 0$ does not point out by its notation which are the independent variables; but $z = f(x, y, \ldots)$ expresses that $x, y, \ldots$ are the independent variables, and that z is the dependent variable. It is for the expression of this circumstance that the two different notations are mainly used.

10. The *limits* of a variable quantity are those values to which it can approach more closely than any assignable quantity, however small, and beyond which it can never pass; that variable being subjected to the conditions under which it was originally introduced into the inquiry.

11. The *incremental ratio* is the value of the fraction which arises from dividing the increment of the function by that of the variable.

Thus, if the function be $u = 2x^2 - 4x$, and x receive any increment h, then u' being put for the *new value* of the function, we have

$$\begin{aligned} u' - u &= \{2(x+h)^2 - 4(x+h)\} - \{2x^2 - 4x\} \\ &= 4hx + 2h^2 - 4h\text{; and hence,} \end{aligned}$$

$$\frac{u'-u}{h} = 4(x-1) + 2h$$

Now $u' - u$ is the increment of the function, and h that of the variable; and the value $4(x-1) + 2h$ is the *incremental ratio*.

It is found convenient (as in *vol. i. p.* 273, *etc.*) to write Δx for h, and Δu for $u' - u$; and in this notation, the incremental ratio is written

$$\frac{\Delta u}{\Delta x} = 4(x-1) + 2\Delta x.$$

When there are several independent variables, the principle of this definition can be easily extended; but in the present stage of our progress we shall only need to consider the case of one independent variable.

12. The *tendent ratio* of a function is that *towards which, as a limit*, the incremental ratio tends, by the *continuous diminution* of h or Δx.

The increments of u and x are in this case designated by the prefix d instead of Δ, the fraction being written $\frac{du}{dx}$.

To ascertain in the preceding example the *tendent ratio*, we see at once that by the continual diminution of Δx, the fraction approximates towards the value $4(x-1)$; whilst for no finite value of Δx, however small, does it become actually equal to that quantity, though we may make it approach to $4(x-1)$ more closely than any quantity which can be assigned. Nor can it ever pass that value by continually diminishing Δx (without changing the sign, which is contrary to our definition); and hence such is properly the tendent ratio.

Had, on the other hand, Δx been originally taken negative, the diminution of its actual value would have led to the same result.

Let us now consider the general form of all the functions in which $f(x+h)$ can be developed [*] in ascending integer powers of h. Put

$$\begin{aligned} u &= fx, \\ u' &= f(x+h) = fx + \mathrm{A}h + \mathrm{B}h^2 + \ldots. \end{aligned}$$

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[*] We say "which can be"—for no others properly come under our consideration. Let those who believe that all functions can be so developed, show it. Lagrange has failed in this, at all events.

where A, B, . . . are functions of x and constants, which do not become infinite; then

$$u' - u = \mathrm{A}h + \mathrm{B}h^2 + \mathrm{C}h^3 + \ldots \text{ or}$$

$$\frac{u' - u}{h} = \mathrm{A} + \mathrm{B}h + \mathrm{C}h^2 + \ldots$$

The tendent or ultimate ratio is therefore

$$\frac{du}{dx} = \mathrm{A}.$$

13. The continually diminishing increments dx, dy, are called the *differentials* of x and y; that is, *minute differences* [*], to distinguish them from the *finite differences* Δx, Δy treated in *vol. i.*

14. The *differential calculus* is that which investigates the properties and results of the tendent ratios of functions, and of their independent variables.

15. The tendent ratio, or ratio of the differentials of the function and its variables is called the *differential co-efficient.*

Thus in the preceding examples,
$4(x - 1)$ and A, are the differential co-efficients of $2x^2 - 4x$, and of u, respectively.

Scholium 1.

It is very important that the student should accustom himself to make a distinction between the *mere addition* of a small quantity to a comparatively large one, and the *ratio* between two very small quantities. Let him consider that though two quantities, each viewed in relation to another quantity, may be very small; and that if the large one be taken as a measure, these quantities may be said by continual diminution according to their respective laws of formation, to approach to equality. Let the one, however, be considered the measure of the other; then they may still have a ratio, which in comparison with either, would become very large.

Take, for instance, the fraction $\frac{(\cdot 1)^n}{(\cdot 1)^{n+1}}$. That is evidently 10, or the numerator is 10 times the denominator, for all values of n.

But if $n = 50$, the *difference* of the numerator and denominator is $(\cdot 1)^{50} \times \cdot 9$, or 9 in the fifty-first decimal place: that is, a quantity extremely small in relation to the ratio itself.

Again, take the fraction $\frac{a^3 - b^3}{a - b}$, whose universal value is $a^2 + ab + b^2$. Now, for all values of b less than a, but approximating more and more closely to it, the numerator and denominator cotemporaneously approach towards zero, and when $b = a$ it becomes actually $\frac{0}{0}$. Yet the fraction itself, or the ratio of $a - b$ to $a^3 - b^3$ may be extremely great. In fact, the smallness of the numerator and denominator depends entirely upon the approximation of the value of b to that of a; whilst the value of the ratio depends upon the actual magnitudes of a and b.

Again, take the expression $\frac{\sin x}{x}$. This, as x diminishes, has both its terms diminished, and when $x = 0$, it takes the form $\frac{0}{0}$. But the series for sin x (*vol. i. p.* 435) being written instead of it, we have

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[*] In their ultimate state of diminution, the differentials become virtually zero: but this notation has the advantage of showing from what considerations these terms have been derived, or rather, perhaps, what functions they are whose diminutions we are considering.

It will be well to keep in mind in this notation, that the numerator expresses the function which is differentiated, and the denominator the variable, according to which the differentiation is performed.

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots}{x} = 1 - \frac{x^2}{1.2.3} + \frac{x^4}{1.2.3.4.5} - \ldots$$

which in the supposed case gives $\frac{\sin 0}{0} = 1$; or 1 is the value to which the fraction approaches by the continual diminution of x.

In this case the terms approach to equality, as in the former cases they were actually 10 and $a^2 + ab + b^2$.

As a fourth example, take $u = x^3$, $u = (x + h)^3$, and $x' = x + h$; then

$$\frac{u' - u}{x - x} = \frac{(x + h)^3 - x^3}{(x + h) - x} = (x + h)^2 + x(x + h) + x^2.$$

Now, let h be diminished, *sine limite*, then we get ultimately the *tendent ratio*, $\frac{u' - u}{x' - x} = \frac{0}{0} = 3x^2$; which shows that whilst the terms of the fraction respectively approximate to 0, the ratio between them tends to $3x^2$.

In all the cases here instanced, *the terms of the fraction* are diminished by diminishing the value of the common measures, viz. of $(\cdot 1)^{50}$ in the first case, of $a - b$ in the second, of x in the third, and of $(x + h) - x$ in the fourth.

But the numerator and denominator may be functions which have no algebraic common measure; and yet continual increase or diminution of some quantity concerned in it may tend to reduce both terms to zero at the same time.

Take, for example, the fraction, $\frac{e^{x+h} - e^x}{(x + h) - x}$ or $\frac{e^x \{e^h - 1\}}{(x + h) - x}$.

In this, as h diminishes, the factor $\frac{e^h - 1}{(x + h) - x}$ approximates to $\frac{0}{0}$, for $e^h - 1 = 0$, and $(x + h) - x = 0$. In the *extreme case only*, then, is there a common measure; and in this case, $\frac{0}{0} = e^x$.

It is unnecessary to multiply instances, as those already adduced are quite competent to show that the terms of a ratio or a fraction may diminish respectively towards the zero of the functions represented by them, and yet that the ratio itself may be of any value whatever.

It may be well to remark that the expressions which for particular values of one or more of the quantities concerned become $\frac{0}{0}$, are called *vanishing fractions*, from the simultaneous evanescence of both their terms.

Scholium 2.

It is to be distinctly kept in mind, that in the extent to which, in a contracted treatise like the present, our investigations can be carried, the only class of functions considered are those which, *á priori*, we show to be capable of development according to *positive integer and ascending powers* of the increment. The *possibility* of the development in other forms is neither affirmed nor denied, since the question does not bear upon the view here taken: and for the same reason, it is equally unimportant to us at present whether all functions whatever admit of such development or not—though, *en passant*, we may remark that there has not yet been given a perfectly logical and convincing proof of the affirmative, notwithstanding the affirmative is the general belief of mathematicians.

Scholium 3.

We embrace this opportunity of making a remark on a subject intimately connected with the present one—in fact, the foundation of *all* development—the doctrine of *indeterminate co-efficients.* The principle upon which we had founded the argument, for the method, (at *page* 237, *vol. i.*,) though strictly logical, is unnecessarily recondite, as it pre-implies the fundamental theorem respecting the roots of an algebraical equation—a principle of which, when most generally considered, the theory of indeterminate co-efficients is altogether independent.

Every function proposed for expansion implies that certain operations are to be performed (which are indicated by the *given form of the function*) upon some combination of the several simple functions; and the development is the expression of the result of the performance of these operations, in monomials, following the powers of the symbol according to which the development is to be made.

Now the general principle on which, in the very outset of algebra, the solution of any problem is made to depend, is:—to suppose the problem solved, there entering into the expression either one or several unknowns; the conditions in order to capability of solution being such as to give as many successively dependent equations as there are unknowns in the system of conditional equations introduced.

When those equations can be fulfilled by real values of the data, their solution constitutes the solution of the problem; but when the solution of these equations involves either imaginary or ambiguous values of the unknowns, the problem is incapable of solution.

The doctrine of indeterminate co-efficients is a direct application of the same principle. It merely *assumes* that the development is effected according to a *certain assigned form;* or, in other words, it is assumed that it is possible to develop the function in that particular form*, the co-efficients being independent of the symbol according to the successive powers of which the development is to be made. If our assumption be correct, and we take any functions of both sides of the equation (between the original function and its development) we must have identical results on both sides.

Taking, then, those *inverse functions* which will reduce the first side to a series of monomes (that is, unravel the function to its monomial elements, and perform the same implied operations on the development), we must have an *identity* between the two results, if the assumed development be a true one. This identity is shown by the equality of the homologous co-efficients; and hence it gives the several required equations, involving the unknown and assumed co-efficients

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* As an instance of the capability of more forms of development than one, we may quote the binomial theorem when the index is fractional. Take the example $(a+x)^{\frac{1}{2}}$, which is identical with $(x+a)^{\frac{1}{2}}$. The two developments are

$$a^{\frac{1}{2}} + \tfrac{1}{2}a^{-\frac{1}{2}}x - \tfrac{1}{8}a^{-\frac{3}{2}}x^2 + \tfrac{1}{16}a^{-\frac{5}{2}}x^3 - \ldots\ldots$$
$$x^{\frac{1}{2}} + \tfrac{1}{2}x^{-\frac{1}{2}}a - \tfrac{1}{8}x^{-\frac{3}{2}}a^2 + \tfrac{1}{16}x^{-\frac{5}{2}}a^3 - \ldots\ldots$$

In the former we proceed according to positive and increasing integer powers of x; and in the latter, according to negative and decreasing fractional powers of x. Yet both are true developments. Innumerable other instances may be quoted.

on the one side, and the known or given co-efficients on the other. Of these, there will be one for each term which involves a separate power of the symbol according to which the development is made, and therefore as many as there are co-efficients to be determined. There are, hence, as many equations as are required for limiting the co-efficients of the development, and none superfluous. The problem is, therefore, reduced to the determination of these values; and if they shall prove to be *real* and *unambiguous*, the problem is solved, and the development effected in the assumed form: but if these values should be imaginary or ambiguous, the development either cannot be effected at all in that form, or else that form is ambiguous and, therefore, useless.

It is understood all through this reasoning, that the co-efficients of the expansion are to be *real*, as well as the constants of the given function. The contrary hypothesis, so far from invalidating our reasoning, subserves it.

The principle, therefore, as a *principle*, is altogether independent of that more recondite but established principle of equations:—that the roots of an equation are equi-numerous with its dimensions. Still more is it independent of the very illogical principle involved in putting $x=0$ in the expansion itself, which has been almost universally employed.

For some simple examples of our meaning, let us turn to *vol. i. p.* 238,

$$\sqrt{1+x^2} = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \dots\dots;$$

this is unravelled, or the inverse function of $\sqrt{\ }$ taken, by squaring both sides.

Again, *page* 238.

$$\frac{(1-x+x^2)(1+x-x^2)}{1+x+x^2} = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \dots\dots;$$

which is unravelled, or the inverse function taken by multiplying out the numerator (it being artificially converted into factors), and multiplying both sides by the denominator. Both sides are thus converted into series of monomes.

Again, in the proof *of synthetic division*, *p.* 524, the form of the quotient is *assumed;* and the synthetic method itself is only a contrivance for conveniently performing the series of operations implied in the solution of the simple equations of condition.

The same view will be found applicable in all cases whatever, though occasionally (as in the binomial theorem and exponential theorem) some address will be required in the application of the principle. The fundamental principle itself is, however, invariable and universal.

The view we have here taken enables us to dispense with much of the complexity of the reasoning at *pp.* 435, 436, *vol. i.:* since we are not required to show that the sine and cosine cannot be developed in other forms than those there given, but merely to establish that they can be so developed. Moreover, there is a flaw in that reasoning, as far as it attempts to establish the needless part of the proposition.

Scholium 4.

For denoting to the eye that the *limiting value*, or *tendent value* of an expression which is written in a finite form, is meant, we shall enclose that finite value in brackets. For instance, when we pass from $\frac{u'-u}{h}$ to its tendent value, we shall express it thus at *pp.* 352, 353.

$$\left[\frac{u'-u}{h}\right] = \frac{du}{dx} = \mathrm{A}; \text{ and } \left[\frac{e^{x+h}-e^{x}}{(x+h)-x}\right] = \frac{de^{x}}{dx} = e^{x}.$$

and so on with other expressions of like nature. The only object of this is brevity and facility of reading.

CHAPTER I.

THE DIFFERENTIATION OF THE DEVELOPABLE FUNCTIONS OF A SINGLE VARIABLE.

PROP. I.

To differentiate algebraic functions composed of a series of monomes.

The general form is,

$$u = ax^m + bx^{m-1} + cx^{m-2} + \dots$$

in which a, b, c, are of any sign and value, or any of them zero.

Let x receive the increment h, and thereby become x'; whilst, in consequence, u is changed to u'. Then, expanding the terms $a(x+h)^m$, $b(x+h)^{m-1}$, *etc.*, by the binomial theorem, we shall have

$$u' - u = \left\{\frac{m}{1} \cdot ax^{m-1} + \frac{m-1}{1} \cdot bx^{m-2} + \dots\dots\dots\dots\dots\right\} h$$

$$+ \left\{\frac{m(m-1)}{1 \cdot 2} ax^{m-2} + \frac{(m-1)(m-2)}{1 \cdot 2} bx^{m-3} + \dots\right\} h^2$$

$$+ \text{ terms in } h^3, h^4, \dots\dots$$

Divide the first side by $x' - x$, and the second by its value h, and take the *tendent ratio* or *limit;* then

$$\text{A} = \left[\frac{u' - u}{x' - x}\right] = max^{m-1} + (m-1)\, bx^{m-2} + \dots\dots = \frac{du}{dx};$$

or in the form of a differential equation,

$$du = \left\{max^{m-1} + (m-1)\, bx^{m-2} + \dots\right\} dx.$$

It will be seen by reference to *vol. i. p* 208, that the differential co-efficient here obtained is precisely the same thing as the *first derived co-efficient* there employed. The rule given in that place applies, therefore, in this; and, with slight verbal alteration, is annexed.

Multiply each term by the index of x in it, and diminish the index by unity: this being done for each of the monomes composing the expression, and the results multiplied by a, b, c, ..., their sum is the differential co-efficient of the entire function.

Corollary 1.

The absolute term, or constant, is eliminated by differentiation. For it is the same in u' and u, and therefore absent from $u' - u$.

Corollary 2.

The differential co-efficient of a constant is 0, whatever its value may be: for both u' and u are each composed of that constant only, and hence $u' - u = 0$.

These results accord with what was shown to take place in *finite differences, vol. i. p.* 275; and as might, *à priori*, have been expected.

Corollary 3.

As the binomial theorem is true for all values of m, whether positive or negative, integer or fractional, this rule applies to all functions composed of monomial terms in x with constant indices.

Corollary 4.

A constant factor of any function is also a factor of the differential or of the differential co-efficient.

EXAMPLES.

Function.	*Value of* $\frac{du}{dx}$.	*Differential equation.*
$u = a$	0	$du = 0dx = 0.$
$u = mx + c$	m	$du = mdx$
$u = ax^2 + bx + c$	$2ax + b$	$du = (2ax + b)\,dx$
$u = ax^n + c$	nax^{n-1}	$du = nax^{n-1}\,dx$
$u = ax^{\frac{m}{n}} + b$	$\frac{m}{n}\,ax^{\frac{m-n}{n}}$	$du = \frac{m}{n}\,ax^{\frac{m-n}{n}}\,dx$
$u = \frac{1}{x^{\frac{2}{3}}} = x^{-\frac{2}{3}}$	$-\frac{2}{3}x^{-\frac{5}{3}}$	$du = -\frac{2}{3}x^{-\frac{5}{3}}dx$
$u = -bx^n$	$-nbx^{n-1}$	$du = -nbx^{n-1}\,dx$
$u = (a^2 - x^2)^{\frac{3}{2}}$	$-3x\,(a^2 - x^2)^{\frac{1}{2}}$	$du = -3x(a^2 - x^2)^{\frac{1}{2}}dx$
$u = -x^{-\frac{1}{2}}$	$\frac{1}{2}\,x^{-\frac{3}{2}}$	$du = \frac{1}{2}\,.\,x^{-\frac{3}{2}}dx$
$u = \sqrt{x} + \frac{1}{\sqrt{x}}$	$\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$	$du = \frac{(x-1)\,dx}{2x\sqrt{x}}$
$u = a - \sqrt{-x^2}$	$-\sqrt{-1}$	$du = -dx\,.\sqrt{-1}.$

PROP. II.

To differentiate exponential and logarithmic functions of a single variable.

1. *The exponential*, $u = a^x$. In this case we have $u' = a^{x+h}$; and hence

$$u' - u = a^{x+h} - a^x = a^x\,(a^h - 1)$$

$$= a^x\{(1 + \frac{\text{A}}{1}\,h + \frac{\text{A}^2}{1\,.\,2}\,.\,h^2 + \ldots) - 1\}$$

$$= a^x\left\{\frac{\text{A}}{1} + \frac{\text{A}^2 h}{1\,.\,2} + \ \ldots\ldots\right\}h;$$ wherefore (i. p. 252),

$$\frac{du}{dx} = \left[\frac{u'-u}{x'-x}\right] = \text{A}a^x = \log_e a\,.\,a^x;$$ and hence, finally,

$$du = da^x = \log_e a\,.\,a^x dx.$$

2. *The logarithmic*, $u = \log_a x$. From this, $x = a^u$; and by the preceding case we get at once

$$\frac{dx}{du} = \mathrm{A}a^u = \mathrm{A}x\text{; or } \frac{du}{dx} = \frac{1}{\mathrm{A}} \cdot \frac{1}{x} = \frac{1}{x \log_e a} = \frac{\mathrm{M}_a}{x}.$$

$$\text{Wherefore, } du = \frac{dx}{x \log_e a} = \frac{\mathrm{M}_a dx}{x}.$$

EXAMPLES.

1. Let $u = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$; then $\frac{du}{dx} = \sqrt{-1}.\left\{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}\right\}$

2. Let $u = \left\{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}\right\}\sqrt{-1}$: then $\frac{du}{dx} = -\left\{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}\right\}$

3. Let $u = \log_e\{2x + 1 + 2\sqrt{1 + x + x^2}\}$. Then,

$$du = \frac{d\{2x + 1 + 2\sqrt{1 + x + x^2}\}}{2x + 1 + 2\sqrt{1 + x + x^2}} = \frac{dx}{\sqrt{1 + x + x^2}}.$$

4. Let $u = x^{\frac{3}{2}}\left(\frac{1 + x}{1 - x}\right)^{\frac{1}{2}}$ be the equation. Take $\log_e$ of both sides: then

$$\log_e u = \tfrac{3}{2}\log_e x + \tfrac{1}{2}\log_e(1 + x) - \tfrac{1}{2}\log_e(1 - x)\text{; and}$$

$$\frac{du}{u} = \frac{3dx}{2x} + \frac{dx}{2(1 + x)} + \frac{dx}{2(1 - x)} = \frac{(3 + 2x - 3x^2)\,dx}{2x(1 - x^2)}.$$

Hence

$$\frac{du}{dx} = \frac{u(3 + 2x - 3x^2)}{2x(1 - x^2)} = \frac{(3 + 2x - 3x^2)}{2(1 - x)} \cdot \left\{\frac{x}{1 - x^2}\right\}^{\frac{1}{2}}.$$

5. Suppose $u = \log^2 x$*. Put $z = \log x$: then $\log^2 x = \log z = u$. Whence

$$du = d\log z = \frac{dz}{z} = \frac{dx}{xz} = \frac{dx}{x\log x}.$$

6. $u = \log\dfrac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}}$: then $\dfrac{du}{dx} = \dfrac{-a}{x\sqrt{a^2 - x^2}}$.

7. If $u = \log^n x$: then $\dfrac{du}{dx} = \dfrac{1}{\log x \log^2 x \log^3 x \ldots \log^{n-1} x}$.

PROP. III.

To differentiate a function composed of any number of factors, which are, each, a function of the same variable.

Let the function be $u = zyvw\ \ldots$, the factors z, y, v, w, being all functions of the same variable x.

Take the $\log_e$ of both sides: then

$$\log u = \log z + \log y + \log v + \log w + \ldots\ ;$$

and differentiating we have, after multiplying by u

$$du = \left\{\frac{dz}{z} + \frac{dy}{y} + \frac{dv}{v} + \frac{dw}{w} + \ldots\right\}.u$$

$$= (yvw\ldots)\,dz + (vwz\ldots)\,dy + (wzy\ldots)\,dv + (zyv\ldots)\,dw + \ldots$$

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* Here $\log^2 x$ is the notation for what is read the logarithm of the logarithm of x; $\log^3 x$ is the logarithm of the last mentioned logarithm, and so on.

That is[*], the differential of the function is composed of the sum formed by multiplying the differential of each factor into all the other factors.

EXAMPLES.

1. Let $u = (2a^2 + 3u^2)(a^2 - x^2)^{\frac{3}{2}}$ be the given function; in which

$$v = 2a^2 + 3x^2, \text{ and } y = (a^2 - x^2)^{\frac{3}{2}}.$$

$$\text{Now } dv = 6x dx, \text{ and } dy = -3x(a^2 - x^2)^{\frac{1}{2}} dx; \text{ whence}$$

$$du = v dy + y dv = (a^2 - x^2)^{\frac{3}{2}} . 6x dx - (2a^2 + 3x^2) . 3x(a^2 - x^2)^{\frac{1}{2}} dx$$
$$= -15x^3 (a^2 - x^2)^{\frac{1}{2}} dx.$$

2. Let $u = (a - x)^{\frac{1}{2}} (b - x)^{\frac{1}{2}} (c - x)^{\frac{1}{2}} = yzw$. Then

$$du = zw dy + wy dz + yz dw \text{ is the general form.}$$

Also, $dy = \dfrac{-dx}{2(a-x)^{\frac{1}{2}}}$, $dz = \dfrac{-dx}{2(b-x)^{\frac{1}{2}}}$, and $dw = \dfrac{-dx}{2(c-x)^{\frac{1}{2}}}$;

whence, dividing by dx, we get the differential co-efficient

$$\frac{du}{dx} = -\frac{1}{2}\left\{\sqrt{\frac{(b-x)(c-x)}{a-x}}\right) + \sqrt{\frac{(c-x)(a-x)}{b-x}} + \sqrt{\frac{(a-x)(b-x)}{c-x}}\right\}.$$

3. Let $u = x^x$: then $\log u = x \log x$, and taking the differentials,

$$d \log u = \log x . dx + x . d \log x,$$

or
$$du = u\{1 + \log x\} dx = x^x \{1 + \log x\} dx.$$

4. Let $u = \log \{(a + x)^m (a_1 + x)^{m_1} (a_2 + x)^{m_2} \ldots\}$ then

$$\frac{du}{dx} = \frac{m}{a + x} + \frac{m_1}{a_1 + x} + \frac{m_2}{a_2 + x} + \ldots$$

5. Let $u = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4) \ldots (1 + x^n)$, to find $\dfrac{du}{dx}$.

6. Let $u = (1 + x)(1 + 2x)(1 + 3x) \ldots\ldots (1 + nx)$, to find $\dfrac{du}{dx}$.

7. Let $u = a_1^{x_1} a_2^{x_2} a_3^{x_3} a_4^{x_4} \ldots\ldots a_n^{x_n}$, to find $\dfrac{du}{dx}$; $x_1, x_2, \ldots$ being given functions of x.

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[*] This might also have been obtained from first principles in the following manner:
Let $u = vy$ be a function of x, in which the factors v, y, are both functions of x; and let h b the increment of x: then

$$v' = v + Ah + Bh^2 + \ldots$$
$$y' = y + A'h + B'h^2 + \ldots$$

Then,

$$u' - u = (v + Ah + Bh^2 + \ldots)(y + A'h + B'h^2 + \ldots) - vy$$
$$= (Ay + A'v) h + \text{terms in } h^2, h^3, \ldots$$

Also, in the separate expansions $A = \dfrac{dv}{dx}$, $A' = \dfrac{dy}{dx}$; hence,

$$\left[\frac{u' - u}{x' - x}\right] = \frac{du}{dx} = Ay + A'v = \frac{y dv}{dx} + \frac{v dy}{dx}.$$

and the differential equation becomes

$$du = y dv + v dy.$$

We might proceed in the same manner when there are more factors, as z, w, *etc.*

PROP. IV.

To differentiate a fraction, the numerator and denominator being functions of one variable.

Let it be $u = \frac{y}{z}$ where y and z are functions of x. This may be written $u = yz^{-1}$; and the problem is reduced to the preceding one. We have, therefore, $du = z^{-1}dy - yz^{-2}dz = \frac{dy}{z} \cdot - \frac{ydz}{z^2} = \frac{zdy - ydz}{z^2}$.

This formula gives rise to the following verbal statement of the rule for solution :—

Multiply the differential of the numerator by the denominator, and then the differential of the denominator by the numerator: then the latter product subtracted from the former, and the difference divided by the square of the denominator is the differential required.

EXAMPLES.

1. Let $u = \frac{ax}{a^2 + x^2}$: then $du = \frac{(a^2 + x^2).\, d\,(ax) - ax.\, d\,(a^2 + x^2)}{(a^2 + x^2)^2}$,

and performing the differentiations and reductions we get the required result,

$$\frac{du}{dx} = \frac{a\,(a^2 - x^2)}{(a^2 + x^2)^2}.$$

2. If $u = \frac{\sqrt[3]{1 + x^2}}{\sqrt{1 + x}}$: then $\frac{du}{dx} = \frac{x^2 + 4x - 3}{6\,(1 + x)^{\frac{3}{2}}\,(1 + x^2)^{\frac{2}{3}}}$.

3. If $u = \frac{x}{x + \sqrt{1 - x^2}}$: then $\frac{du}{dx} = \frac{1}{\{x + \sqrt{1 - x^2}\}^2 \cdot \sqrt{1 - x^2}}$.

4. If $u = \log_e \left\{\frac{x}{x + \sqrt{1 + x^2}}\right\}$: then $\frac{du}{dx} = \frac{1}{x} - \frac{1}{\sqrt{1 + x^2}}$.

5. If $u = x^{\frac{1}{x}}$: then $\frac{du}{dx} = (1 - \log_e x)\; x^{\frac{1-2x}{x}}$.

6. If $u = e^x \left(\frac{1 + x}{1 - x}\right)^{\frac{1}{2}}$: then $\frac{du}{dx} = \frac{(2 - x^2)e^x}{(1 - x)\,\sqrt{1 - x^2}}$.

PROP. V.

If a series of expressions be so related that the first is a function of the variable in the second, the second a function of the variable in the third, the third a function of the variable in the fourth, and so on: it is required to differentiate the first function as one of the last variable.

Let, for instance, u be a function of y, and y a function of x; it is required to find $\frac{du}{dx}$.

Since y is a function of x, we have

$$y' - y = \mathrm{A}h + \mathrm{B}h^2 + \mathrm{C}h^3 + \ldots. = k, \text{ suppose.}$$

Also, since u is a function of y, and $y' = y + k$, we get

$$\begin{aligned} u' - u &= \mathrm{A}'k + \mathrm{B}'k^2 + \mathrm{C}'k^3 + \ldots \\ &= \mathrm{A}'(\mathrm{A}h + \mathrm{B}h^2 + \ldots) + \mathrm{B}'(\mathrm{A}h + \mathrm{B}h^2 + \ldots)^2 + \ldots. \\ &= \mathrm{A}'\mathrm{A}h + \mathrm{A}'\mathrm{B}h^2 + \ldots + \mathrm{B}'\mathrm{A}^2h^2 + \ldots. + \ldots \end{aligned}$$

Hence, $\left[\frac{u'-u}{h}\right]=\frac{du}{dx}=\text{A}'\text{A}$. But $\text{A}'=\frac{du}{dy}$, and $\text{A}=\frac{dy}{dx}$;

$$\text{wherefore, } \frac{du}{dx}=\frac{du}{dy}\cdot\frac{dy}{dx}.$$

Were there any number of such dependent equations and corresponding variables, we should, by following out an analogous course of investigation, arrive at the general result,

$$\frac{du}{dx}=\frac{du}{dv}\cdot\frac{dv}{dz}\cdot\frac{dz}{dy}\;\ldots\ldots\ldots\ldots\frac{dy}{dx}.$$

EXAMPLES.

1. Let $u=(a+bx^n)^m$; then $y=a+bx^n$, and $u=y^m$.

$$\text{But } \frac{du}{dy}=my^{m-1}, \text{ and } \frac{dy}{dx}=nbx^{n-1}; \text{ whence,}$$

$$\frac{du}{dx}=\frac{du}{dy}\cdot\frac{dy}{dx}=my^{m-1}\times nbx^{n-1}=mnbx^{n-1}(a+bx^n)^{m-1}.$$

2. Let $u=\sqrt{x+\sqrt{1+x^2}}=\{x+(1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}$; then,

$$\frac{du}{dx}=\frac{\sqrt{x+\sqrt{1+x^2}}}{2\sqrt{1+x^2}}.$$

3. If $u=e^{x^n}$; then $\frac{du}{dx}=nx^{n-1}e^{x^n}$.

4. If $u=\log^n x$; then $\frac{du}{dx}=\frac{1}{x\log x\log^2 x\log^3 x\ldots\ldots\log^{n-1}x}$.

5. If $u=\log_e\frac{\sqrt{1+x^2}+x\sqrt{2}}{\sqrt{1-x^3}}$; then $\frac{du}{dx}=\frac{\sqrt{2}}{(1-x^2)\sqrt{1+x^2}}$.

6. If $ye^{ny}=ax^m$ be given; then $\frac{dy}{dx}=\frac{ma^{\frac{1}{m}}y^{\frac{m-1}{m}}}{(y+n)e^{\frac{ny}{m}}}$.

7. If $u^2=\frac{a^2+x^2}{a^2-x^2}$; then $\frac{du}{dx}=\frac{2a^2x}{(a^2-x^2)\sqrt{a^4-x^4}}$.

PROP. VI.

To differentiate an implicit algebraic function of two variables.

1. When the equation can be resolved for one of the variables, the expression to be differentiated will take one of the preceding forms; and the solution may be effected by the rules already laid down.

2. It will rarely be the case that the rule just given can be applied to the actual cases which occur; and even in most cases where it can be so, the solution obtained by such means will not be that which is most conducive to the successful investigation of the problems to which we shall have occasion to apply it. The following very obvious rule applies to all cases:—

Reduce the function to the simplest algebraic form, by the elimination of denominators and radicals; and equate the whole of the terms to zero or to a constant. By differentiating this expression, we shall have an equation containing the first powers of the differentials of the two variables, each multiplied by certain functions of those variables; and the whole equated to zero, (the differential of zero or of the constant,) gives an expression from which $\frac{du}{dx}$ can be obtained in functions of u and x.

It is to be remarked, however, that the differential co-efficient will generally be a function of both variables in the result. When, therefore, it is essential to give the result in terms of either of the variables above, it will also be essential to resolve the given equation for the other variable, and substitute its value in the expression for the differential co-efficient spoken of above.

Scholium.

If we differentiate the function $u = 0$ separately, as functions of y and x, considering in the two cases respectively x and y as constants, the differential coefficient will be

$$\frac{dy}{dx} = -\frac{dy}{du} \cdot \frac{du}{dx}.$$

This is derived from the consideration that y is a function of u, and u a function of x; precisely as in *prop. v. p.* 360. The employment of this formula is, however, often attended with more trouble than the principle laid down above, on account of its generally requiring additional reductions. The student will, however, do well to familiarise himself with it, on account of its uses hereafter.

Another proof of this will be given by means of *Taylor's theorem*, a few pages further on. See *page* 373.

EXAMPLES.

1. Let $u^2 - 2ux + a^2 = 0$ be an implicit function of u and x. Then differentiating we have,

$$2udu - 2udx - 2xdu = 0, \text{ and hence}$$

$$\frac{du}{dx} = \frac{u}{u - x}, \quad \text{or} \quad \frac{dx}{du} = \frac{u - x}{u};$$

which is an implicit function of u and x.

If, however, we require to have it expressed as a function of u or x solely, we must resolve the given equation. From this we have

$$\frac{du}{dx} = \frac{x \pm \sqrt{x^2 - a^2}}{\pm \sqrt{x^2 - a^2}}, \quad \text{or} \quad \frac{dx}{du} = \frac{u^2 - a^2}{2u^2}.$$

Now it will depend upon the nature and conditions of the problem, which of these variables it will be best to consider the independent one; but, *primâ facie*, it is obvious that the latter is the preferable one from its greater simplicity. Circumstances, however, must, in all cases, decide which should be taken, as the respective values of the two will depend upon the ulterior operations to be performed.

2. Let $x \log y - y \log x = 0$. Then, differentiating, we have

$$\log y \,.\, dx - y \,.\, \frac{dx}{x} + x \,.\, \frac{dy}{y} - \log x \,.\, dy = 0, \text{ or}$$

$$\frac{dy}{dx} = \frac{y\,(y - x \log y)}{x\,(x - y \log x)}.$$

3. If $y^n \log y = ax$; then $\dfrac{dy}{dx} = \dfrac{a}{(1 + n \log y)\, y^{n-1}}$.

4. If $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$; then $\dfrac{dy}{dx} = \dfrac{x\{a^2 - 2\,(x^2 + y^2)\}}{y\{b^2 + 2\,(x^2 + y^2)\}}$

5. If $x\sqrt{1 + y} + y\sqrt{1 + x} = 0$; then $\dfrac{dy}{dx} = \dfrac{y}{x} \cdot \dfrac{y + 2\sqrt{(1 + x)\,(1 + y)}}{x + 2\sqrt{(1 + x)\,(1 + y)}}$.

6. If $u = \sqrt{ax + \sqrt{bx + \sqrt{ax + \,.\, ad\ inf}}}$; or $u^4 - 2au^2x - u + a^2x^2 - bx = 0$; then $\dfrac{du}{dx} = \dfrac{2au^2 - 2a^2x + b}{4u^3 - 4aux - 1}$.

PROP. VII.

The tendent ratios *of an arc to its sine and to its tangent, are ratios of equality, the arc being diminished sine limite.*

For, recurring to *vol. i. p.* 436, we have generally,

$$\frac{\sin h}{h} = 1 - \frac{h^2}{1\,.\,2\,.\,3} + \frac{h^4}{1\,.\,2\,.\,3\,.\,4\,.\,5} - \ldots.$$

$$\frac{\tan h}{h} = 1 + \frac{2h^2}{1\,.\,2\,.\,3} + \frac{16h^4}{1\,.\,2\,.\,3\,.\,4\,.\,5} + \ldots.$$

Now as the values of h in these equations are diminished, *sine limite*, the values of the fractions on the left side, or of the tendent ratios, are in both cases those of equality. That is,

$$\left[\frac{\sin h}{h}\right] = 1, \text{ and } \left[\frac{\tan h}{h}\right] = 1.$$

Corollary.

$$[\cos h] = 1\,; \text{ for} \left[\frac{\sin h}{h}\right] = 1, \text{ and } \left[\frac{\tan h}{h}\right] = 1\,; \text{ wherefore,}$$

$$[\cos h] = \left[\frac{\sin h}{\tan h}\right] = \frac{1}{1} = 1.$$

PROP. VIII.

To differentiate the sine, cosine, tangent, etc., of an arc, as a function of the arc.

(1). $u' - u = \sin(x + h) - \sin x = 2\cos(x + \frac{1}{2}h)\sin\frac{1}{2}h$; or

$$\frac{u' - u}{h} = \cos(x + \tfrac{1}{2}h)\,\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}.$$

In the limit the last factor is unity (*prop. vii.*); and

$$\frac{du}{dx} = \cos x, \text{ or } du = \cos x\,dx.$$

(2). $u' - u = \cos(x + h) - \cos x = -\sin(x + \frac{1}{2}h)\sin\frac{1}{2}h$;
whence, as in the preceding case,

$$\frac{du}{dx} = -\sin x\,; \text{ or } du = -\sin x\,dx.$$

We may deduce the whole of the other functions by corresponding processes: but they will be more simply obtained after the manner of the tangent in the next example.

(3). $\tan x = \dfrac{\sin x}{\cos x}$. Differentiate this as a fraction (*prop. iv. p.* 360.); then

$$du = \frac{\cos x\ d\sin x - \sin x\ d\cos x}{\cos^2 x} = \frac{(\cos^2 x + \sin^2 x)\,dx}{\cos^2 x} = \sec^2 x\,dx.$$

We shall leave the investigation of the remaining functions for the student's exercise; and merely tabulate the results as follows.

Scholium.

The preceding investigation supposes the radius to be unity, a circumstance of comparative unfrequent occurrence in those researches to which the calculus is applied. The radius a being introduced to render the expressions homogeneous, will convert the expressions obtained into the required form, precisely as in ordinary trigonometry. In order to this, it is requisite to keep in mind that $\frac{x}{a}$ is to be written for x; and the table referred to in the proposition, may be written as follows:

$d \sin x = \frac{1}{a} \cos x \, dx$	$d \cos x = -\frac{1}{a} \sin x \, dx$
$d \tan x = \frac{1}{a^2} \sec^2 x \, dx$	$d \cot x = -\frac{1}{a^2} \operatorname{cosec}^2 x \, dx$
$d \sec x = \frac{1}{a^2} \tan x \sec x \, dx$	$d \operatorname{cosec} x = -\frac{1}{a^2} \cot x \operatorname{cosec} x \, dx$
$d \operatorname{vers} x = \frac{1}{a} \sin x \, dx$	$d \operatorname{covrs} x = -\frac{1}{a} \cos x \, dx.$

When the radius is unity, we have only to omit the factors $\frac{1}{a}$ and $\frac{1}{a^2}$ from these expressions.

EXAMPLES.

1. Let $u = \log_e \tan x$. Then we have (*prop. ii.*),

$$\frac{du}{dx} = \frac{d \tan x}{\tan x} = \frac{\sec^2 x}{\tan x} = \frac{1}{\sin x \cos x} = \frac{2}{\sin 2x}.$$

2. Let $u = \log_e (\cos x + \sin x \sqrt{-1})$: then as before

$$\frac{du}{dx} = \frac{d(\cos x + \sin x \sqrt{-1})}{\cos x + \sin x \sqrt{-1}} = \frac{-\sin x + \cos x \sqrt{-1}}{\cos x + \sin x \sqrt{-1}} = \sqrt{-1}.$$

3. If $u = \sin (\log x)$: then $\frac{du}{dx} = \frac{\cos (\log x)}{x}$.

4. If $u = e^x \cos x$: then $\frac{du}{dx} = e^x (\cos x - \sin x)$.

5. If $u = \cos x \, . \, e^{\sin x}$: then $\frac{du}{dx} = (1 - \sin x - \sin^2 x) e^{\sin x}$.

6. If $u = \log \sqrt{\sin x} + \log \sqrt{\cos x}$: then $\frac{du}{dx} = \cot 2x$.

7. If $u = a(z + \sin z)$ and $x = a(1 - \cos z)$: then $\frac{du}{dx} = \frac{\sqrt{2ax - x^2}}{x}$.

8. If $u = a(\sin x \pm \cos x)$: then $\frac{du}{dx} = a \sqrt{1 \mp \sin 2x}$.

PROP. IX.

To differentiate an arc as a function of its trigonometrical functions.

THIS case belongs to the developable functions; as since we have the arc in terms of its sine, cosine, *etc.*, we can revert the series (*vol. i. p.* 272), and thus obtain the arc in terms of its trigonometrical functions. Availing ourselves, however, of the results of the preceding proposition, we can obtain the required differentials in a more brief and elegant manner.

Let $u = \sin^{-1} x$: then $x = \sin u$; and by the preceding proposition we have $\frac{dx}{du} = \cos u$, or $du = \frac{dx}{\cos u} = \frac{dx}{\cos \sin^{-1} x} = \frac{dx}{\sqrt{1-x^2}}$.

If this be required to radius a instead of unity, we have only to render the terms of the numerator and denominator homogeneous. This changes the preceding into

$$\frac{du}{dx} = \frac{a}{\sqrt{a^2-x^2}}.$$

The other functions are left for the student's exercise, and the results put down.

$d \sin^{-1} x = \frac{adx}{\sqrt{a^2-x^2}}$	$d \cos^{-1} x = -\frac{adx}{\sqrt{a^2-x^2}}$
$d \tan^{-1} x = \frac{a^2dx}{a^2+x^2}$	$d \cot^{-1} x = -\frac{a^2dx}{a^2+x^2}$
$d \sec^{-1} x = \frac{a^2dx}{x\sqrt{x^2-a^2}}$	$d \operatorname{cosec}^{-1} x = -\frac{a^2dx}{x\sqrt{x^2-a^2}}$
$d \operatorname{vers}^{-1} x = \frac{adx}{\sqrt{2ax-x^2}}$	$d \operatorname{covrs}^{-1} x = -\frac{adx}{\sqrt{2ax-x^2}}$

When the radius is unity, we have only to drop the a and a^2 as factors of these expressions.

EXAMPLES.

1. Let $u = \sin^{-1} \frac{x}{y}$. Then $\frac{x}{y} = \sin u$; and differentiating,

$$\frac{ydx - xdy}{y^2} = \cos u \, du = du \, . \sqrt{1 - \frac{x^2}{y^2}}; \text{ and reducing}$$

$$du = \frac{ydx - xdy}{y\sqrt{y^2-x^2}}.$$

2. If $u = \tan^{-1} \frac{x}{y}$: then $du = \frac{ydx - xdy}{x^2+y^2}$.

3. If $u = \sin^{-1} (2x\sqrt{1-x^2})$ then $du = \frac{2dx}{\sqrt{1-x^2}}$.

4. If $u = xe^{\tan^{-1} x}$; then $\frac{du}{dx} = \left(\frac{1+x+x^2}{1+x^2}\right) e^{\tan^{-1} x}$.

5. If $u = \tan^{-1} \frac{2x}{1-x^2}$: then $\frac{du}{dx} = \frac{2}{1+x^2}$.

6. If $u - \sin^{-1} x = \sqrt{1-x^2}$: then $\frac{du}{dx} = \sqrt{\frac{1-x}{1+x}}$.

7. If $y = \cos^{-1} \sqrt{\frac{b^2-x^2}{a^2-x^2}}$: then $\frac{dy}{dx} = \frac{x}{a^2-x^2} \cdot \sqrt{\frac{a^2-b^2}{b^2-x^2}}$.

8. If $\sin^{-1} \frac{x}{h} + \sin^{-1} \frac{y}{k} = c$: then $\frac{dy}{dx} = -\sqrt{\frac{k^2-y^2}{h^2-x^2}}$.

9. If $\log^{-1}\left(u - \sec^{-1} \frac{x}{a}\right) = x + \sqrt{x^2-a^2}$: then $\frac{du}{dx} = \frac{1}{x}\left(\frac{x+a}{x-a}\right)^{\frac{1}{2}}$.

CHAPTER II.

ON SUCCESSIVE DIFFERENTIATION, AND THEOREMS DEDUCED BY MEANS OF IT.

The differential co-efficient of any function may be, and generally is, a function of the variables concerned. This, therefore, viewed as a new function of those variables, may be differentiated, and its differential co-efficient assigned in the same manner that the first was found. In like manner a third differential co-efficient may be found from the second, a fourth from the third, and so on, as long as the successive differential co-efficients contain either one or more variables.

The actual differentiation itself is easily and directly performed by means of the rules already given. We shall take one or two instances.

(1). Let $u = ax^3$: then,

1st. Diff. co-eff. .... $= 3ax^2$

2nd. .... $= 6ax$

3rd. diff. co-eff. .... $= 6a$

4th. .... $= 0$

and all succeeding ones $= 0$.

(2). Let $u = a^x$: then we have

1st. diff. co-eff. $= \log_e a \, . \, a^x$

2nd. .... $= (\log_e a)^2 \, a^x$

3rd. diff. co-eff. $= (\log_e a)^3 \, a^x$

and so on.

(3). Let $y = \log_e x$; then the co-efficients are

1st diff. co-eff. $= + \frac{1}{x}$

2nd .... $= - \frac{1}{x^2}$

3rd .... $= + \frac{1 \, . \, 2}{x^3}$

4th diff. co-eff. $= - \frac{1 \, . \, 2 \, . \, 3}{x^4}$

5th .... $= + \frac{1 \, . \, 2 \, . \, 3 \quad 4}{x^5}$

and so on.

(4). Let $y = \sin x$; then the co-efficients are

1st diff. co-eff. $= + \cos x$

2nd .... $= - \sin x$

3rd .... $= - \cos x$

4th diff. co-eff. $= + \sin x$

5th .... $= + \cos x$

and so on.

To render this view of the subject useful, we must fix upon a notation for the successive differential co-efficients. Moreover, as our present employment of it is only where there is either one explicit function of a principal variable or an implicit function of two variables, we are at liberty, consistent with our general principles, to consider the differential of either of these variables as constant. Under this condition we shall take dx as the constant.

Moreover, let d^2u denote $d\,(du)$, d^3u denote $d\,(d^2u)$, and so on: also let the second, third, *etc.*, powers of dx be written dx^2, dx^3, *etc.* Then,

$$d\left(\frac{du}{dx}\right)=\frac{dx \, . \, d^2u-du \, . \, d^2x}{dx^2}=\frac{d^2u}{dx}\text{; or }\frac{d\left(\frac{du}{dx}\right)}{dx}=\frac{d^2u}{dx^2}=\text{2nd diff. co-eff.}$$

By proceeding similarly, we get for the succeeding differential co-efficients,

$$\frac{d^3u}{dx^3},\ \frac{d^4u}{dx^4},\ \textit{etc.}$$

EXAMPLES FOR EXERCISE.

1. If $u = x^n$: then $\frac{d^ru}{dx^r} = n\,(n-1) \ldots\ldots (n - \overline{r-1})\, x^{n-r}$.

2. If $u = x^{-n}$: then $\frac{d^ru}{dx^r} = (-1)^r n\,(n+1) \ \ldots (n + \overline{r-1})\, x^{-(n+r)}$.

3. If $u = a^x$: then $\frac{d^ru}{dx^r} = (\log_e a)^r a^x$.

4. If $u = \sin nx$: then $\frac{d^ru}{dx^r} = n^r \sin\,(nx + \frac{1}{2} r\pi)$.

5. If $u = \cos nx$: then $\frac{d^ru}{dx^r} = n^r \cos\,(nx + \frac{1}{2}\, r\pi)$.

6. If $u = \log x$: then $\frac{d^ru}{dx^r} = (-1)^{r-1}\,(r-1)\,(r-2) \ldots\ldots 3 \,.\, 2 \,.\, 1 \,.\, x^{-r}$.

7. If $u = \frac{1+x}{1-x}$: then $\frac{d^ru}{dx^r} = 2\,(1-x)^{-(r+1)}\, r\,(r-1)\,(r-2) \ldots 3 \,.\, 2 \,.\, 1$.

8. If $u = e^{nx}$: then $\frac{d^ru}{dx^r} = n^r \,.\, e^{nx}$.

9. If $u = \cos mx$: find the value of $\frac{d^nu}{dx^n}$, and assign the law of the signs + and – in the several successive differential co-efficients.

10. If $u = e^{x\cos\theta} \cos\,(x \sin\theta)$: then $\frac{d^nu}{dx^n} = e^{x\cos\theta} \cos\,(x\sin\theta + n\theta)$.

11. If $u = e^{ax} \cos nx$; then $\frac{d^mu}{dx^m} = (a^2 + n^2)^{\frac{m}{2}}\, e^{ax} \cos\,\left(nx + m \tan^{-1}\frac{n}{a}\right)$.

For further discussions and methods relative to successive differentiation, Mr. Gregory's valuable *Examples in the Differential and Integral Calculus* is strongly recommended. Also, for an excellent praxis, Hind's *Digested Series of Examples on the Differential Calculus*.

PROP. I.

If there be a developable function of the sum of two quantities given, one only of which is supposed to vary at the same time; then the respective differential co-efficients on each supposition as to which is the variable quantity will be equal.

Let $f\,(x+y) = 0$ be the given function; then, supposing x constant, $\frac{df\,(x+y)}{dy}$, $\frac{d^2f\,(x+y)}{dy^2}$, *etc.*, will be the same as $\frac{df\,(x+y)}{dx}$, $\frac{d^2f\,(x+y)}{dx^2}$, *etc.* when y is considered constant.

For, since $f(x+y), =f(y+x)$, we have

$$\frac{df(x+y)}{d(x+y)} = \frac{df(y+x)}{d(y+x)}.$$

Consider in the two sides of this equation, the former letter to be constant; then, since the quantities are the same in both cases, their developments are the same, and hence their second terms are the same, except the interchange of y and x; also, the differentials of the denominators are dy and dx. Whence

$$\frac{df(x+y)}{dy} = \frac{df(y+x)}{dx} = \frac{df(x+y)}{dx}.$$

It is readily shown that the successive differential co-efficients must follow the same law. Indeed, the entire principle is almost an axiom.

For put $df(x+y) = u$: then, as just found, $\frac{du}{dx} = \frac{du}{dy}$; whence,

$$\frac{d\left(\frac{du}{dx}\right)}{dx} = \frac{d^2u}{dx^2}, \text{ and } \frac{d\left(\frac{du}{dy}\right)}{dy} = \frac{d^2u}{dy^2}; \text{ or } \frac{d^2u}{dx^2} = \frac{d^2u}{dy^2}; \text{ and so on.}$$

PROP. II.

To prove Taylor's theorem; viz., that if $u = f(x)$, then, for all developable functions we shall have,

$$u' = f(x+h) = u + \frac{du}{dx} \cdot \frac{h}{1} + \frac{d^2u}{dx^2} \cdot \frac{h^2}{1.2} + \frac{d^3u}{dx^3} \cdot \frac{h^3}{1.2.3} + \ldots\ldots$$

Let us assume the series in which u, A, B, C, ... are functions of only x, and constants, or independent of h.

$$u' = u + Ah + Bh^2 + Ch^3 + \ldots\ldots$$

Then differentiating both sides, both as functions of x and h we have the series

$$\frac{du'}{dx} = \frac{du}{dx} + \frac{dA}{dx}.h + \frac{dB}{dx}.h^2 + \frac{dC}{dx}.h^3 + \ldots\ldots$$

$$\frac{du'}{dh} = A + 2Bh + 3Ch^2 + 4Dh^3 + \ldots\ldots$$

Also by the preceding, $\frac{du'}{dx} = \frac{du'}{dh}$, since $u' = f(x+h)$, and we have supposed each of the quantities, x and h respectively to be the variable. Now, as these two series are identical, we may equate the homologous co-efficients of h. We thus obtain,

$$A = \frac{du}{dx}$$

$$B = \frac{1}{3} \cdot \frac{dA}{dx} = \frac{1}{2} \cdot \frac{d\left(\frac{du}{dx}\right)}{dx} = \frac{d^2u}{dx^2} \cdot \frac{1}{2}$$

$$C = \frac{1}{3} \cdot \frac{dB}{dx} = \frac{1}{3} \cdot \frac{d\left(\frac{d^2u}{dx^2}\right)}{dx} = \frac{d^3u}{dx^3} \cdot \frac{1}{1.2.3}.$$

and so on.

Substituting these values of the co-efficients A, B, C, *etc.*, in the assumed series, we get the proposition above, known as *Taylor's theorem*.

PROP. III.

To investigate Maclaurin's theorem : viz.

If $y = f(x)$ be a function of x, which admits of development in ascending integer powers of x, then it will be expressed by

$$y = U_0 + U_1 . \frac{x}{1} + U_2 . \frac{x^2}{1 . 2} + U_3 . \frac{x^3}{1 . 2 . 3} + \ldots\ldots$$

where U_0, U_1, U_2, *etc.*, denote the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, *etc.*, when x is put equal to zero.

Let $y = A + Bx + Cx^2 + Dx^3 + \ldots.$; then

$$\frac{dy}{dx} = B + 2Cx + 3Dx^2 + \ldots.$$

$$\frac{d^2y}{dx^2} = 2C + 2.3.Dx + \ldots.$$

$$\frac{d^3y}{dx^3} = 2.3.D + \ldots\ldots \text{and so on.}$$

Let us now make $x = 0$ in these values of the co-efficients: whence

$$U_0 = A, \qquad U_2 = 2C, \text{ or } C = \frac{1}{2} . U_2,$$

$$U_1 = B, \qquad U_3 = 2.3.D, \text{ or } D = \frac{1}{2.3} . U_3, \textit{ etc.}$$

Proceeding thus to any extent, we find the succeeding co-efficients to be as stated in the theorem.

Scholium.

Professor Young (*Diff. Calc. p.* 36.) has deduced from this, a very elegant and valuable rule for the ready development of a function, when we know that of its first differential co-efficient. It is as follows :—

Develope $\frac{dy}{dx}$ in a series of ascending powers of x; increase the exponent of each power of x by unity, and divide by the exponent thus increased.

EXAMPLES ON DEVELOPMENT, BY TAYLOR'S AND MACLAURIN'S THEOREMS.

It will frequently happen that we have functions to develope in which the employment of such methods as have already been explained (indeterminate co efficients mainly), would lead to intermediate expressions of great complexity: whereas, by these theorems, the results can be obtained with great readiness and simplicity.

1. To find sin h and cos h in terms of h by *Taylor's theorem.* Put $y = \sin x$: then

$$\frac{dy}{dx} = \cos x, \frac{d^2y}{dx^2} = - \sin x, \frac{d^3y}{dx^3} = - \cos x, \text{ and so on.}$$

Then the development of sin $(x + h)$ being expressed by Taylor's theorem, by giving the values above to the differential co-efficients, we get

$$\sin (x + h) = \sin x + \cos x . \frac{h}{1} - \sin x . \frac{h^2}{1 . 2} - \cos x . \frac{h^3}{1 . 2 . 3} + \ldots\ldots..$$

$$= \sin x \left(1 - \frac{h^2}{1 . 2} \ldots.\right) + \cos x \left(h - \frac{h^3}{1 . 2 . 3} + \ldots\ldots\right)$$

But $\sin(x+h) = \sin x \cos h + \cos x \sin h$: whence also

$$\sin h = h - \frac{h^3}{1.2.3} + \dots \quad \text{and} \quad \cos h = 1 - \frac{h^2}{1.2} + \dots\dots$$

as determined in *vol. i. p.* 435.

2. To develop $\tan^{-1}(x+h)$ in trigonometrical functions of x and h.

Put $y = \tan^{-1} x$: then, $\frac{dy}{dx} = \cos^2 y$,

$$\frac{d^2y}{dx^2} = -2\sin y \cos y \frac{dy}{dx} = -\sin 2y \cos^2 y,$$

$$\frac{d^3y}{dx^3} = -2(\cos 2y \cos^2 y - \sin 2y \sin y \cos y)\frac{dy}{dx} = -2\cos 3y \cos^3 y$$

$$\frac{d^4y}{dx^4} = +2.3(\sin 3y \cos^3 y + \cos 3y \sin y \cos^2 y)\frac{dy}{dx} = 2.3 \sin 4y \cos^4 y$$

$$\frac{d^5y}{dx^5} = +2.3.4(\cos 4y \cos^4 y - \sin 4y \sin y \cos^3 y)\frac{dy}{dx} = 2.3.4 \cos 5y \cos^5 y$$

etc. *etc.*

wherefore,

$$\tan^{-1}(x+h) = y + \cos^2 y \,.\, h - \tfrac{1}{2}\sin 2y \cos^2 y \,.\, h^2 - \tfrac{1}{3}\cos 3y \cos^3 y \,.\, h^3$$
$$+ \tfrac{1}{4}\sin 4y \cos^4 y \,.\, h^4 + \tfrac{1}{5}\cos 5y \cos^5 y \,.\, h^5 - \dots\dots$$

which is the development required.

If $y = \tan^{-1} x = 0$, then $x = 0$; and this is converted into

$$\tan^{-1} h = h - \tfrac{1}{3}h^3 + \tfrac{1}{5}h^5 - \tfrac{1}{7}h^7 + \dots\dots$$

If, moreover, we take $h = 1$, then $\tan^{-1} h = 45^\circ$, and we get Machin's series for the length of the eighth part of the circumference (*vol. i. p.* 475, *note*).

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\dots\dots$$

3. To develop $\tan^{-1} x$ by Maclaurin's theorem.

$$\begin{array}{ll|l}
\text{Put } y & = \tan^{-1} x & U_0 = 0; \\
\frac{dy}{dx} & = \frac{1}{1+x^2} & U_1 = 1; \\
\frac{d^2y}{dx^2} & = -\frac{2x}{(1+x^2)^2} & U_2 = 0; \\
\frac{d^3y}{dx^3} & = -\frac{2}{(1+x^2)^2} + \frac{2.4x^2}{(1+x^2)^3} & U_3 = -2; \\
\frac{d^4y}{dx^4} & = \frac{2^3 x}{(1+x^2)^3} + \frac{2^4.x}{(1+x^2)^3} - \frac{2^4.3.x^3}{(1+x^2)^4} & U_4 = 0; \\
\frac{d^5y}{dx^5} & = \frac{2^3.3}{(1+x^2)^3} - \frac{2^5\ 3^2x^2}{(1+x^2)^4} + \frac{2^7.3x^4}{(1+x^2)} & U_5 = 2^3.3.\text{etc.}
\end{array}$$

whence putting $\tan y$ for x, and these values of the co-efficients in Maclaurin's series, we get as above.

$$y = \tan y - \tfrac{1}{3}\tan^3 y + \tfrac{1}{5}\tan^5 y - \dots\dots$$

4. To find the same development by Young's rule *p.* 368.

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \tan^0 y - \tan^2 y + \tan^4 y - \tan^6 y + \dots$$

Increase the index of each tan y by 1, and divide by the index so increased: then we get as above,

$$y = \tan y - \tfrac{1}{3} \tan^3 y + \tfrac{1}{5} \tan^5 y - \tfrac{1}{7} \tan^7 y + \ldots.$$

(*Young's D. Calculus, p.* 50).

5. Find a form of the function f which fulfils the equation.

$$f(x) \,.\, f(h) = f(x + h) + f(x - h).$$

Let $u = f(x)$: then

$$f(x + h) = u + \frac{du}{dx} \cdot \frac{h}{1} + \frac{d^2u}{dx^2} \cdot \frac{h^2}{1 \,.\, 2} + \frac{d^3u}{dx^3} \cdot \frac{h^3}{1 \,.\, 2 \,.\, 3} + \ldots.$$

$$f(x - h) = u - \frac{du}{dx} \cdot \frac{h}{1} + \frac{d^2u}{dx^2} \cdot \frac{h^2}{1 \,.\, 2} - \frac{d^3u}{dx^3} \cdot \frac{h^3}{1 \,.\, 2 \,.\, 3} + \ldots.$$

Whence adding, equating the result to the first side of the equation, and dividing all by u, we get

$$f(h) = 2 \left\{1 + \frac{1}{u} \cdot \frac{d^2u}{dx^2} \cdot \frac{h^2}{1 \,.\, 2} + \frac{1}{u} \cdot \frac{d^4u}{dx^4} \cdot \frac{h^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \ldots \right\}$$

Also, since h is independent of x (the given equation being unrestricted as to dependence of x and h), the co-efficients $\frac{1}{u} \cdot \frac{d^2u}{dx^2}$, $\frac{1}{u} \cdot \frac{d^4u}{dx^4}$, *etc.*, which do not contain h, may be considered as constants. Let us, therefore, put

$$\frac{1}{u} \cdot \frac{d^2u}{dx^2} = - a^2, \text{ or } \frac{d^2u}{dx^2} = - a^2u; \text{ then, also,}$$

$$\frac{d^4u}{dx^4} = - a^2 . \frac{d^2u}{dx^2} = + a^4u; \ \frac{d^6u}{dx^6} = - a^6u; \ etc.$$

Hence $f(h) = 2 \left\{1 - \frac{a^2h^2}{1 \,.\, 2} + \frac{a^4h^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} - \ldots \right\} = 2 \cos ah;$

which gives the form of the function f as required; and the function is

$$2 \cos ah \,.\, \cos ax = \cos a\,(x + h) + \cos a\,(x - h),$$

which is at once verified by trigonometry.

6. Show that $\log\,(x + h) = \log x + \frac{h}{x} - \frac{1}{2} \cdot \frac{h^2}{x^2} + \frac{1}{3} \cdot \frac{h^3}{x^3} - \ldots.$

7. Find $\frac{d^6u}{dx^6}$, where $u = \sqrt{\cos x}$.

8. Prove that $\tan\,(x + h) = \tan x + h \sec^2 x + h^2 \sec^2 x \tan x + \ldots.$

9. $\sin^{-1}(x+h) = \sin^{-1} x + \frac{h}{\sqrt{1-x^2}} + \frac{h^2x}{2\sqrt{(1-x^2)^3}} + \frac{(1 + 2x^2)\,h^3}{2.3\sqrt{(1-x^2)^5}} + \ldots$

10. $\sec x = 1 + \frac{x^2}{1 \,.\, 2} + \frac{5x^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \frac{61\,x^6}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6} + \ldots.$

11. $\tan x = x + \frac{x^3}{1 \,.\, 3} + \frac{2x^5}{3 \,.\, 5} + \frac{17\,x^7}{3 \,.\, 5 \,.\, 7} + \ldots.$

12. $\sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \,.\, 3}{2 \,.\, 4} \cdot \frac{x^5}{5} + \frac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6} \cdot \frac{x^7}{6} + \ldots.$

PROP. IV.

To differentiate an implicit function of two, three, or more variables, each of which is a function of another and principal variable.

First. LET there be two independent functions of x, viz., p and q; it is required to find $\frac{df(p,\, q)}{dx}$, When x becomes $x + h$, let p and q become $p + k$, or p', and $q + k_{,}$, or q'. Then the function $f(p, q)$ becomes $f(q', p + k)$; in which, q' is not affected by the increment k of the variable p.

We have, therefore, by *prop. ii.*

$$(1).\ f(q', p + k) = f(q', p) + \frac{d.f(q'.p)}{dp}\cdot\frac{k}{1} + \frac{d^2f(q', p)}{dp^2}\cdot\frac{k^2}{1.2} + \dots\dots \quad (1).$$

Next, for $f(q', p)$ write its value $f(p, q + k_{,})$: then

$$(2).\ f(q', p) = f(p, q + k_{,}) = u + \frac{du}{dq}\cdot\frac{k_{,}}{1} + \frac{d^2u}{dq^2}\cdot\frac{k_{,}^2}{1.2} + \dots\dots\dots\dots \quad (2),$$

$$3.\ \frac{df(q'.p)}{dp} = \frac{du}{dp} + \frac{d\left(\frac{du}{dq}\right)}{dp}\cdot\frac{k_{,}}{1} + \dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (3);$$

and so on, for the successive differential co-efficients of $f(q', p)$.

Substitute in (1) for the first two terms, their values from (2) and (3); then

$$f(q + k_{,}, p + k) = u + \frac{du}{dq}\cdot\frac{k_{,}}{1} + \dots\dots + \frac{du}{dp}\cdot\frac{k}{1} + \dots\dots\dots\dots\dots \quad (4).$$

But, k is the increment of p, and k' that of q, when x becomes $x + h$: whence by Taylor's Theorem,

$$k = \frac{dp}{dx}\cdot\frac{h}{1} + \frac{d^2p}{dx^2}\cdot\frac{h^2}{1.2} + \dots\dots \;\Big|\; k_{,} = \frac{dq}{dx}\cdot\frac{h}{1} + \frac{d^2q}{dx^2}\cdot\frac{h^2}{1.2} + \dots\dots$$

Substitute these in (4), and there results, since $u = f(q, p)$,

$$f(q + k_{,}, p + k) - f(q, p) = u = \left(\frac{du}{dq}\cdot\frac{dq}{dx} + \frac{du}{dp}\cdot\frac{dp}{dx}\right)h + \dots\dots$$

$$\text{or}\quad \frac{du}{dx} = \frac{du}{dq}\cdot\frac{dq}{dx} + \frac{du}{dp}\cdot\frac{dp}{dx}.$$

Second. Let there be three functions, p, q, r, of x, such that $f(p, q, r) = 0$, to find $\frac{df(p, q, r)}{dx}$. When x becomes $x + h$, let p, q, r, become $p + k$, or the function p', $q + k_1$ or q', $r + k_2$ or r'. Then r' enters as a constant in $f(p + k, q + k_1, r')$. Hence, if $u = f(p, q, r')$, we have, as above,

$$f(p + k, q + k_1, r') = u' + \left\{\frac{du}{dq}\cdot\frac{dq}{dx} + \frac{du}{dp}\cdot\frac{dp}{dx}\right\}h + \dots$$

Also, let u' be the value of u when x becomes $x + h$; then

$$u' = u + \frac{du}{dr}\cdot\frac{k_2}{1} + \frac{d^2u}{dr^2}\cdot\frac{k_2^2}{1.2} + \dots\dots$$

$$\frac{du'}{dq} = \frac{du}{dq} + \frac{d\left(\frac{du}{dr}\right)}{dq}\cdot\frac{k_2}{1} + \dots\dots\dots$$

$$\frac{du'}{dp} = \frac{du}{dp} + \frac{d\left(\frac{du}{dr}\right)}{dp}\cdot\frac{k_2}{1} + \dots\dots$$

For a reason similar to that employed in the final step of the last case, we have

$$k_2 = \frac{dr}{dx}\cdot\frac{h}{1} + \frac{d^2r}{dx^2}\cdot\frac{h^2}{1.2} + \dots\dots$$

Hence, substituting as before, we get for our final expression

$$f(p+k, q+k_1, r+k_2) = u + \left\{\frac{du}{dr}\frac{dr}{dx} + \frac{du}{dq}\frac{dq}{dx} + \frac{du}{dp}\cdot\frac{dp}{dx}\right\}\frac{h}{1} + \ldots .$$

and finally *,

$$\left\{\frac{du}{dx}\right\} = \frac{du}{dr}\cdot\frac{dr}{dx} + \frac{du}{dq}\cdot\frac{dq}{dx} + \frac{du}{dp}\cdot\frac{dp}{dx};$$

and the same reasoning may be extended to any number of functions whatever.

Hence the rule :—

Differentiate the function for each of the variables singly as the only variable Also differentiate each of the functions as a function of the principal variable. With these co-efficients form the right side of the above equation; which will be the total differential of the general function with respect to the principal variable.

Corollary 1.

If one of the functions of x, as p for instance, be x simply, then the expressions become simplified; for then $\frac{dp}{dx} = 1$; and we have, when $u = f(x, q)$, and $u = f(x, q, r)$ respectively,

$$\frac{du}{dx} = \frac{du}{dx} + \frac{du}{dq}\cdot\frac{dq}{dx};$$

$$\left\{\frac{du}{dx}\right\} = \frac{du}{dx} + \frac{du}{dq}\cdot\frac{dq}{dx} + \frac{du}{dr}\cdot\frac{dr}{dx}.$$

Corollary 2.

The former of these equations, since $u = 0$ is an implicit function, gives $\frac{du}{dx} = 0$, and hence

$$\frac{du}{dx} = -\frac{du}{dq}\cdot\frac{dq}{dx}.$$

This is the formula alluded to at *p*. 362.

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* It will be observed that the function on the right side does not differ in any respect as to *form*, except that it is enclosed in braces, from one of the terms of its value on the right. The braces are used to express that it is the *total differential co-efficient* which is meant: viz. that composed of the several *partial differential co-efficients*, taken in succession with respect to the successive variables. The complexity of the form *in print* is the only objection, that I am aware of, to the following; and in writing it may therefore be advantageously employed.

Denote by $d_x u$, the differential of the function u, on the hypothesis that x is the only variable, $d_x y$ that of y on the same hypothesis, $d_y u$ that of u supposing y alone to vary, $d_{x,y,z,\ldots} u$ that of u supposing all to vary together; and so on for other functions and variables. Then the above equation would be expressible in either of the two following ways :—

$$d_{p,q,r,} u = \frac{d_x u}{dx} + \frac{d_q u}{dq}\cdot\frac{d_x q}{dx} + \frac{d_r u}{dr}\cdot\frac{d_x r}{dx},$$

$$\text{or } d_{p,q,r,[x} u = d_x u + \frac{d_q u}{dq}\cdot d_x q + \frac{d_r u}{dr}\cdot d_x r.$$

The notation in the text is, however, no doubt fully adequate (taken in conjunction with the remark at the foot of page 352) to express the composition of the equation, as to the dependence of the variables.

Scholium.

Taylor's theorem is said *to fail*, and the cases in which it fails are called *failing cases*, when the function $f(x)$ is of some particular forms. The meaning of these phrases is, that the differential co-efficients become in these cases illusory or ambiguous, either under the general form, or for particular values of the variables; and the immediate inference is, that the expansion cannot be *fully* effected according to positive and ascending integer powers of h.

As the process of differentiation is a *direct* one, it can introduce no radical forms which were not in the original function; and hence the impossibility of such expansions cannot be expressed by imaginary radicals, such as $a \pm b\sqrt{-1}$. The only other ways in which we can conceive the indications of impossibility to appear are these two:

(1). That some of the co-efficients of the integer powers of h which do not enter into the development are *zero*.

(2). That there always occur two or more co-efficients of integer powers of h which are *infinite*.

Into the theory of these failing cases we cannot here enter; but must refer the reader to the very able discussion of them given by Professor Young, in his *Differential Calculus, chap. ix*. We have only to remark:

1. That when the co-efficients of any powers of h appear in the development of finite values, the terms which contain them are terms of the true development.

2. That the terms with cipher co-efficients are absent from the development.

3. That the terms with infinite co-efficients imply the extreme of terms in powers of h with fractional co-efficients.

4. That when the development is required, recourse must be had to other methods depending on the nature of the function itself; as the binomial theorem, indeterminate co-efficients, *etc.*

PROP. V.

To change the independent variable in a differential equation.

LET $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, *etc.* be differential co-efficients of u, with regard to the independent variable x; it is required to express them as functions of another independent variable t.

Since $u = f(x)$, and $x = \phi\,(t)$, we have (*page* 373)

$$\frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt}\,;\ \text{or}\ \frac{du}{dx} = \frac{du}{dt} \div \frac{dx}{dt} = -\frac{(du)}{(dx)};$$

where we put $(du) = \frac{du}{dt}$ and $(dx) = \frac{dx}{dt}$ for brevity of writing.

Again,

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2}\cdot\frac{dx^2}{dt^2} + \frac{du}{dx}\,\frac{d^2x}{dt^2}\,;\ \text{and hence}$$

$$\frac{d^2u}{dx^2} = \left\{\frac{d^2u}{dt^2} - \frac{(du)}{(dx)}\cdot(d^2x)\right\}\ (dx) = \frac{(d^2u)\ (dx) - (d^2x)\ (dy)}{(dx)^3}.$$

and so on for higher orders of differential co-efficients. Such transformations as we proceed, however, become extremely laborious; but above the second here given, they are never required in ordinary practice.

EXAMPLES.

1. Let $u = \sin x$ and $x = \cos^{-1} t$: then

$$\frac{du}{dx} = \cos x \text{ and } \frac{du}{dt} = -\frac{1}{\sqrt{1-t^2}}; \text{ hence we have}$$

$$\frac{du}{dt} = -\frac{\cos x}{\sqrt{1-t^2}} = -\frac{t}{\sqrt{1-t^2}}$$

2. Change the formula $\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx} + (y-a)\frac{d^2y}{dx^2} = 0$, in which x is the principal variable into one where y is the principal variable.

$$\textit{Ans. } 1 + \frac{dx^2}{dy^2} - (y-a)\frac{d^2x}{dy^2} = 0.$$

3. The expressions $\dfrac{\left\{1 + \frac{dy^2}{dx^2}\right\}^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}}$, and $\dfrac{\left\{1 + \frac{dx^2}{dy^2}\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}$ have identical values, x being the principal variable in the former, and y in the latter case.

4. If $\frac{ds^2}{dx^4} = 1 + \frac{dy^2}{dx^2}$, it is required to change the expression $\dfrac{\frac{d^2y}{}}{\sqrt{1 + \frac{dy^2}{dx^2}}}$ into one where s is an independent variable. *Ans.* $\frac{d^2y}{ds^2}\cdot\frac{du}{ds} - \frac{d^2x}{ds^2}\cdot\frac{dy}{ds}$.

5. Show that if $x = r\cos\theta$ and $y = r\sin\theta$; then the expressions in *Ex.* 3 will be changed as functions of θ into $\dfrac{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r.\frac{d^2r}{d\theta^2}}$.

CHAPTER III.

VANISHING OR AMBIGUOUS FRACTIONS.

This term applies to such fractions, $\frac{F(x)}{f(x)}$, as for some one or more specific *finite* values of x in both its terms vanish simultaneously, or the fraction to take the form $\frac{0}{0}$; and thereby render the value of the fraction ambiguous. Instances have been given in the introduction to this subject, such as $\frac{x^4 - a^4}{x - a}$, $\frac{\sin\theta}{\theta}$, *etc.*, which, when $x = a$, and $\theta = 0$, take, respectively, the values $3a^3$ and 1. The differential calculus furnishes a ready method of finding the values of these fractions under such conditions.

1. In the general development of $f(x + h)$ by Taylor's series, it is impossible that all the differential co-efficients shall vanish with the same value of x.

For in such case, if a be the value of x, we should also have

$$f(a + h) = 0$$

for all values of h; and as in this case $f(a) = 0$, we have

$$f(a+h) = fa, \text{ or } h = 0;$$

which is incompatible with the hypothesis of the equations, being independent of the values of h. Whence all the differential co-efficients cannot *simultaneously vanish.*

2. Develop the fraction $\frac{F(x+h)}{f(x+h)}$; and write the differential co-efficients, for the sake of brevity, as follows:—

$$\frac{df(x)}{1 \, . \, dx} = f_1(x), \ \frac{d^2f(x)}{1 \, . \, 2 \, . \, dx^2} = f_2(x_2), \ \textit{etc.},$$

(as in the derived equations, *vol. i. p.* 208), and similarly $F_1(x)$, $F_2(x)$, *etc.*, for the differential co-efficients of $F(x)$. Then,

$$\frac{F(x+h)}{f(x+h)} = \frac{F(x) + F_1(x) \, . \, h + F_2(x) \, . \, h^2 + F_3(x) \, . \, h^3 + \dots}{f(x) + f_1(x) \, . \, h + f_2(x) \, . \, h^2 + f_3(x) \, . \, h^3 + \dots}$$

If a value a be given to x such that $F(a) = 0$, and $f(a) = 0$, then, dividing all the terms of both remaining series by h, we get

$$\frac{F(x+h)}{f(x+h)} = \frac{F_1(a) + F_2(a)\,h + F_3(a)\,h^2 + \dots}{f(a) + f_2(a)\,h + f_3(a)\,h^3 + \dots} = \frac{F_1(a)}{f_1(a)};$$

when h is taken zero.

Should this also give the ambiguous form, let us still further divide by h, and in the result make $h = 0$: then,

$$\frac{F(x)}{f(x)} = \frac{F_2(a)}{f_2(a)}.$$

Or if, again, the second side of this be $\frac{0}{0}$, we proceed onward, till at last we arrive at terms either $F_n(a)$ and $f_n(a)$, which are one or both not zero. Such values as are furnished by the quotient are those of the original fraction. Whence the rule.

Substitute the successive differentials of the numerator, and the corresponding ones of the denominator, for the original numerator and denominator: then the quotients which are thus formed, and are not of the ambiguous form, are the values of the fraction itself.

EXAMPLES.

1. Find the value of $\frac{x^3 - a^3}{x - a}$, when $x = a$.

Here $F_1(x) = 3x^2$, and $f_1(x) = 1$; hence, putting $x = a$, $\frac{F_1(a)}{f_1(a)} = 3a^2$.

2. What is the value of $\frac{a^x - b^x}{x}$ when x is made equal to zero?

$\frac{F_1(x)}{f_1(x)} = a^x \log_e a - b^x \log_e b$. Hence $\frac{F_1(0)}{f_1(0)} = a^0 \log_e a - b^0 \log_e b = \log_e \frac{a}{b}$.

3. If $x = \frac{1}{2}\pi$ in $\frac{\cos x - \sin x + 1}{\cos x + \sin x - 1}$; then, $\frac{F_1(\frac{1}{2}\pi)}{f_1(\frac{1}{2}\pi)} = -\frac{\cos \frac{1}{2}\pi + \sin \frac{1}{2}\pi}{\cos \frac{1}{2}\pi - \sin \frac{1}{2}\pi} = 1$.

4. To find the value of $\frac{a - \sqrt{a^2 - x^2}}{x^2}$ when $x = 0$. $\frac{F_2(0)}{f_2(0)} = \frac{1}{2a}$

5. Find $\frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)\, x^{n+2} - n^2 x^{n+3}}{(1-x)^3}$ when $x = 1$.

$$\frac{F_3(1)}{f_3(1)} = \frac{n\,(n+1)\,(2n+1)}{1 \, . \, 2 \, . \, 3}.$$

It will often be advantageous, and sometimes essential, to develop without the aid of *Taylor's* theorem; and, if possible, to eliminate the common factor upon which the expression depends for its ambiguous form. We have not, however, room to dilate here.

Thus if $\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$ be required when $x=a$, we may write it

$$\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}}+\frac{1}{\sqrt{x+a}}=\frac{1}{\sqrt{2a}}.$$

On this subject, treated under a different and more general aspect, see Young's *Diff. Calc.* pp. 76—81.

6. The value of the differential co-efficient of $u=x^4+3a^2x^2-4a^2xy-a^2y^2=0$, when $x=0$, is $-2\pm\sqrt{7}$, the function being explicit.

7. The differential co-efficient of $ay^2-2a^2y-2x^3+3ax^2=0$, when $x=a$, is $\pm\sqrt{3}$, the function being an explicit one.

8. The value of $e^{-\frac{1}{x}}(1-\log x)$ is 0, when $x=0$;

9. The value of $e^{\frac{1}{x}}\sin x$ is infinity when $x=0$.

10. The value of $x^n e^{-x}$ when x is infinite is $\frac{n(n-1)\ldots 3.2.1}{e^x}=0$.

11. When $x=0$, the value of $(\cos \alpha x)(\text{cosec } \beta x)^2$ is $e^{\frac{\alpha^2}{2\beta^2}}$.

12. When $x=0$, the value of $(1+nx)^{\frac{1}{x}}$ is e^n.

CHAPTER IV.

THE THEORY AND APPLICATION OF MAXIMA AND MINIMA.

When one variable of an equation depends upon the variations undergone by one or more independent variables, its value will vary according to the variation of those upon which it depends. For any specified values of the independent variable, it will have one or more specific values, dependent solely upon the form of the equation which connects all the variables. This value, (or values) can be always found, provided we admit the possibility of resolving the equation which connects the variables.

During the progressive changes, however, of those independent variables, there will be some values of the function which are greater or less than others; and as we suppose the independent variables to change progressively, *according to specified laws*, the value of the function itself must undergo its changes progressively, *also according to a specified law*. The laws of change are also implied in the form of the equation itself; and hence may be considered as given for all the uses we shall have occasion to make of them.

Those changes may be of various kinds; either sometimes increasing up to a certain value, and then commencing to decrease; or first decreasing to a certain value, and then commencing to increase. The function may be of such a form as successively to do both, even many times in succession: or it may, either after doing both any number of times or doing either, increase or decrease without finite limit, or to infinity.

(1). As an example let us take $\sin \theta$. This may increase from 0 to $\sin \frac{1}{2}\pi$ or unity: after which it commences to decrease till $\sin \pi = 0$. Continuing to decrease (for a *negative* value will represent decrease below 0) it arrives at $\sin \frac{3}{2}\pi$ or -1; and then it commences its increase, by diminishing its negative value, till it arrives again at the value $\sin 2\pi$ or 0.

(2). If we consider $y = f(x)$ to be the relation of the ordinate of a line of the second order to its abscissa, precisely the same results will follow from an examination of the equation of the conic section $y^2 = hx + kx^2$. This the student can follow out for himself. He will see that when $k = 0$, there is neither a greatest nor a least value of y; that is, such as the values for x immediately preceding and succeeding are both greater or both less than for that at any specified value of x. Whilst for real values of k there are values of y which are either both greater or both less on each side (at the smallest difference in the value of x) of that particular value.

Other instances will presently occur; the present, however, will suffice for explaining the following definitions:—

Def. 1. A *maximum* is that value of the function of one or more variables corresponding to specific values of those variables, such that if compared with the values of the function dependent upon two contiguous values of the independent variables, it shall be greater than either of them.

Def. 2. A *minimum* is that value of the function, two contiguous ones of which are greater than it.

The algebraic expression of these definitions will obviously be:—

Let $u = f(x)$ be a function of x, and a be such a value of x that for any finite value of h, however small, and for all less values of h down to 0,

$$f(a - h) - f(a) = \pm \text{ and } f(a + h) - f(a) = \pm,$$

that is, shall have the *same sign:* then

(1). If that sign be $-$ the function $u_1 = f(a)$ is a *maximum.*

(2). .... $+$ .... $u_2 = f(a)$ .. *minimum.*

PROP. I.

Those values of x which render u a maximum or a minimum, render also $\frac{du}{dx} = 0$.

Let $u_1 = f(x + h)$ and $u_2 = f(x - h)$: then generally

$$u_1 - u = + \frac{du}{dx} \cdot \frac{h}{1} + \frac{d^2u}{dx^2} \cdot \frac{h^2}{1 \cdot 2} + \frac{d^3u}{dx^3} \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \ldots.$$

$$u_2 - u = - \frac{du}{dx} \cdot \frac{h}{1} + \frac{d^2u}{dx^2} \cdot \frac{h^2}{1 \cdot 2} - \frac{d^3u}{dx^3} \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \ldots.$$

Now in every series of ascending powers of h, in which all the co-efficients are finite, a value so small may be given to h, as shall render the aggregate value of all the terms after the first, less than the value of that first term [*]; and hence

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[*] This proposition has been virtually proved at *p.* 221. *vol. i. theorem xi.*, as the method of reasoning there employed may be extended to an infinite series as well as to a finite number of terms. For the sake of completeness, however, we shall restate the argument in a slightly modified form.

Let the given series be

$$Ah + Bh^2 + Ch^3 + \ldots\ldots\ldots;$$

that value of h will render the aggregate value of the entire series of the *same sign* as that prefixed to the first term. But in these two series, $\frac{du}{dx} \cdot \frac{h}{1}$ is the same value with different signs; and hence, except $\frac{du}{dx} = 0$, the condition of the maximum and minimum (viz. that the functions $u_1 - u$ and $u_2 - u$ shall have the same signs) cannot be fulfilled. Whence the proposition follows: for then the third terms, and hence the aggregate values of the series, will have the same signs.

PROP. II.

To investigate the significations of the subsequent terms of the development.

1. Since h^3 is essentially positive, the sign of the term $\frac{d^2u}{dx^2} \cdot \frac{h^2}{1 \cdot 2}$ depends on the factor $\frac{d^2u}{dx^3}$.

When $\frac{d^2u}{dx^2} = +$ the function $f(a)$ is a *minimum*.

When $\frac{d^2u}{dx^2} = - \dots\dots\dots\dots f(a) \dots$ *maximum.*

The maxima and minima can therefore be distinguished by this criterion: whilst the equation from which a is to be determined is that $\left(\frac{df(x)}{dx} = 0\right)$ established in the preceding proposition.

2. Should it also happen, that for $x = a$, $\frac{d^2u}{dx^2} = 0$, then $\frac{du}{dx} = 0$ is insufficient for determining the maxima and minima, as the following terms of the two series $\pm \frac{d^3u}{dx^3} \cdot \frac{h^3}{1 \cdot 2 \cdot 3}$, have different signs, and the analytical condition ceases to be fulfilled. Whence, that the condition should be fulfilled we must further have $\frac{d^3u}{dx^3} = 0$; and then the separate cases of maxima and minima will be determined by the sign of $\frac{d^4u}{dx^4}$ as they were before by $\frac{d^2u}{dx^2}$.

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and let the ratio of any one of the co-efficients to the one immediately preceding it, $H \div G$ be finite; then if r be greater than the greatest of all these ratios, we shall have

$$\frac{B}{A} \angle \rho, \text{ or } B \angle A\rho; \ \frac{C}{B} \angle \rho, \text{ or } C \angle B\rho, \text{— or } C \angle A\rho^2;$$

$$\frac{D}{C} \angle \rho, \text{ or } D \angle C\rho, \text{— or } D \angle A\rho^3; \textit{ etc.}$$

Whence

$$\begin{aligned} Ah + Bh^2 + Ch^3 + \dots\dots\dots &\angle Ah + A\rho h^2 + A\rho^2 h^3 + \dots\dots\dots \\ &\angle Ah \left\{1 + \rho h + \rho^2 h^2 + \dots\dots\dots\dots\right\} \\ &\angle Ah \cdot \frac{1}{1 - \rho h}. \end{aligned}$$

Let $h = \frac{1}{2\rho}$; then $\frac{1}{1 - \rho h} = 2$; and we have

$$Ah + Bh^2 + Ch^3 + \dots\dots\dots \angle 2Ah, \text{ or } Bh^2 + Ch^3 + \dots\dots\dots \angle Ah.$$

The sum of the whole of the remaining series is, in this case, rendered less in value than the first term; and hence the sign of the entire value for this value of h is the same as the sign of the first term.

3. Should the differential co-efficients for the value $x = a$ become zero through any number of stages, it will follow from similar reasoning, that the maxima and minima will only be indicated by an even-numbered co-efficient, as the $2n^{\text{th}}$. being the first one of the series which has a finite value; and the separate cases will be determined by the same criterion as to the + or — value of that co-efficient, as was given above.

4. If u be a maximum or minimum for a given value $x = u$, then any constant multiple, sub-multiple, root, or power of u, will also be a maximum or minimum respectively.

5. If u be a maximum, then $\frac{1}{u}$ is a minimum; and conversely.

6. If u be a maximum or a minimum, $\log u$ is so too.

7. If in the equation $\frac{du}{dx} = 0$, there be *factors of the variable which, for all values of that variable, remain positive*, these may be rejected in the same way that constants are, in determining the *sign* of $\frac{d^2u}{dx^2}$; for as it is the sign alone with which we are here concerned, it is obvious that the rejection of these factors does not affect, in any way, this particular object of our search. Such factors are $(a - x)^{2n}$, $\frac{1}{x^{2n}}(x - x^2)^{-2n}$, *etc.*, which are positive for all integer values of n. This will often greatly simplify the work. We must not, however, forget that we thus determine only a part of the maxima and minima, viz. those which depend upon the compound expression which is multiplied by these factors. In such cases, however, the determination of the maxima and minima corresponding to the roots of $f_1(x) = 0$, which these expressions imply, is very simple, and generally done by inspection, on the principle illustrated in *Ex.* 2.

Scholium.

If $u = f(x, y, \ldots)$, the same reasoning will apply; and it will follow that the several conditions already deduced must be fulfilled by the differential coefficients of Taylor's theorem, taken in respect to *each* of the variables, $x, y, \ldots$; but in the present work, it would be impossible to enter upon the requisite discussions for their establishment. The student is referred to *Gregory's Examples, p.* 107, *etc.*, for the requisite information.

PROP. III.

The signs of the differential co-efficient for values immediately preceding and following a maximum or minimum are different.

For since the function is a max. or a min. we have

$$f(x - h) = f(x) - f_1(x).\frac{h}{1} + f_2(x).\frac{h^2}{1.2} - \ldots.$$

$$f(x + h) = f(x) + f_1(x).\frac{h}{1} + f_2(x).\frac{h^2}{1.2} + \ldots.$$

both less or both greater than $f(x)$. that is, the two series

$$-f_1(x).\frac{h}{1} + f_2(x).\frac{h^2}{1.2} - f_3(x).\frac{h^3}{1.2.3} + \ldots.$$

$$+f_1(x).\frac{h}{1} + f_2(x).\frac{h^2}{1.2} + f_3(x)\frac{h^3}{1.2.3} + \ldots.$$

must have the same sign; viz. — when the function $f(x)$ is a maximum, and + when a minimum.

Now, with values of h sufficiently small, the signs of these series will be the same with those of their first terms, — and + respectively; and hence the condition of the values of the series having the same signs, requires that in passing through the maximum or the minimum, the sign of $f_1(x)$ shall be changed.

Scholium.

The criterion of the maxima and minima laid down in the previous proposition (*ii.*) is often of difficult application, owing to the great complexity assumed by the successive differential co-efficients. It also becomes inapplicable in certain cases from the illusory form which the expansion of the series by Taylor's theorem assumes. For these reasons, this proposition (*iii*) is often useful, and sometimes essential for the determination of maxima and minima. Its praxis is comprised in the following *rule.*

Make $\frac{du}{dx}$ [or $\frac{dx}{du}$, if more convenient] $= 0$, and find the values of x [or of u] which fulfil this: insert successively this value decreased and increased by a positive quantity h, separately in the given function, and observe the signs belonging to both results.

If both results be — the value a of x in $\frac{du}{dx} = 0$ makes the function a maximum: if both be + that value of a makes the function a minimum: and if the signs of both results be unlike, that value renders the function neither a maximum nor a minimum.

EXAMPLES.

1. Find the maximum and minimum values of the function u, in

$$u = x^4 - 8x^3 + 22x^2 - 24x + 12.$$

Here,

$$\frac{du}{dx} = 4x^3 - 24x^2 + 44x - 24 = 0; \text{ roots } 1, 2, 3.$$

$$\frac{d^2u}{dx^2} = 12x^2 - 48x + 44; \text{ values for the roots } +, -, +.$$

Hence for $x = 1$, the value of u is a *min.;* for $x = 2$, a *max.;* and for 3, a *min.*

2. To divide the number a into two parts, x and $a - x$, such that $u = x^m (a - x)^n$ shall be a maximum.

$$\frac{du}{dx} = 0 = mx^{m-1} (a-x)^n - nx^m (a-x)^{n-1} = x^{m-} (a-x)^{n-1} \{ma - (m+n)x\}$$

$$\frac{d^2u}{dx^2} = x^{m-2} (a-x)^{n-2} \{m(m-1)(a-x)^2 - 2mn\,x(a-x) + n(n-1)x^2\}$$

Now the values of x in $\frac{du}{dx} = 0$ are 0, a, and $\frac{ma}{m+n}$.

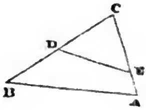
(1). When $x = 0$, u is a *min.* if m be even; but neither *max.* nor *min.* if m be odd. For it is clear, that by continuing the differentiation, the factor x^{m-2p}, which renders the first even-numbered differential co-efficient of the form 0^0 (or 1) depends upon $m - 2p$ being an even number, and therefore m even. Otherwise, the term 0^0 is an odd-numbered differential co-efficient; which is neither indicative of a *max.* or a *min.*

(2). When $x = a$, it will similarly depend on n being even or odd, whether

$(a-x)^{n-2p}$ becomes 0^0 in an even or an odd-numbered differential co-efficient; and the same conclusion follows for n as before for m.

(3). Let $x=\frac{am}{m+n}$: then $\frac{d^2u}{dx^2}$ becomes $-(m+n)\left(\frac{ma}{m+n}\right)^{m-1}\left(\frac{na}{m+n}\right)^{n-1}$ indicative of u being a *max.*

3. To bisect a given triangle, ABC, by the shortest line, DE, possible.



Denote the sides and angles of the given triangle ABC as usual; and let CD $=x$, CE $=y$, and DE $=u$. Then $\Delta ABC=2\Delta CDE$; or $\frac{1}{2}ab\sin C=2\cdot\frac{1}{2}xy\sin C$, or $ab=2xy$. But

$$u^2=x^2-2xy\cos C+y^2=x^2+\frac{a^2b^2}{4x^2}-ab\cos C.$$

Whence, $2u\cdot\frac{du}{dx}=0=2x-\frac{a^2b^2}{2x^3}$; or $x^4=\frac{1}{4}a^2b^2$, and $x=\sqrt{\frac{1}{2}ab}$.

Wherefore also, $y=\frac{ab}{2x}=\sqrt{\frac{1}{2}ab}$; and, substituting,

$$u^2=\tfrac{1}{2}ab+\tfrac{1}{2}ab-ab\cos C=ab(1-\cos C)=\tfrac{1}{2}(a-b+c)(-a+b+c).$$

4. To find the quadrilateral which, formed by four given lines, a_1, a_2, a_3, a_4, contains the greatest area.

Let θ be the angle included by a_1, a_2; and ω that contained by a_3, a_4; also let x be the diagonal, subtending these angles, and u be the area of the figure. Then we have

$$u=\tfrac{1}{2}a_1a_2\sin\theta+\tfrac{1}{2}a_3a_4\sin\omega;$$

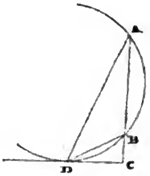
$$\frac{du}{d\theta}=\tfrac{1}{2}\left(a_1a_2\cos\theta+a_3a_4\cos\omega\frac{d\omega}{d\theta}\right)=0.$$

But $a_4{}^2-2a_4a_3\cos\omega+a_3{}^2=x^2=a_1{}^2-2a_1a_2\cos\theta+a_2{}^2$;

and differentiating we get, $a_3a_4\frac{d\omega}{d\theta}=a_1a_2\frac{\sin\theta}{\sin\omega}$; whence, substituting,

$\cos\theta+\cos\omega\cdot\frac{\sin\theta}{\sin\omega}=0$; or $\sin\omega\cos\theta+\cos\omega\sin\theta=0$; or $\sin(\omega+\theta)=0$; or lastly, $\theta+\omega=\pi$. The quadrilateral, therefore, is inscriptible in a circle.

5. There is a mark on a maypole of a feet high, at the distance of b feet from the bottom; at what distance on the horizontal plane from the foot of the maypole must I stand to see the upper part subtend the greatest angle?



Put AC $=a$, BC $=b$, CD $=x$, the distance sought, and ADB $=\theta$, the angle subtended by AB at D. Then

$$u=\tan\theta=\tan\left\{\tan^{-1}\frac{a}{x}-\tan^{-1}\frac{b}{x}\right\}=\frac{(a-b)x}{x^2+ab};$$

which is to be a *max.*, or $\frac{1}{u}=\frac{ab+x^2}{(a-b)x}=v$, a *min.* Hence, dropping the constant $a-b$, we have

$$0=\frac{dv}{dx}=1-\frac{ab}{x^2}, \text{ and } \frac{d^2v}{dx^2}=\frac{2ab}{x^3}.$$

From the former of these $x^2=ab$, showing that D is the point of contact of a circle through A, B, with the line CD; and the second being + shows that v is a *min.*, and therefore $u=\tan\theta$, is a *max.*

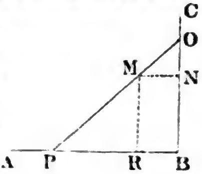
6. Find the greatest ordinate in a given ellipse, referred to a given diameter.

Let the equation of the ellipse be referred to the given diameter $2a$, and its conjugate $2b$, the origin being the centre. Then we have

$$a^2y^2 + b^2x^2 = a^2b^2, \text{ and } y \text{ being a function of } x,$$

then differentiating and applying the criterion, the condition is found to be fulfilled by $x = 0$, $y = a$, or the greatest ordinate is the semi-diameter conjugate to the given one.

7. Determine the longest pole that can be put up a chimney, the height RM from the floor to the mantel being a, and the depth RB from front to back being b.



Let OP be the position of the pole, and the angle MPR $= \theta$; then OP $= u = a \operatorname{cosec} \theta + b \sec \theta$.

Differentiate; then

$$\frac{du}{d\theta} = -\frac{a \cos \theta}{\sin^2\theta} + \frac{b \sin \theta}{\cos^2\theta} = 0, \text{ or } \tan \theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}.$$

Whence,

$$u = a \operatorname{cosec} \theta + b \sec \theta, \text{ or PO} = \pm \{a^{\frac{2}{3}} + b^{\frac{2}{3}}\}^{\frac{3}{2}};$$

the double sign signifying that the pole is measured from *either* end, O or P.

Again, the first differential co-efficient gives both

$$\begin{array}{l|l} \tan \theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}, \text{ and} & \cot \theta = \left(\frac{a}{b}\right)^{\frac{1}{3}}, \text{ and} \\ \\ \frac{d^2u}{d\theta^2} = \sec^2\theta & \frac{d^2u}{d\theta^2} = -\operatorname{cosec}^2\theta. \end{array}$$

The results respectively show that u is both a *max.* and a *min.* We must, therefore, recur to the *geometrical* conditions, to assign which is to be taken.

Now, the condition which we have used is that the line PO passes through M and is terminated by the lines AB and BC. But that line may be increased without limit, since when it becomes parallel to either of the lines AB or BC, it is infinite. The line found above is, then, the *least* that can be drawn through M.

If a pole longer than that now found were placed in the chimney, it must in its progress have coincided in length and position with PO; and as these two conditions are contradictory, PO is the longest pole that could fulfil the conditions.

8. The four edges of a rectangular piece of lead, a inches in length and b in breadth, are to be turned up perpendicularly so as to form a vessel which shall hold the greatest quantity of water: what breadth is to be turned up to form the sides?

Let it be x inches; then

$$u = x\,(a - 2x)\,(b - 2x) = abx - 2\,(a + b)\,x^2 + 4x^3 = max.$$

$$\frac{du}{dx} = ab - 4\,(a + b)\,x + 12x^2 = 0, \text{ or}$$

$$x = \tfrac{1}{6}\{a + b \pm \sqrt{a^2 - ab + b^2}\}$$

$$\frac{d^2u}{dx^2} = -4\,(a + b) + 24x;$$

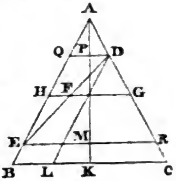
in which, inserting the previously found value of u, we get

$$\frac{d^2u}{dx^2} = \pm 4 \sqrt{a^2 - ab + b^2}.$$

This solution is double, like the preceding one, and similar remarks apply to it; but without this discussion, as only the maximum is sought, we have taken the sign of the radical—to fulfil the condition, and hence

$$x = \tfrac{1}{3}\{a + b - \sqrt{a^2 - ab + b^2}\} \text{ gives the } max. \text{ vessel.}$$

9. Cut the greatest ellipse from a given right cone.



Let A be the vertex, and EK the section made in the base of the cone by the transverse plane Then, put AE $= a$, AD $= x$, AER $=$ A, and ED, $\sqrt{\text{HF}\,.\,\text{FG}}$ the transverse and conjugate diameters. (See *pages* 124 and 144). Then by trigonometry,

$$\text{ED}^2 = a^2 - 2ax\cos\text{A} + x^2,$$

$$\text{HF} = \text{PD} = x\sin\tfrac{1}{2}\text{A},\ \text{EM} = \text{FG} = a\sin\tfrac{1}{2}\text{A}.$$

Also, area of ellipse $= \text{ED}.\sqrt{\text{HF}.\text{FG}}\,\varpi$; or substituting

$$= \varpi\sqrt{(a^2 - 2ax\cos\text{A} + x^2).ax\sin^2\tfrac{1}{2}\text{A}}.$$

This being a max., its square and submultiples will be a *max.* Whence, for our present purpose, we have

$$u = a^2x - 2ax^2\cos\text{A} + x^3, \text{ a } max.,$$

$$\frac{du}{dx} = 3x^2 - 4ax\cos\text{A} + a^2 = 0\,;\ \text{from which}$$

$$x = \tfrac{1}{3}a\,\{2\cos\text{A} \pm \sqrt{\cos^2\text{A} - 3\sin^2\text{A}}\}$$

$$\text{Again, } \frac{d^2u}{dx^2} = 6x - 4a\cos\text{A} = \pm\,2a\sqrt{\cos^2\text{A} - 3\sin^2\text{A}},$$

the negative sign betokening the *max.*; and hence the solution is

$$x = \tfrac{1}{3}\,a\,\{2\cos\text{A} - \sqrt{\cos^2\text{A} - 3\sin^2\text{A}}\}$$

The limit of possibility is evidently $\tan\text{A} = \frac{1}{3}\sqrt{3}$.

9. Given the base and sum of the sides of a spherical triangle, to find it so that the vertical angle shall be the greatest possible.

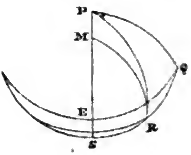
Let 2α be the given base, 2β the given sum of the sides, and $\beta + \theta$, $\beta - \theta$ the sides themselves. Then if C be the vertical angle, we shall have

$$\cos\text{C} = \frac{\cos 2\alpha - \cos(\beta + \theta)\cos(\beta - \theta)}{\sin(\beta + \theta)\sin(\beta - \theta)} = \frac{\cos 2\alpha - \cos^2\beta + \sin^2\theta}{\sin^2\beta - \sin^2\theta}$$

$$\frac{d\cos\text{C}}{d\theta} = \frac{4(\cos 2\alpha - \cos 2\beta)\sin 2\theta}{(\cos 2\theta - \cos 2\beta)^2} = 0\,;\ \text{or } \theta = 0.\ \text{ Omit the constants : then}$$

$$\frac{d^2\cos\text{C}}{d\theta^2} = \frac{2\{1 - \cos 2\beta\cos 2\theta + \sin^2 2\theta\}}{(\cos 2\theta - \cos 2\beta)^3} = \frac{2}{(1 - \cos 2\beta)^2}\ \text{when } \theta = 0.$$

The value of $\frac{d\cos\text{C}}{d\theta}$ is therefore a *min.*; and hence C itself is a *max.* as was required.



10. (*Edin. Trans. vol. xii. p.* 333). To find the sun's declination ϕ when in a given co-latitude λ, the ratio of the sun's motion in right ascension to his motion in longitude is a maximum, and to assign that ratio.

Let P be the pole of the equinoctial, M the pole of the ecliptic, Q the first point of capricorn, and R the sun's place at the time in question.

Put $PR = \phi$, $SPR = \theta$, $PM = \lambda$, $RS = \chi$; then we have

$$(\Delta PSR) \ldots\ldots \cos\chi = -\cos\phi\sin\lambda + \sin\phi\cos\lambda\cos\theta \ldots\ldots\ldots\ldots (1),$$

$$(\Delta PMR) \ldots\ldots \cot\phi = -\tan\lambda\cos\theta, \text{ (since } MR = \tfrac{1}{2}\pi) \ldots\ldots\ldots\ldots (2).$$

$$\frac{d\chi}{d\theta} = u, \text{ (} u \text{ being the variable ratio)} \ldots\ldots\ldots\ldots (3).$$

From (1, 2) we get

$$\cos\chi = -\frac{\cos\phi}{\sin\lambda}, \text{ or } \frac{d\chi}{d\phi} = -\frac{\sin\phi}{\sqrt{\sin^2\lambda - \cos^2\phi}} \ldots\ldots\ldots\ldots (4),$$

Differentiating (2), we get

$$\frac{d\theta}{d\phi} = -\frac{\cos\lambda}{\sin\phi\sqrt{\sin^2\lambda - \cos^2\phi}} \ldots\ldots\ldots\ldots (5).$$

Insert (4, 5) in (3), and reducing, we have

$$\sin^2\phi = u\cos\lambda \ldots\ldots\ldots\ldots (6),$$

which solves the problem when the ratio of the two motions, u, is given.

To have u a maximum, let us differentiate (6); then we get the equation

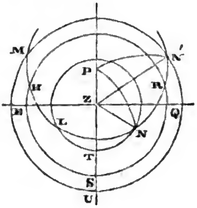
$$0 = \frac{du}{d\phi} = \frac{2\sin\phi\cos\phi}{\cos\lambda} = \frac{\sin 2\phi}{\cos\lambda}; \text{ or } \phi = \tfrac{1}{2}\pi.$$

This substituted in (6) gives $u = \dfrac{1}{\cos\lambda} = \sec\lambda$, as the greatest or least value of the ratio in question. The criterion gives

$$\frac{d^2u}{d\phi^2} = \frac{2\cos 2\phi}{\cos\lambda} = -\frac{2}{\cos\lambda} = -2\sec\lambda,$$

in the case before us. Hence the value just determined is a maximum.

11. In a given latitude to find the time when the sun passes from one given almacantar to another.



Let EHRQ be the horizon of the place, Z the zenith, P the pole of the equinoctial; N′, N, the points in which the sun's path LNR, cuts the given almacantars MUN′, LSN: it is required to find the polar distance PN′ or PN (this or the declination being supposed to not change during the time of passing through NN′), so that the arc NN′, (and consequently the time of its description,) shall be the least possible.

Put PZ (the co-latitude) $= \lambda$; $PN' = PN = \rho$, the spherical radius of the circle LNR; ZN and ZN′ those of LTN, MUN′ $= \rho_1, \rho_2$, respectively, and the angles ZPN, ZPN′ $= \theta_1, \theta_2$, respectively. Then, putting $\theta_2 - \theta_1 = \theta$,

$$\theta_1 = \cos^{-1}\frac{\cos\rho_1 - \cos\rho\cos\lambda}{\sin\rho\sin\lambda}, \text{ and } \theta_2 = \cos^{-1}\frac{\cos\rho_2 - \cos\rho\cos\lambda}{\sin\rho\sin\lambda}; \text{ or}$$

$$NN' = \theta_2 - \theta = \cos^{-1}\frac{\cos\rho_2 - \cos\rho\cos\lambda}{\sin\rho\sin\lambda} - \cos^{-1}\frac{\cos\rho_1 - \cos\rho\cos\lambda}{\sin\rho\sin\lambda}.$$

Now as the sun moves uniformly through NN′, the time of describing NN′ will be a minimum when NN′ is a minimum. Hence this, as a function of ρ, is to be a minimum. Take, therefore, the differential; and we get

$$0 = \frac{d\theta}{d\rho} = \frac{(\cos\lambda - \cos\rho_1\cos\rho)\operatorname{cosec}\rho}{\sqrt{(\sin^2\lambda - \cos^2\rho_1) + 2\cos\lambda\cos\rho_1\cos\rho - \cos^2\rho}}$$

$$-\frac{(\cos\lambda - \cos\rho_2\cos\rho)\operatorname{cosec}\rho}{\sqrt{(\sin^2\lambda - \cos^2\rho_2) + 2\cos\lambda\cos\rho_2\cos\rho - \cos^2\rho}};$$

which upon reduction by ordinary trigonometrical processes becomes,

$$\cos^2\rho - 2\cos\lambda \,.\, \frac{1+\cos\rho_1\cos\rho_2}{\cos\rho_1+\cos\rho_2}\cos\rho + \cos^2\lambda = 0 \;\ldots\ldots\ldots\; (1),$$

$$\text{or, } \cos\rho = \frac{\cos^2\frac{1}{2}(\rho_1 \mp \rho_2)\cos\lambda}{\cos\frac{1}{2}(\rho_1+\rho_2)\cos\frac{1}{2}(\rho_1-\rho_2)}.$$

$$= \frac{\cos\frac{1}{2}(\rho_1 \mp \rho_2)}{\cos\frac{1}{2}(\rho_1 \pm \rho_2)}\cos\lambda. \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

Differentiating (1) and inserting the values of ρ from (2), we get

$$\frac{d^2\theta}{d\rho^2} = \mp\frac{\sin\rho_1\sin\rho_2}{\cos\rho_1+\cos\rho_2}.\cos\lambda \;\ldots\ldots\ldots\ldots\; (3).$$

First, take the *upper signs* (which are correspondent in all the equations), and we have as the condition of the minimum, that $\sin\rho_1 \sin\rho_2 \cos\lambda$ shall have the same sign as $\cos\rho_1 + \cos\rho_2$, as an essential relation amongst the data. The result is otherwise a maximum.

Second, take the *lower signs*, and we must have the values of $\sin\rho_1 \sin\rho_2 \cos\lambda$ and $\cos\rho_1 + \cos\rho_2$ of different signs. The result is otherwise a maximum.

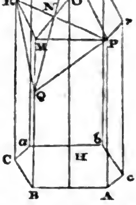
These conditions being those amongst the same functions of the same quantities, cannot be co-existent.

One of the expressions, therefore, gives the maximum, and the other the minimum time of describing the arc NN′ of right ascension; and the expressions for the maxima and minima change according to the values of λ, ρ_1, ρ_2, passing from the fulfilling of one or other of the above conditions to its opposite.

This is a generalization of a problem of some celebrity—more for the illustrious geometers who have engaged in its discussion, and the imperfection of the results generally deduced by analysis, than for any inherent difficulty in the investigation or importance to be attached to the result—*to find the time when the duration of twilight is shortest.* The problem here solved becomes that of the shortest twilight when ρ_1, $= \frac{1}{2}\pi$; and by substituting this value of ρ_1 in (2) we get the usual results.

The reader who wishes to see a further discussion of the general problem, as well as its general history, may consult a paper by the editor of this work in the *Phil. Mag.* for *Sept.* and *Oct.* 1833.

12. From a point in the axis of a right hexagonal prism, three planes are drawn, each passing through two alternate angles of the prism: it is required to find the inclinations of these planes to the axis of the prism, in order that the prism may have a given volume, and the entire surface of the figure be the least possible.



Let ABC*abc* be the base of the hexagonal prism, and PMR*pmr* the upper parallel surface; and let S be a point in the axis of the prism, from which planes, of which SRP is one, be drawn through the alternate angles P, R, to cut the edge MB, in Q. Let O be the intersection of the axis SH with the surface PMR*pmr*; and join OM, cutting RP in N, and SN, NQ.

Then, O is the centre of the hexagon PMR*pmr*; and hence MR = RO = OP = PM, and OM, RP bisect each other in N, and are at right angles to one another, by the known properties of the rhombus MROP.

Again, since SO, MQ are perpendicular to the plane PMR*pmr*, and that from O and M perpendiculars to the line RP in that plane are drawn, the lines SN, NQ will also be perpendicular to RP (*vol. i. p.* 351); and they are by construction in one plane PQRS. Whence the points S, N, Q are in one line. Also, the lines SQ, MO being in the same plane (that of the parallels SO, MQ) they intersect at N, making the angles ONS, MNQ equal.

Wherefore the triangles ONS, MNQ have ONS = MNQ, SON = QMN, and ON = NM; and hence OS = MQ.

It follows, therefore, that the triangular pyramids, QRMP, SORP, having their bases RMP. POR equal, and their altitudes QM, OS also equal, have equal volumes. And the same holds with respect to the other two pairs of pyramids.

Whence, the volume of the figure is the same, wherever in the axis of the prism, the point S be taken; and it remains to investigate when the entire surface is the least possible.

Let the side of the hexagonal base AB $= 2a$; the height of the prism AP $= b$; and the angle made by one of the cutting planes with the axes of the prism, OSN $= \theta$. Then the following equations are at once derived.

$$\mathrm{ON} = \mathrm{NM} = a;\quad \mathrm{SN} = a \operatorname{cosec} \theta;\ \text{and } \mathrm{QM} = a \cot \theta.$$

Surf. ABPQ $= \frac{1}{2}\,\mathrm{AB}\,(2\mathrm{AP} - \mathrm{MQ}) = a(2b - a \cot \theta)$

Surf. PQRS $= \mathrm{PR}\,.\,\mathrm{SN} = 4a \cos 30^\circ.\, a \operatorname{cosec} \theta = 2a^2 \sqrt{3}.\operatorname{cosec} \theta.$

Hence the surface of the entire solid being

6 ABPQ + 3 PQRS, we have to find

$$u = 6a\,(2b - a \cot \theta) + 6a^2\sqrt{3}\,.\operatorname{cosec} 0,\ \text{a } \mathit{min}.$$

$$\frac{du}{d\theta} = \operatorname{cosec}^2\theta - \sqrt{3}\,.\operatorname{cosec} \theta \cot \theta = 0,\ \text{or } \cos \theta = \frac{1}{\sqrt{3}}$$

Again,

$$\frac{d^2u}{d\theta^2} = \operatorname{cosec} \theta \left\{ \sqrt{3}\,(\operatorname{cosec}^2\theta + \cot^2\theta) - 2 \operatorname{cosec} \theta \cot \theta \right\},$$

and inserting the above value of $\cos \theta$, recollecting that the angle θ is acute, and therefore $\operatorname{cosec} \theta$ and $\cot \theta$ both +, we have

$$\frac{d^2u}{d\theta^2} = \frac{3}{\sqrt{2}}.$$

The function is, therefore, a *minimum* as required by the question.

"This is the celebrated problem of the form of the cells of bees. Maraldi was the first who measured the angles of the faces of the terminating solid angle (BQP and QPA), and he found them to be 109° 28′ and 70° 32′ respectively. It occurred to Réaumur, that this might be the form, which for the same solid content gives the minimum of surface, and he requested Kœnig to examine the question mathematically. That geometer confirmed the conjecture, the result of his calculations agreeing with Maraldi's measurements within 2′. Maclaurin and L'Huillier, by different methods verified the preceding results, excepting that they showed that the difference of 2′ was due to the calculations of Kœnig—not to a mistake on the part of the bees." *Gregory's Ex.* p. 104.

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[*] I would add, however, that I have found *many* cells which are far from being perfectly formed, even in the prismatic parts. I have seen them formed as little more than adhering cylinders—very slightly flattened at the edges of contact; and the interstices sometimes entirely void, but more frequently crossed obliquely by filmy leaves, so to say, of dry wax. The same occasionally, but less frequently, occurs with respect to the PQRS of the figure, and I have seen the plates often not plane, nor with their opposite faces parallel. These observations particularly apply (so far as I have made them) to the first year's honeycomb; but they may often be seen, though less distinctly, in combs of bees of all ages. I had no means of accurate measurement, so as to ascertain whether any variation in the inclination of the planes of the faces of the prism take place.

But this is not the place to discuss such a subject; though the circumstance is worthy of the attention of the natural historian.

EXAMPLES FOR EXERCISE.

1. The greatest rectangle inscriptible in a triangle whose base is a and altitude b, is when the breadth of the rectangle is $\frac{1}{2}b$.

2. The greatest rectangle inscriptible in a semicircle of radius a, has the sides $r\sqrt{2}$ and $\frac{1}{2}r\sqrt{2}$.

3. The fraction which the most exceeds its square is $\frac{1}{2}$; and its cube is $\sqrt{\frac{1}{3}}$.

4. To divide the angle a into two parts, such that the mth power of the sine of one part multiplied by the nth power of the sine of the other shall be a maximum.

Ans. θ is found from $\frac{\sin(a - 2\theta)}{\sin a} = \frac{n - m}{n + m}$.

5. To divide the number a into a number of equal parts, such that their continued product shall be a maximum.

Ans. x (the number of parts) $= \frac{a}{e}$, or $\left(\frac{a}{x}\right) = e^{\frac{a}{e}}$.

6. Where must a balloon rest between the earth and moon (they being spheres of radii r_1, r_2, and the distance of their centres a) so that the person in the car shall see the greatest possible quantity of spherical surface?

Ans. Dist. from centre of earth $= \frac{ar_1^{\frac{3}{2}}}{r_1^{\frac{3}{2}} + r_2^{\frac{3}{2}}}$.

7. The content of a right cone on a circular base being given, to find its form when the surface is a maximum.

Ans. $\frac{\text{altitude of cone}}{\text{radius of base}} = 2\sqrt{2}$.

8. If $a^{mx} b^{ny} = c$, then $u = (mx + n)(ny + m)$ is a maximum, when $x = \frac{\log\left(\frac{cb^m}{a^n}\right)}{\log(a^{2m})}$.

9. The greatest parabola that can be cut from a given cone is that whose vertex is distant from the vertex of the cone one-fourth of the slant side of the cone.

10. Find the point of a given ellipse most remote from the extremity of the conjugate diameter.

11. Let AQ, BR, be the opposite sides of a leaf of a book, AB, the bottom, being $= a$: it is required to crease it down so that PQ, the crease, being a minimum, the corner A shall rest upon the opposite edge BQ at R.

Ans. $AP = x = \frac{3}{4}a$.

12. When the area turned down is a minimum, then $x = \frac{2}{3}a$.

13. In the quadrilateral ABCD the base $AB = a$, $AD = BC = b$: find CD (parallel to AB), so that the area shall be a maximum.

Ans. $CD = \frac{1}{2}\{\sqrt{8b^2 + a^2} - a\}$.

14. Two ships, A and B, 360 fathoms asunder, begin to sail at the same instant, A towards the place where B starts from at the rate of 80 fathoms per minute, and B towards C, at right angles to AB, at the rate of 50 fathoms per minute: it is required to find when, after the time of starting, they will be nearest to each other.

Ans. $3^{m}\ 14{\cdot}157^{s}$.

15. The azimuth of a star, when six hours from the meridian, was 49° 20′ from the north, and the difference between the star's co-altitude and zenith distance was a maximum: required, the latitude of the place, and the star's altitude and declination.

Ans. lat. = 51° 5′ 15″ N.; alt. = 38° 54′ 45″; decl. = 53° 49′ 42″ N.

16. The difference of the expense of paving an octagonal court at a penny a foot, and palisading it at a guinea a yard was a maximum: find the area.

Ans. side = 23·1959 yards.

17. Generally for n sides, at a pence for paving and b pence for palisading, the side will be $x = \frac{b}{a} \tan \frac{\pi}{n}$.

18. Find the value of x which renders $(x^m)^{x^n}$ a minimum. *Ans.* $x = e^{-\frac{1}{n}}$.

19. Lines are drawn from the angles of a triangle to a point within it, so that their sum is a minimum: find their lengths, and the angles under which they meet.

Ans. They make angles of 120° with each other, and their values are given by the solution of *Ex.* 8, *p.* 193, *vol. i.*

20. Required the dimensions of a cylindrical tankard, to hold one quart of ale measure, that can be made of the least possible quantity of silver, of a given thickness.

21. Find that system of diameters in an ellipse which have the most oblique conjugate angle.

22. Find those diameters, the sum of which is a maximum and a minimum.

23. The radii of the cylinders which form the inner and outer surfaces of the shaft of a winding stair-case are r_1, r_2, the inclination of the ceiling of the stair-case to the horizon is α, and the height of the ceiling b: show that the longest pole that can be carried up the stair-case, is the square root of

$$\left\{b + 2r_2\varpi \tan \alpha \left(\frac{\cos^{-1} \frac{r_1}{r_2}}{\pi}\right)\right\}^2 + 4\left(r_2^2 - r_1^2\right);$$

ϖ signifying 3·14159 . . .

24. Given the length of a circular arc ($2a$) to find the radius x, such that the corresponding segment shall be a maximum. *Ans.* $x = \frac{2a}{\varpi}$.

25. Two bodies start from A and B at the same time, and move with uniform given velocities along two given lines situated in the same plane, when will they be nearest to each other, and what will their distance then be? Give also a geometrical construction.

26. Inscribe the greatest parallelogram in a *parabola;* and likewise in the *double cissoïd.*

27. The least parabola that can circumscribe a given circle, has its parameter equal to the radius of the circle.

28. Of all pyramids of the same altitude and standing on the same triangular base, to show that the one which has the greatest vertical solid angle is that which is contained by three equal plane angles.

29. If a conical glass whose altitude is a, and the generating angle be α, be filled with water, find the radius, r, of the sphere which being put into it shall cause the greatest quantity of water to overflow.

$$Ans.\ r = \frac{a \sin \alpha}{2 \sin \frac{1}{2}\left(\frac{1}{2}\pi - \alpha\right) \sin \frac{3}{2}\left(\frac{1}{2}\pi - \alpha\right)}.$$

30. From two points, both within or both without a circle, lines are drawn to a point in the circle whose sum is a *max.* or a *min.*: show that these lines make equal angles with the radius drawn to the same point in the circle.

31. Show that the angle contained by the lines drawn as in the last proposition is greatest when the sum of the lines is least, and least when the sum of the lines is the greatest.

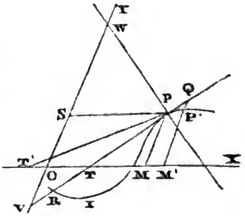
CHAPTER V.

TANGENTS, NORMALS, AND RECTILINEAR ASYMPTOTES TO CURVE LINES, REFERRED TO RECTILINEAR CO-ORDINATES.

DEFINITION.

A tangent to a curve is a straight line which meets it in one point P, and being produced does not meet it again, except (which it may possibly do) at a finite distance from that point.

Thus if P be a point in the curve P′PR, and PT a line which meets it at P, and does not meet it again, except at a finite distance PR, this line is a tangent to the curve at the point P.



PROP. I.

To find the equation of the tangent.

Let OX, OY, be the axes of rectilinear co-ordinates to which the curve $y = f(x)$ is referred.

Put $OM = x$, $MP = y$, $MM' = h$, $M'P' = y'$; also let XY be the current co-ordinates of the line PP′. Its equation will be (*p.* 256) since it passes through $(xy,\ x'y')$, $Y - y = \frac{y' - y}{x' - x}(X - x)$

$$\text{But } y' = f(x + h) = y + \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\,.\,2} + \ldots$$

$$\text{Whence } \frac{y' - y}{x' - x} = \frac{\frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\,.\,2} + \ldots}{h} = \frac{dy}{dx} + \frac{d^2y}{dx^2}\cdot\frac{h}{1\,.\,2} + \ldots$$

which at the limit, when h vanishes, becomes reduced simply to the equation

$$\frac{y' - y}{x' - x} = \frac{dy}{dx}:$$

and the equation of the tangent at the point xy of the curve, referred to the curve at co-ordinates XY of that tangent is

$$Y - y = \frac{dy}{dx}(X - x).$$

For the sake of uniformity and future convenience, the letters denoting the co-ordinates of P are marked, and the plain xy written for XY. The result thus takes the usual form given to it; viz.

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1).}$$

Corollary 1.

The quantity $\frac{dy_1}{dx_1}$ is obviously that which in the earlier part of this work has been designated by p, or $\frac{\sin\beta}{\sin(\alpha-\beta)}$, where α is the angle of ordination, and β the angle made by the line with the axis of x. When $\alpha = \frac{1}{2}\pi$, $\frac{dy_1}{dx_1}$ is the tangent of the angle PTM, which we shall call $\tan\omega_1$.

Corollary 2.

The distance MT at which the ordinate and tangent cut the axis of x, (or the subtangent,) is found as before in the conic sections, by putting $y = 0$, and deducing the value of x which corresponds to it. Thus, in (1)

If $y = 0$, we have $x - x_1 = -\frac{y_1 dx_1}{dy_1} = \text{MT}$, and $\text{OT} = x_0$[*] $= x_1 - \frac{y_1 dx_1}{dy_1}$ .... (2).

Similarly the sub-tangent on the axis of y is found to be

$$y - y_1 = -\frac{x_1 dy_1}{dx_1} = \text{SV}, \text{ and } \text{OV} = y_0 = y_1 - \frac{x_1 dy_1}{dx_1} \quad \ldots\ldots\ldots\ldots \quad (3).$$

Corollary 3.

The length of the tangent is found thus:

$$\text{PT}^2 = \text{MP}^2 - 2\text{MT}.\text{MP}\cos\text{TMP} + \text{TM}^2,$$

$$= y_1^2 + 2.\frac{y_1^2 dx_1}{dy_1}\cos\alpha + \frac{y_1^2 dx_1^2}{dy_1^2},$$

$$\text{PT} = y_1\sqrt{1 + 2\left(\frac{dx_1}{dy_1}\right)\cos\alpha + \left(\frac{dx_1}{dy_1}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots \quad (4);$$

and when the angle of ordination is $\frac{1}{2}\pi$, this becomes

$$\text{PT} = y_1\sqrt{1 + \frac{dx_1^2}{dy_1^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

Corollary 4.

The sub-tangents on the two axes of co-ordinates being found by these methods, viz. MT, SV, we readily obtain two points, T, V, in the tangent; and the line VT being drawn through the points T, V, it will be the tangent required.

Corollary 5.

When $\frac{dy_1}{dx_1} = 0$, the tangent is parallel to the axis of x, and when $\frac{dy_1}{dx_1} = \frac{1}{0}$, or $\frac{dx_1}{dy_1} = 0$, it is parallel to the axis of y; and the ordinate or abscissa respectively has attained to a *maximum or minimum value.*

---

[*] This convenient notation x_0, y_0, for the values of x and y in the equation of the *tangent* when it cuts the axes X and Y, was proposed by Mr. Gregory. In analogy to this we may employ $x_{\ominus}$, $y_{\ominus}$, for the corresponding points of the *normal*.

Corollary 6.

If p_1 denote the perpendicular from the origin of co-ordinates on the tangent to a curve $y = f(x)$ at the point $x_1 y_1$, and t_1 the portion of that tangent cut off by it, estimated from the point $x_1 y_1$, then we shall have their values expressed by

$$p_1 = \frac{y_1\, dx_1 - x_1\, dy_1}{\sqrt{dx_1^2 + dy_1^2}}, \text{ and } t_1 = \frac{x_1\, dx_1 + y_1\, dy_1}{\sqrt{dx_1^2 + dy_1^2}}.$$

Both these results are deducible from what is shown respecting the expressions for the corresponding cases at *pp*. 253, 262.

It may, moreover, be remarked, that if $u = f(x, y)$; then these lines will take the following forms for their expressions:

$$p_1 = \frac{\frac{x_1\, du_1}{dx_1} + \frac{y_1\, du_1}{dy_1}}{\sqrt{\left(\frac{du_1}{dx_1}\right)^2 + \left(\frac{du_1}{dy_1}\right)^2}}; \text{ and } t_1 = \frac{\frac{x_1\, du_1}{dy_1} - \frac{y_1\, du_1}{dx_1}}{\sqrt{\left(\frac{du_1}{dx_1}\right)^2 + \left(\frac{du_1}{dy_1}\right)^2}}.$$

PROP. II.

To find the equation of a normal at the point $x_1 y_1$.

The normal is the perpendicular PN to the tangent at $x_1 y_1$.

By *p*. 260, *eq*. 18, we have, putting a the angle of ordination, and $\frac{dy_1}{dx_1} = \tan \omega_1$,

$$y - y_1 = - \frac{1 + \frac{dy_1}{dx_1} . \cos a}{\frac{dy_1}{dx_1} + \cos a} (x - x') \ldots\ldots\ldots\ldots\ldots\ldots (6);$$

and when the angle of ordination $a = \frac{1}{2}\pi$, this reduces to, simply,

$$y - y_1 = - \frac{dx_1}{dy_1} (x - x') \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Corollary 1.

The sub-normals on the axes of x and y are found in the same way as the sub-tangents were determined. Their expressions are,

When a is not $\frac{1}{2}\pi$,	When $a = \frac{1}{2}\pi$.
$x - x_1 = y_1 \dfrac{\frac{dy_1}{dx_1} + \cos a}{1 + \frac{dy_1}{dx_1} \cos a}$	$x - x_1 = y_1 . \dfrac{dy_1}{dx_1}$
$y - y_1 = x_1 \dfrac{1 + \frac{dy_1}{dx_1} \cos a}{\frac{dy_1}{dx_1} + \cos a}$	$y - y' = x_1 \dfrac{dx_1}{dy_1}.$

Corollary 2.

The normal at $x_1 y_1$ might, therefore, be traced in the same way by the sub-normals that it was by the sub-tangents; viz. by finding x and y.

Corollary 3.

The lengths of the normals on the axes of x and y may be expressed in the same way as those of the tangents; but the result is seldom required except for rectangular ordination. They are then expressed by

$$y_1\sqrt{1+\frac{dx_1^2}{dy_1^2}}, \text{ and } x_1\sqrt{1+\frac{dy_1^2}{dx_1^2}}.$$

PROP. III.

To find whether a curve has rectilinear asymptotes, and to assign the positions of such as exist.

DEF. An *asymptote* is a tangent to the curve at a point infinitely distant from the origin, but which cuts one or both the axes of co-ordinates at a finite distance from that origin.

First method.

The equation of the tangent at x_1y_1 is $y - y_1 = \frac{dy_1}{dx_1}(x - x_1)$; and this tangent cuts the axes X, Y, in points whose distances x_0y_0 from the origin, have the values

$$x_0 = \frac{x_1dy_1 - y_1dx_1}{dy_1}, \text{ and } y_0 = \frac{y_1dx_1 - x_1dy_1}{dx_1}.$$

If when the values of x, or y, or both, are increased without limit, x_0, y_0, or both x_0 and y_0 have finite values, the straight line defined by the equation is an asymptote. If no such values of x_0 and y_0 can be found, there are no asymptotes.

This method is in practice sometimes inconvenient, and hence we give the

Second (or *Stirling's*) *method.*

If $y = f(x)$ or $x = F(y)$ can be expanded in descending integer powers of x or y respectively, so that

$$y = a_mx^m + a_{m-1}x^{m-1} + \ldots a_1x + a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \ldots$$
$$\text{or } x = a_my^m + a_{m-1}y^{m-1} + \ldots a_1y + a_0 + a_{-1}y^{-1} + a_{-2}y^{-2} + \ldots$$

then (attending, for illustration, only to the first series) when $x = \frac{1}{0}$, all the terms after a_0 become zero; and the curve

$$y = a_mx^m + a_{m-1}x^{m-1} + \ldots a_1x + a_0$$

is an asymptote to the proposed curve $y = f(x)$.

When $y = a_1x + a_0$, it is a rectilinear asymptote; and where it can be applied, it is a very simple method. It has already been employed at *p.* 322, in showing that the common hyperbola has asymptotes.

Scholium.

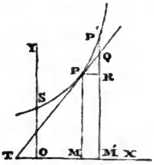
Whenever the asymptote is parallel to one of the axes of co-ordinates, Stirling's method ceases to apply; but in general, to this case the other method applies very simply. However, this can generally be detected from inspection, as in this case a particular value of one variable renders the other infinite, whilst a value greater or less, but *continuous*, renders it finite. For instance, if the curve be $y = \frac{a^2 + ax + x^2}{a - x}$, the value $x = a$ renders $y = \frac{3a^2}{0}$ = infinity: thus showing that there is an asymptote parallel to the axis of y at the distance a from the origin.

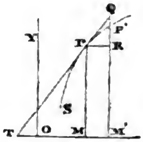
The case here cited is, in fact, that of the hyperbola, one of whose asymptotes is parallel to the axes of y.

PROP. IV.

To find the position of a curve in relation to its concavity or convexity to the axes.

Let OM, PM, be the co-ordinates of the point P in the curve P′PS, and take MM′ $= h$, and draw M′P′ to cut the tangent at Q, and PR parallel to X. Then we shall have the following systems of reducing equations:





$$\text{M}'\text{P}' = y' = y + \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\,.\,2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\,.\,2\,.\,3} + \ldots.$$

$$\text{or RP}' = y' - y = \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\,.\,2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\,.\,2\,.\,3} + \ldots.$$

$$= \text{PR tan QPR} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\,.\,2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\,.\,2\,.\,3} + \ldots.$$

$$= \text{QR} \qquad + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\,.\,2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\,.\,2\,.\,3} + \ldots.$$

$$\text{Whence P}'\text{R} - \text{QR} = \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\,.\,2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\,.\,2\,.\,3} + \ldots.$$

And by diminishing h, *sine limite,* the value of the series will have the same sign as the first term $\frac{d^2y}{dx^2}$.

Whence, when $\frac{d^2y}{dx^2}$ is $+$, P′R is greater than QR, and the curve lies above the tangent, or *is convex to the axis of x* (*fig.* 1).

When $\frac{d^2y}{dx^2}$ is $-$, P′R is less than QR, and the point P′ lies below the tangent, or the *curve is concave to the axis of x* (*fig.* 2).

EXAMPLES.

1. Find the equation of the tangent at a given point x_1y_1 of the *general curve* denoted by $\text{A}y^2 + \text{B}xy + \text{C}x^2 + \text{D}y + \text{E}x + \text{F} = 0$.

Differentiate and dash the variables to mark the point of contact $x'y'$; then

$$p = \frac{dy_1}{dx_1} = -\frac{2\text{C}x_1 + \text{B}y_1 + \text{E}}{2\text{A}y_1 + \text{B}x_1 + \text{D}};$$

which is precisely the result obtained at *page* 301, by a much longer and more laborious process.

2. The equation of the *parabola* referred to two tangents as axes of co-ordinates is known to be

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1;$$

show that if x_0 and y_0 are the sub-tangents on X and Y, then we shall have

$$\frac{x_0}{a} + \frac{y_0}{b} = 1.$$

3. The sub-tangent and subnormal of the *cissoïd* on the axis of x are respectively $x_0 = \frac{x_1\,(2a - x_1)}{3a - x_1}$, and $x_{\ominus} = \frac{(3a - x_1)\,x_1^2}{(2a - x_1)^2}$.

4. The sub-tangent and sub-normal of the *witch* are $-\frac{2ax_1 - x_1^2}{a}$, and $-\frac{4a^3}{x_1^2}$.

5. Show that the sub-tangent of the *parabola* on its principal axis is double the abscissa.

6. The equation of a tangent and the value of the sub-tangent to the *conchoïd* at x_1y_1 are respectively,

$$y - y_1 = -\frac{ab^2 + y_1^3}{y_1^2\sqrt{b^2 - y_1^2}}\,(x - x_1)\,;\ \text{and}\ x_0 = \frac{ab^2 + y_1^3}{y_1\sqrt{b^2 - y_1^2}}.$$

7. Show that any tangent to the hyperbola cuts from the asymptotic space, a triangle of constant area.

8. The sub-tangent of the *logarithmic curve* is $x - x_1 = a$; the tangent $= \sqrt{a^2 + y_1^2}$; the subnormal $= \frac{y_1^2}{a}$; the normal $= \frac{y_1\sqrt{a^2 + y_1^2}}{a}$.

$$\text{Also}\ p_1 = \frac{y_1\,(a - x_1)}{\sqrt{a^2 + y_1^2}},\ \text{and}\ t_1 = \frac{y_1^2 + ax_1}{\sqrt{a^2 + y_1^2}}.$$

9. The sub-tangent to the *common cycloïd* referred to its vertex is $\frac{y_1x_1}{\sqrt{2ax_1 - x_1^2}}$; and the tangent itself is parallel to the chord of the circle on the axis which the abscissa x_1 has been taken.

Also, $y_0 = y_1 - \sqrt{2ax_1 - x_1^2}$; $p_1 = y_0\left(\frac{x_1}{2a}\right)^{\frac{1}{2}}$; normal $= y_0\left(\frac{2a}{x_1}\right)^{\frac{1}{2}}$

10. The normal to the curve $y^2 = 4ax$ is a tangent to the curve $27ay^2 = 4\,(x - 2a)^3$; and the two curves intersect at a point where $x = 8a$.

11. Find the asymptotes of the following curves:

The *witch;* the *cissoïd;* the *conchoïd;* the *figures of tangents, co-tangents, secants, and co-secants.*

12. Find the asymptotes of the curves expressed by the equations:

$$y^2 = \frac{x^3 + ax^2}{x - a}\,;\ y = \frac{x^3 - 3ax^2 + a^3}{x^2 - 3bx + b^2}\,;\ \text{and}\ y = \frac{x^3}{a\,(x - b)}\,;$$

also, of those expressed by

$$ay^3 - bx^3 + c^2xy = 0;\ \text{and}\ a^2y^2 - 2b^3y - x^4 = 0.$$

13. Show that the following curves have no rectilinear asymptotes:

The *lemniscate;* the *cycloïd family;* the *logarithmic curve;* the *figures of sines and cosines;* the *catenary;* and *Watts's parallel motion.*

14. Show that in the catenary whose equation is $y = \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}$, we have

$$\text{tangent} = \frac{y_1^2}{\sqrt{y_1^2 - a^2}} \qquad \text{sub-tangent} = \frac{ay_1}{\sqrt{y_1^2 - a^2}}$$

$$\text{normal} = \frac{y_1^2}{a} \qquad \text{sub-normal} = \frac{a}{4}\left\{e^{\frac{2x}{a}} + e^{-\frac{2n}{a}}\right\}.$$

15. The curve $y^4 - x^4 + 2bx^2y = 0$, has asymptotes $y = \frac{1}{2}\,b \pm x$.

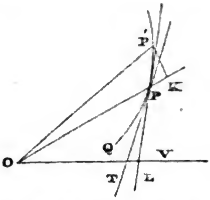
16. Also, $ay^3 - bx^3 + c^2xy = 0$; has asymptotes $y = \left\{\frac{b}{a}\right\}^{\frac{1}{3}}\left\{x - \frac{c^2}{3(ab^2)^{\frac{1}{3}}}\right\}$

CHAPTER VI.

THE TANGENTS, NORMALS, AND ASYMPTOTES OF CURVES REFERRED TO POLAR CO-ORDINATES.

PROP. I.

To find the angle made by the radius-vector and tangent at a given point in a curve referred to polar co-ordinates.



Let P be the given point, and P' another near it, in the curve P'PQ; and let $r\theta$, $r'\theta'$, be co-ordinates of the points P, P' in reference to the pole O and polar axis OV. Then putting P'OP $= h$, we have

$$r' = r + \frac{dr}{d\theta} \cdot \frac{h}{1} + \frac{d^2r}{d\theta^2} \cdot \frac{h^2}{1 \,.\, 2} + \dots.$$

Draw P'K perpendicular to OP (produced when necessary): then we have

$$\tan \text{OPL} = \tan \text{P'PK} = \frac{\text{KP}'}{\text{KP}} = \frac{\text{KP}'}{\text{OK} - \text{OP}} = \frac{\text{OP}' \sin h}{\text{OP}' \cos h - \text{OP}} = \frac{r' \sin h}{r' \cos h - r}$$

$$= \frac{r' \sin h}{r' - r - 2r' \sin^2 \frac{1}{2}h} = \frac{r'}{\dfrac{r' - r}{\sin h} - r' \tan \frac{1}{2}h}.$$

Now the limit of this value of tan OPL, or that where P'P coalesce and P'P becomes a tangent, is evidently

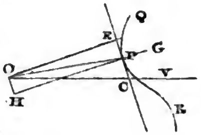
$$\tan \omega = \frac{r}{\dfrac{dr}{d\theta}} = \frac{r d\theta}{dr}.$$

Whence, accenting $r_1\theta_1$ to denote a given point on the curve, and putting OPT $= \omega$, we have

$$\tan \omega = \frac{r_1 d\theta_1}{dr_1}.$$

PROP. II.

To find the equation of the tangent of the point $r_1\theta_1$ of a curve referred to polar co-ordinates.



Let EC be the tangent at the point P, referred as in the preceding proposition. Then the general form of its equation is (*p.* 264),

$$r = p \sec (\theta - \beta) \dots\dots\dots\dots (1).$$

Also, by the preceding,

$$\text{OPE} = \tan^{-1} \left(- \frac{r_1 d\theta_1}{dr_1} \right); \text{ and hence,}$$

$$\text{OE} = r_1 \sin \tan^{-1} \left(- \frac{r_1 d\theta_1}{dr_1} \right) = p \dots\dots (2).$$

Again,

$$\text{EOC} = \tfrac{1}{2}\pi - \text{EPO} + \theta_1 = \beta, \text{ or}$$

$$\beta = \tfrac{1}{2}\pi - \tan^{-1}\left(-\frac{r_1 d\theta_1}{dr_1}\right) + \theta_1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Expand (1), and in the result write the values of sin β, cos β obtained from (3): then after slight reduction

$$\frac{p}{r} = -\frac{r_1 d\theta_1 \cos(\theta - \theta_1) - dr_1 \,.\, \sin(\theta - \theta_1)}{\sqrt{r_1{}^2 d\theta_1{}^2 + dr_1{}^2}} \ldots\ldots\ldots\ldots\ldots (4).$$

But from (2) we get

$$p = \frac{-r_1{}^2 d\theta_1}{\sqrt{r_1{}^2 d\theta_1{}^2 + dr_1{}^2}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

This inserted in (4) gives at once the equation required,

$$r\left\{\cos(\theta - \theta_1) - \sin(\theta - \theta_1)\frac{dr_1}{r_1 d\theta_1}\right\} = r_1 \ldots\ldots\ldots\ldots (6),$$

$$\text{or } \frac{r_1}{r} = \cos(\theta - \theta_1) - \sin(\theta - \theta_1) \,.\, \frac{dr_1}{r_1 d\theta_1} \quad\ldots\ldots\ldots\ldots\ldots (7),$$

both of which equations are in a form convenient for use.

Corollary 1.

The perpendicular on the tangent is $\text{OE} = \dfrac{r_1 d\theta_1}{\sqrt{r_1{}^2 d\theta_1{}^2 + dr_1{}^2}}$.

Corollary 2.

If p be the perpendicular on the tangent, we have by simple substitution,

$$p = \frac{r^2}{\sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}}, \text{ or } \frac{d\theta}{dr} = \frac{p}{r\sqrt{r^2 - p^2}}.$$

If, therefore, we can find $\dfrac{d\theta}{dr}$ from the equation of the curve, in terms of r and constants only, we shall have the equation of the curve expressed as a relation between r and p. In many cases this is useful; and is, in fact, the only method of treating polar curves which has been generally known and employed.

PROP. III.

To find the equation of the normal at the point $r_1\theta_1$, a curve referred to polar co-ordinates.

This might be found by employing *eq.* 18, *p.* 267, for the equation of the perpendicular through $r_1\theta_1$ to the tangent whose equation has been just found: but as it may be derived at once, and independently, by a method analogous to the preceding, and gives somewhat less trouble in reducing, the latter method is here employed. *See figure to last proposition.*

Draw OH perpendicular to the normal GPH: then we have

$$\text{HOP} = \tan^{-1}\frac{dr_1}{r_1 d\theta_1}, \text{ (since HOP + OPC} = \pi.\text{), or as before,}$$

$$\beta = \tfrac{1}{2}\pi - \tan^{-1}\frac{dr_1}{r_1 d\theta_1} + \theta_1.$$

Substitute sin β, cos β, derived from this in the general equation of a line $r = p \cos(\theta - \beta)$; and there results

$$\frac{p}{r} = \frac{\cos(\theta - \theta_1)\, dr_1 + \sin(\theta - \theta_1)\, r_1 d\theta_1}{\sqrt{r_1^2 d\theta_1^2 + dr_1^2}} \text{ .... (1).}$$

But $p = r_1 \sin \tan^{-1} \dfrac{dr_1}{r_1 d\theta_1} = \dfrac{r_1 dr_1}{\sqrt{r_1^2 d\theta_1^2 + dr_1^2}}$; which converts (1) into

$$r \left\{ \sin(\theta - \theta_1) + \cos(\theta - \theta_1) \frac{dr_1}{r_1 d\theta_1} \right\} = r_1 \cdot \frac{dr_1}{r_1 d\theta_1} \text{ .... (2),}$$

$$\text{or, } \frac{r_1}{r} = \cos(\theta - \theta_1) + \sin(\theta - \theta_1) \cdot \frac{r_1 d\theta_1}{dr_1} \text{ .... (3),}$$

both of which are convenient forms for use.

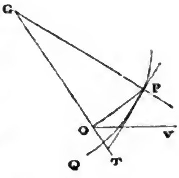
Scholium.

This and the preceding proposition were first solved by the editor of this work in a paper on *Spherical Co-ordinates* in the *Edinburgh Transactions, vol. xii. pp.* 407—9, dated February 1833. The former of them was subsequently deduced in a somewhat different manner by Professor Avery, in Gill's *Mathematical Miscellany,* No. *vii,* 1839; but of the latter, there has not been, it is believed, any other investigation than the one above, yet published.

These are the equations referred to in the note, *p.* 331 of this volume.

PROP. IV.

To find the subtangent and subnormal of a given point in a curve referred to polar co-ordinates.



Def. If a perpendicular GT be drawn through the pole O, to meet the tangent and normal in T and G respectively; then PT is the *tangent;* PG is the *normal;* OT is the *subtangent;* and OG is the *subnormal.*

We have from the right angled triangles of the figure, and the value of tan OPT already found;—

$$\textit{Subtangent, } \mathrm{OT} = \mathrm{OP} \tan \mathrm{OPT} = r_1 \cdot \frac{r_1 d\theta_1}{dr_1} = \frac{r_1^2 d\theta_1}{dr_1} \text{ .... (1).}$$

$$\textit{Subnormal, } \mathrm{OG} = \mathrm{OP} \cot \mathrm{OPT} = r_1 \cdot \frac{dr_1}{r_1 d\theta_1} = \frac{dr_1}{d\theta_1} \text{ .... (2).}$$

$$\textit{Tangent, } \quad \mathrm{PT} = \sqrt{\mathrm{OP}^2 + \mathrm{OT}^2} = r_1 \sqrt{1 + \left(\frac{r_1 d\theta_1}{dr_1}\right)^2} \text{ .... (3).}$$

$$\textit{Normal, } \quad \mathrm{PG} = \sqrt{\mathrm{OP}^2 + \mathrm{OG}^2} = \sqrt{r_1^2 + \left(\frac{dr_1}{d\theta_1}\right)^2} \text{ .... (4).}$$

Corollary.

By these equations we can draw a tangent or normal to any given point $r_1\theta_1$ of a polar curve, in nearly the same manner as when rectilinear co-ordinates are employed.

Scholium.

The subnormal and subtangent on OV, and its perpendicular through O, might have been found with equal facility; but it would, in fact, be only giving polar expressions for the magnitudes already found by rectangular co-ordinates, and hence would, probably, in most cases, be superfluous.

PROP. V.

To transform the equation of a curve referred to rectilinear co-ordinates to an equation referred to polar co-ordinates.

As far as the equation is concerned with the co-ordinates, apart from their differentials, this problem has already been solved at *pp.* 250—1. We shall, therefore, suppose that the two preliminary steps are taken: viz. transformation to the origin of polar co-ordinates, and transformation to rectangular axes, one of which is the original X.

Let then β be the angle which the polar axis makes with X, and supposed positive: then we have, by what is there shown,

$$y = r\sin(\theta-\beta),\text{ and } x = r\cos(\theta-\beta),$$
$$dy = r\cos(\theta-\beta)\,d\theta + \sin(\theta-\beta)\,dr,$$
$$dx = -r\sin(\theta-\beta)\,d\theta + \cos(\theta-\beta)\,dr;\text{ and hence}$$
$$\frac{dy}{dx} = \frac{rd\theta + \tan(\theta-\beta)\,dr}{dr - \tan(\theta-\beta)\,rd\theta} = \frac{\dfrac{rd\theta}{dr} + \tan(\theta-\beta)}{1 - \tan(\theta-\beta)\,.\,\dfrac{rd\theta}{dr}}.$$

By differentiating again, we have

$$d^2y = \sin(\theta-\beta)\,d^2r + 2\cos(\theta-\beta)\,drd\theta - r\sin(\theta-\beta)\,d^2\theta,$$
$$d^2x = \cos(\theta-\beta)\,d^2r - 2\sin(\theta-\beta)\,drd\theta - r\cos(\theta-\beta)\,d^2\theta:$$

and the process may evidently be continued to any required extent.

Scholium.

By a corresponding method, though a little more complex in the reductions, the converse problem may be solved, making use of the fundamental equation (14), *p.* 252; and still better by equation (15), making the transformation for β afterwards in the rectangular equation thus deduced.

PROP. VI.

To find the asymptotes of polar curves.

1. *Rectilinear asymptotes.* These will evidently depend upon the circumstance of the subtangent OT being finite when the radius vector OP is infinite.

Hence assume r infinite, and find the values of θ and $\frac{r^2d\theta}{dr}$ which simultaneously fulfil the condition and are infinite. The first gives the position of the line to which the asymptote is parallel, and the second gives the distance upon the perpendicular OT from the origin, at which that asymptote is situated.

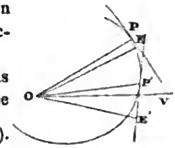
2. *Circular asymptotes.* These depend upon θ being infinite and giving r a finite value. Then r is the radius of the asymptotic circle.

PROP. VII.

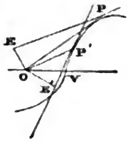
To determine the direction of curvature of a polar curve with respect to its radius vector.

Since p, the perpendicular on the tangent, is a function of $r\theta$, and that θ is, a function of r, therefore p is a function of r, or $p = f(r)$.

When the spiral is *concave towards the pole*, p and r (as is obvious from the slightest attention to the figure) increase or decrease together; and hence $\frac{dr}{dp}$ must be *positive* (fig. 1).



In like manner, when it is convex towards the pole, r increases while p diminishes, and *vice versa*, and hence $\frac{dr}{dp}$ must be *negative* (fig. 2.)



EXAMPLES.

1. In the *spiral of Archimedes*, expressed by $r = a\theta$, we have the equation

$$\omega_1 = \tan^{-1}\frac{r_1 d\theta_1}{dr_1} = \tan^{-1}\frac{r_1}{a}.$$

$$\text{subtangent} = r_1^2 \cdot \frac{d\theta_1}{dr_1} = \frac{r_1^2}{a}.$$

2. The *hyperbolic spiral* is expressed by $r = a\theta^{-1}$; and we have

$$\text{subtangent} = r_1^2 \frac{d\theta_1}{dr_1} = a^2\theta^{-2}\frac{d\theta}{-a\theta^{-2}d\theta} = -a.$$

This negative value signifies that the subtangent is measured the contrary way from that which in the figure, we employed in the general investigation, it was designed to represent it. This is analogous to the negative values of the rectilinear co-ordinates.

It is often, however, found convenient in *spirals*, properly so called, to put $r = \frac{1}{u}$; as it frequently simplifies the dependent reductions. In such case the value of the subtangent is simply $r^2\frac{d\theta}{dr} = -\frac{d\theta}{du}$.

Applying this to the example before us, we have

$$u = \frac{\theta}{a}; \text{ and the subtangent, } -\frac{d\theta}{du} = -a, \text{ as before.}$$

Since the subtangent is constant, the locus of its extremity is a circle; and $\theta = 0$ renders $r = \frac{1}{0}$ or infinite, it follows that a line drawn parallel to the axis at the distance a is an asymptote.

3. The *lituus* is expressed by $r = a\theta^{-\frac{1}{2}}$, or $u = \frac{\theta^{\frac{1}{2}}}{a}$. Then $\tan\omega_1 = -2\theta_1$, and the subtangent $= 2a\theta_1^{\frac{1}{2}}$.

The value $\theta = 0$ gives r infinite; and hence the polar axis is an asymptote.

4. *The logarithmic spiral* is expressed by $r = ae^{\frac{\theta}{a}}$: show that

(1). $\omega = \tan^{-1} a$, or constant: whence it is often called the *equiangular spiral.*

(2). $p = r \sin \tan^{-1} a = \dfrac{ra}{\sqrt{1 + a^2}}$.

(3). Subtangent $= ra$.

In spirals of the class already noticed: viz. the algebraic and exponential, the equations of the tangent and normal lead to no results worthy of notice. It is otherwise with the following curves.

5. The *lemniscate of Bernoulli* is expressed by $r^2 = a^2 \cos 2\theta$. At the point $r_1\theta_1$, we have

(1). $\omega_1 = \tan^{-1} \dfrac{r_1 d\theta_1}{dr_1} = \tan^{-1} (-\cot 2\theta_1) = 2\theta_1 - \frac{1}{2}\pi$.

Also, $\dfrac{dr_1}{r_1 d\theta_1} = -\dfrac{a^2 \sin 2\theta_1}{r_1^{\,2}} = -\dfrac{a^2 \sin 2\theta_1}{a^2 \cos 2\theta_1} = -\tan 2\theta_1$; and the equations of the *tangent* and *normal*, and the length of the perpendicular become respectively,

$$(2).\ r \cos (\theta - 3\theta_1) = r_1 \cos 2\theta_1.$$

$$(3).\ r \sin (\theta - 3\theta_1) = r_1 \sin 2\theta_1.$$

$$(4).\ p = r_1 \cos 2\theta_1 = r_1 . \frac{r_1^{\,2}}{a^2} = \frac{r_1^{\,3}}{a^2}.$$

6. The *Cardioïd* is expressed by $r = a (1 - \cos \theta) = 2a \sin^2 \frac{1}{2}\theta_1$, where a is the diameter of one of the circles. Then we have by a similar process,

(1). $\omega_1 = \frac{1}{2}\theta_1$.

(2). $r \sin (\frac{3}{2}\theta_1 - \theta) = r_1 \sin \frac{1}{2}\theta_1$ is the equation of *tangent.*

(3). $r \cos (\frac{3}{2}\theta_1 - \theta) = r_1 \cos \frac{1}{2}\theta_1$ .... *normal.*

(4). $p = r_1 \sin \frac{1}{2}\theta_1 = 2a \sin^3 \frac{1}{2}\theta_1$.

(5). If r' be the radius-vector produced backwards to meet the curve: then $r' = a \{1 - \cos (\theta + \pi)\} = a (1 + \cos \theta)$; and hence $r + r' = 2a =$ twice the diameter of either circle. Hence all lines through the pole, which are terminated both ways by the curve, are equal: a property which it possesses in common with the circle.

For an investigation of the general family of curves which possesses this property, the reader may consult Gergonne's *Annales des Mathematiques*, or Leybourn's *Mathematical Repository.*

7. Let the spiral be that, whose equation is $r^n = a^n \sin n\theta$.

(1). $\tan \omega_1 = \tan n\theta_1$, or $\omega_1 = n\theta_1$.

(2). $r \cos \{\theta - (n + 1)\theta_1\} = \dfrac{r_1^{\,n+1}}{a^n}$ is the *tangent.*

(3). $r \sin \{\theta - (n + 1)\theta_1\} = r_1 \cot n\theta_1$...*normal.*

(4). $p = -\dfrac{r^{n+1}}{a^n}$.

(5). If the radius vector be produced, so as to change θ_1 into $\pi + \theta_1$, have $\omega_2 = n (\theta_1 + \pi)$, or $\omega_2 - \omega_1 = n\pi$. The difference of the angles made by any chord through the focus with the tangents at its extremities is, therefore, constant.

8. Let the *ellipse* referred to its focus be taken as the spiral.

In the same way we may obtain the other equations of the conic sections referred to the centre or focus, which are tabulated at *p.* 330.

9. In the *hyperbola* referred to its focus as pole, to find the asymptote.

If $r = \frac{1}{0}$, then $r = \frac{a\,(e^2 - 1)}{1 - e\cos\theta}$ becomes $\cos\theta = \frac{1}{e}$, which gives the position of the infinite radius vector.

The polar sub-tangent in this case becomes $-a\sqrt{c^2 - 1}$, and hence the asymptote is determined.

10. In the *cissoïd*, viewed as a spiral curve, viz. $r = \frac{2a\sin^2\theta}{\cos\theta}$, we have, when $r = \frac{1}{0}$, $\cos\theta = 0$, or $\theta = \frac{1}{2}\pi$.

The sub-tangent, when r is infinite, viz. $\frac{2a\sin^2\theta}{2 - \sin^2\theta}$, becomes, since $\theta = \frac{1}{2}\pi$, simply $2a$; whence the asymptote can be drawn.

11. Is the spiral of Archimedes convex or concave to the radius vector?

Here $p = \frac{r^2}{\sqrt{r^2 + a^2}}$; $\frac{dp}{dr} = \frac{r\,(2a^2 + r^2)}{(a^2 + r^2)^{\frac{3}{2}}}$; which is always positive for r positive, and negative for r negative. Under these circumstances respectively, it is concave or convex towards the radius vector.

Note. In the ordinary discussion of this curve, there are only considered the positive values of r; but negative ones being admissible, the entire curve is not, therefore, usually described.

12. Is the hyperbola referred to its centre convex or concave to the radius vector?

$$r = a\sqrt{\frac{e^2 - 1}{e^2\cos^2\theta - 1}};\ p = \frac{ab}{\sqrt{r^2 - a^2 + b^2}};\ \frac{dp}{dr} = -\frac{abr}{(r^2 - a^2 + b^2)^{\frac{3}{2}}};$$

which being negative, the curve is *convex* to the radius vector throughout.

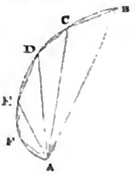
13. The tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ intercepted by the axis of co-ordinates, is of constant length, and less than any other line drawn through the point of contact.

CHAPTER VII.

THE DIFFERENTIALS OF THE LENGTHS AND AREAS OF CURVES BOTH IN REFERENCE TO RECTANGULAR AND POLAR CO-ORDINATES.

PROP. I.

Any finite portion of a curve line is greater than the chord joining its extremities: but the arc is less than the sum of the two tangents at its extremities.



1. Let AB be the chord, and take any point C in the curve AB, and join BC, CA: then AC + CB is greater than AB.

Take a point D in the curve AC, and join CD, DA: then CD + DA is greater than CA; and hence still more is BC + CD + DA greater than AB.

Take other points E, F, *etc.*: then successively the expressions

BC + CD + DE + EA,
BC + CD + DE + EF + FA, *etc.*,

more and more exceed the chord BA.

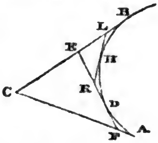
Hence, as we divide the arc AB into more and more minute portions, we find that the inscribed polygons acquire greater and greater perimeters, or that the sum of all the sides but AB exceed AB by greater and greater magnitudes. If, therefore, we suppose the summits of the polygon to become continuously consecutive, these two circumstances will simultaneously take place:—

(1). The polygon will become identical with the curve, and therefore equal to it in length.

(2). The perimeter of the polygon will be the greatest possible under the given conditions.

Wherefore it follows that the arc of the curve, which is equal to the greatest polygon, is greater than the chord AB.

2. Let AC, BC be the tangents at the extremities of the arc AB: then the arc AB is less* than AC + CB.



For draw any tangent EF at a point D intermediate to A, B, as in the figure: then

BC + CA is greater than BE + EF + FA.

Again, draw LK a tangent at a point H intermediate to DB; then

BE + EF + FA is greater than BL + LK + KF + FA;

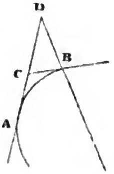
and so on, the perimeter of the new polygon being continually less than that of the preceding one, and hence becoming more and more deficient from AC + CB.

But if we conceive, as in the preceding case, the summits of the polygon to become continuously consecutive, the perimeter of the polygon will be the least possible, and during the progress towards that state, becoming more and more defective from the sum of the two tangents AC, CB.

Corollary.

If a line BD be drawn from one extremity B, to cut the tangent AC at the other extremity in D, and not to meet the arc AB in the interval, then AD + DB is greater than the arc itself. For

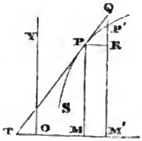
$$AD + DB = AC + CD + DB > AC + CB.$$



PROP. II.

Of the arc, chord, and tangent the tendent ratio of any one to each of the others is a ratio of equality.

Let O be the origin of rectangular co-ordinates, P a point in the curve, PT the tangent at P, and P′ another point in the curve. Draw the ordinates PM, P′M′, the latter meeting PT in Q; and draw PR parallel to the axis of x.



Denote the arc, generally estimated from any specified point in it, by s, and the increment MM′ of x by h.

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* In this the arc is supposed to have no changes of flexure, (that is, turning its concavity different ways with respect to the tangent at any point intermediate to A and B) within the given finite limits. For the cases in which we shall require the use of the proposition, this condition is *always* fulfilled, as our object will only lead us to consider it *at the limit.*

Also, let tan QPR = tan QTM = tan $\omega = v$. Then, $\mathrm{RP'} = y_1 - y$; and we get successively.

$$\mathrm{RQ} = hv, \text{ and } \mathrm{PQ} = \sqrt{h^2 + h^2v^2} = h\sqrt{1 + v^2}, \text{ and}$$

$$\frac{\mathrm{PQ}+\mathrm{QP'}}{\mathrm{PP'}} = \frac{h\sqrt{1 + v^2} + hv - (y_1 - y)}{\sqrt{h^2 + (y_1 - y)^2}} = \frac{v + \sqrt{1 + v^2} - \frac{y_1 - y}{h}}{\sqrt{1 + \left(\frac{y_1 - y}{h}\right)^2}}.$$

Now, at the *limit* this becomes sinply, since $\left[\frac{y_1 - y}{h}\right] = \frac{dy}{dx} = v$,

$$\left[\frac{\mathrm{PQ} + \mathrm{QP'}}{\mathrm{PP'}}\right] = \frac{v + \sqrt{1 + v^2} - v}{\sqrt{1 + v^2}} = 1.$$

Now PQ + QP′ is greater, and the chord PP′ is less than the arc PP′ for all *finite* values of the arc PP′; and as at the limit they become equal to each other, the arc which always in respect of magnitude lies between them, will then also be in a ratio of equality with each of them. Wherefore

$$\left[\frac{\text{arc PP'}}{\text{chord PP'}}\right] = \left[\frac{\text{arc PP'}}{\text{tangent PQ}}\right] = \left[\frac{\text{chord PP'}}{\text{tangent PQ}}\right] = 1.$$

PROP. III.

To find the differential of the arc of a curve line in rectangular co-ordinates.

SINCE the chord $\mathrm{PP'} = \sqrt{(x_1 - x)^2 + (y_1 - y)^2}$;
and taking the limits

$$[\mathrm{PP'}] = [\sqrt{(x_1 - x)^2 + (y_1 - y)^2}], \text{ or } ds = \sqrt{dx^2 + dy^2}$$

which is the differential required: and it may be written

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

PROP. IV.

To find the differential of the area of a curve line in rectangular co-ordinates.

BECAUSE, the increment $\mathrm{MM'} = x' - x = h$, therefore we have $\mathrm{M\overset{\frown}{PP'}M'}$ the increment of the area. Or in symbols, S representing the area, we have

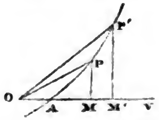
$$\left[\frac{\mathrm{S}_1 - \mathrm{S}}{x_1 - x}\right] = \left[\frac{\mathrm{M\overset{\frown}{PP'}M'}}{\mathrm{MM'}}\right] = \left[\frac{\text{quadrilateral MPP'M'}}{\mathrm{MM'}}\right]$$

$$= \left[\frac{(\mathrm{PM} + \mathrm{P'M'})\mathrm{MM'}}{\mathrm{MM'}}\right] = \left[\frac{y_1 + y}{2}\right] = y.$$

$$\text{Whence } \frac{d\mathrm{S}}{dx} = y, \text{ or } d\mathrm{S} = ydx.$$

PROP. V.

To find the differential of the length of a curve referred to polar co-ordinates.



LET O be the pole, P a point in the curve, and P′ a point adjacent to it; and draw PM, P′M′ perpendicular to the polar axis OV; and let A, the intersection of the curve with the polar axis, be the origin of the arc.

Then,

$\mathrm{OP} = r$, $\mathrm{OP}' = r_1$, $\mathrm{POV} = \theta$, $\mathrm{P'OV} = \theta_1$, $\theta_1 - \theta = h$, $\mathrm{AP} = \mathrm{S}$, $\mathrm{AP'} = \mathrm{S}'$; and we have

$$\left[\frac{\mathrm{S}' - \mathrm{S}}{\theta' - \theta}\right] = \left[\frac{\text{arc PP}'}{h}\right] = \left[\frac{\text{chord PP}'}{h}\right]$$

$$= \left[\frac{\sqrt{r_1^2 - 2rr_1 \cos h + r^2}}{h}\right] = \left[\frac{\sqrt{(r_1 - r)^2 + 2r_1 r\,(1 - \cos h)}}{h}\right]$$

$$= \left[\sqrt{\left(\frac{r_1 - r}{h}\right)^2 + rr_1\left(\frac{\sin \frac{1}{2}h}{\frac{1}{2}h}\right)^2}\right]; \text{ and hence}$$

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dr^2}{d\theta}\right) + r^2}, \text{ and } ds = \sqrt{dr^2 + r^2 d\theta^2}.$$

PROP. VI.

To find the differential of the area of a polar curve.

For $\left[\frac{\mathrm{S}_1 - \mathrm{S}}{\theta_1 - \theta}\right] = \left[\frac{\text{area POP}'}{h}\right] = \left[\frac{\text{trian POP}'}{h}\right] = \left[\frac{rr_1 \sin h}{2h}\right]$;

or $\frac{d\mathrm{S}}{d\theta} = \frac{r^2}{2}$, and $d\mathrm{S} = \frac{1}{2}r^2 d\theta$.

EXAMPLES.

1. Find the differential of the arc of a circle, to radius a, in rectangular co-ordinates.

Here, referred to the centre, $y = \sqrt{a^2 - x^2}$; whence

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \frac{x^2}{a^2 - x^2}} = \frac{adx}{\sqrt{a^2 - x^2}}.$$

2. Find the differential of the area of the parabola referred to rectangular co-ordinates.

Here $y^2 = 4ax$; and $d\mathrm{S} = ydx = 2a^{\frac{1}{2}}x^{\frac{1}{2}}dx$.

3. Find differentials of the length and area of the spiral of Archimedes.

$$ds = d\theta\sqrt{r^2 + a^2} = ad\theta\,.\sqrt{1 + \theta^2}; \text{ and } d\mathrm{S} = \tfrac{1}{2}a^2\theta^2 d\theta.$$

4. The differentials of the arcs of an ellipse and parabola are

$$ds = dx\left\{1 + \frac{x^2(1 - e^2)}{a^2 - x^2}\right\}^{\frac{1}{2}} \text{ and } \frac{dy}{2a}\left\{4a^2 + y^2\right\}^{\frac{1}{2}}.$$

5. The length of a cycloïdal arc has the differential

$$ds = dx\left(\frac{2r}{x}\right)^{\frac{1}{2}}.$$

6. Find the differentials of the conchoïdal area, and those of the lengths and areas of the witch and cissoïd, the first and third in reference to polar and rectangular co-ordinates.

Scholium.

It is always to be understood, that the differential which can be most easily traced back to its primitive function is the most desirable in all the applications of these inquiries: and it is likewise to be kept in mind, that the differential expression is to be converted into a function of only one of the variables, either x or y, in rectangular co-ordinates, or either r or θ (or r or p) in polar ones. Moreover, the expressions given in the text in x and y are referred to rectangular co-ordinates only: but if they be required for an oblique angle of ordination a, they must undergo the following obvious changes:

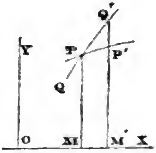
$$ds = \sqrt{dx^2 + 2dxdy \cos a + dy^2}, \text{ and } d\mathrm{S} = ydx \sin a.$$

CHAPTER VIII.

OSCULATION AND CURVATURE, WITH INVOLUTES AND EVOLUTES OF CURVES, REFERRED TO RECTANGULAR CO-ORDINATES.

PROP. I.

To find an expression for the difference of the ordinates of two curves at a point in the region of their intersection.



Let two curves PP', QQ', which are referred to the same rectangular axes, and intersect in P, be denoted respectively by the general equations,

$$y = f(x), \text{ and } y = \phi(x).$$

Let a, b be the co-ordinates of the point of intersection P. Then $b = f(a)$ and $b = \phi(a)$: and hence $f(a) = \phi(a)$.

Now, if we take another absciss OM', the ordinates are no longer equal: let, then, M'P' $= y_1$ and M'Q' $= y_2$: then supposing MM' $= h$, we have the general form

$$y_1 = f(x) + \frac{df(x)}{dx} \cdot \frac{h}{1} + \frac{d^2f(x)}{dx^2} \cdot \frac{h^2}{1 \cdot 2} + \ldots\ldots\ldots$$

$$y_2 = \phi(x) + \frac{d\phi(x)}{dx} \cdot \frac{h}{1} + \frac{d^2\phi(x)}{dx^2} \cdot \frac{h^2}{1 \cdot 2} + \ldots\ldots\ldots$$

Whence, since as above $f(x) = \phi(x)$ in the case supposed, we get by subtraction

$$\text{P'Q'} = \left(\frac{df(x)}{dx} - \frac{d\phi(x)}{dx}\right) \cdot \frac{h}{1} + \left(\frac{d^2f(x)}{dx^2} - \frac{d^2\phi(x)}{dx^2}\right) \cdot \frac{h^2}{1 \cdot 2} + \ldots\ldots$$

Def. I. Curves $y = f(x)$ and $y = \phi(x)$ are said to *meet* one another, when $f(x) = \phi(x)$.

Def. II. These curves are said to have a *contact of the nth order*, when, besides meeting, they have their first n differential co-efficients equal, each to each.

Thus, the conditions

$f(x) = \phi(x)$, and $\frac{df(x)}{dx} = \frac{d\phi(x)}{dx}$, indicate a contact of the *first order*.

$f(x) = \phi(x)$, $\frac{df(x)}{dx} = \frac{d\phi(x)}{dx}$, and $\frac{d^2f(x)}{dx^2} = \frac{d^2\phi(x)}{dx^2}$, indicate a contact of the *second order*; and so on.

Def. III. Such curves are said to *osculate*, and the orders of contact are sometimes called *orders of osculation*.

PROP. II.

To determine the highest degree of contact which two curves can have, one of which contains n arbitrary constants, and the other of which is either of a higher degree, or contains a greater number of given co-efficients.

LET $y=f(x)$ be that curve which contains n arbitrary constants, and $y=\phi(x)$ the other curve. Then there can be formed the following n successive equations.

$$f(x)=\phi(x),\ \frac{df(x)}{dx}=\frac{d\phi(x)}{dx}\ \ldots\ldots\ \frac{d^nf(x)}{dx^n}=\frac{d^n\phi(x)}{dx^n}.$$

Now, since $x=a$, a given or known quantity at the point where the curves meet, the second side of each of these n equations is given. Whence the n arbitrary co-efficients can be determined, and the special form of the essential terms of $y=f(x)$ can be found.

EXAMPLE 1.

To find all the possible orders of contact of a straight line with a given curve.

In this case $y=ax+b$ is the equation of the line, and it contains two arbitrary constants: whence the contact is of the *first order*.

Hence we have $\frac{df(x)}{dx}=\frac{d\phi(x)}{dx}=a$; and putting x_1, y_1, the values of x, y, at the point of intersection, we get

$$b=y-ax=y-\frac{dy_1}{dx_1}\cdot x;\ \text{or}\ y-y_1=\frac{dy_1}{dx_1}(x-x_1)$$

which is the ordinary equation of the tangent. The tangent, therefore, has a contact of the first order with the curve.

EXAMPLE 2.

To find the highest order of contact of a circle with a given curve, and to assign the magnitude and position of the circle.

Let $y=\phi(x)$ be the given curve, and denote the circle by

$$\rho^2=(x-x_1)^2+(y-y_1)^2;$$

of which ρ is the radius, and x_1, y_1, the co-ordinates of the centre.

As this contains three arbitrary constants, the highest contact will be of the second order, or have the first two differential co-efficients equal to the two first in $y=\phi(x)$.

Now differentiating the equation of the circle, and recollecting that the point xy is common to both curves, we have

$$\frac{dy}{dx}=-\frac{x-x_1}{y-y_1},\ \text{or}\ (x-x_1)^2=(y-y_1)^2.\frac{dy^2}{dx^2}.$$

$$\frac{d^2y}{dx^2}=-\frac{1+\frac{dy^2}{dx^2}}{y-y_1},\ \text{or}\ (y-y_1)^2=\frac{\left\{1+\frac{dy^2}{dx^2}\right\}^2}{\left(\frac{d^2y}{dx^2}\right)^2}.$$

Substituting the first and second of these equations in the equation of the circle, we get

$$\rho^2=\frac{\left\{1+\frac{dy^2}{dx^2}\right\}^2}{\left(\frac{d^2y}{dx^2}\right)^2}\cdot\left\{1+\frac{dy^2}{dx^2}\right\}=\frac{\left\{1+\frac{dy^2}{xd^2}\right\}^3}{\left(\frac{d^2y}{dx^2}\right)^2},\ \text{or}$$

$$\rho = \frac{\left\{1 + \frac{dy^2}{dx^2}\right\}^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}} = \frac{(1 + p^2)^{\frac{3}{2}}}{-q};$$

where xy are the co-ordinates of the curve $y = \phi(x)$ at which the contact takes place, and p, q, are the first and second differential co-efficients.

The double sign in the value of ρ, which arises from the extraction, has been generally considered to designate the direction of the concavity of the curvature. It has, however, been proved by Professor Young (*Math. Diss. pp.* 21—30), that it is superfluous, the concavity being naturally determinable by another criterion, viz. the sign of q or $\frac{d^2y}{dx^2}$.

We have yet to find x_1, y_1, the co-ordinates of the centre; which, however, flow at once from the preceding.

For $y - y_1 = -\frac{1 + p^2}{q}$, and hence $y_1 = y + \frac{1 + p^2}{q}$,

$$x - x_1 = -(y - y_1)\, p, \text{ and hence } x_1 = x - \frac{1 + p^2}{q} \,.\, p.$$

Corollary.

The radius drawn from the centre of the circle of complete contact (or centre of curvature, *def. iv.* below) is a normal to the curve at the point of contact: for both the circle and curve have a common tangent, and the radius is perpendicular to the tangent of the circle, and therefore to the tangent of the curve; that is, a normal to the curve.

Scholium.

The curvature of the circle being uniform, it is the most convenient measure of the curvature of any other curve at any specified point. The *curvature* itself is estimated by the *value of* ρ^{-1}.

EXAMPLE 3.

When the radius of curvature is a maximum or a minimum, the circle has a contact of the third order with the curve.

For $\rho = -\frac{(1 + p^2)^{\frac{3}{2}}}{q}$; $\frac{d\rho}{dx} = -\frac{3(1 + p^2)^{\frac{1}{2}} pq^2 + (1 + p^2)^{\frac{3}{2}} r}{q^2}$.

Whence at the *max.* and *min.* $\frac{d\rho}{dx} = 0$, or $-3pq^2 + (1 + p^2)\, r = 0$. Divide by q, and substitute $-q(y - y_1)$ for $1 + p^2$; then

$$(y - y_1)\, r + 3pq = 0.$$

But this is the third differential co-efficient $\frac{d^3y}{dx^3}$ of the equation of the circle of curvature; or the circle and the curve have a contact of the third order at the maximum and minimum values of the radius of a curvature.

Def. IV. The circle so determined is called the *circle of curvature*, or *osculating circle;* and its centre and radius, the *centre of curvature* and the *radius of curvature*, respectively.

Def. V. The *chords of curvature* are chords drawn in the circle from the extremities of the diameter through the point of contact, parallel to the axes of co-ordinates.

Def. VI. The *evolute of a curve* is the locus of its centre of curvature, varying as the position of P in the given curve varies.

Def. VII. The *involute of a curve* is that whose consecutive centres of curvature are situated in the curve.

PROP. III.

To find the evolute of a curve.

Denote by $x_1 y_1$ the co-ordinates of the evolute corresponding to xy of the given curve. These are the co-ordinates of the centre of curvature.

Then we have the three equations from which to eliminate x, y, p, q, (the two latter being functions of x, y); viz.

$$y = f(x),\ y - y_1 = -\frac{1+p^2}{q},\ \text{and}\ x - x_1 = \frac{1+p^2}{q}\,.\,p.$$

The resulting equation in $x_1 y_1$ and constants will be that of the evolute.

The difficulty of this elimination is, however, often very considerable; and in fact, there is but a comparatively small number of curves, in which it can be actually performed. A few of such as admit of it will be given amongst the examples at the end of the chapter.

PROP. IV.

The radius of curvature of any point in a proposed curve is a tangent to the corresponding point of its evolute.

We have already seen that the equations of the evolute are

$$x - x_1 + (y - y_1)\frac{dy}{dx} = 0,\ \text{and}\ 1 + \frac{dy^2}{dx^2} + (y - y_1)\frac{d^2y}{dx^2} = 0.$$

Now this being the locus of the centres of curvature, x_1 and y_1 are functions of x, and therefore variable. Also, y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$, are functions of x.

Differentiate the former; then we get

$$1 - \frac{dx_1}{dx} + \frac{dy^2}{dx^2} - \frac{dy_1}{dx}\cdot\frac{dy}{dx} + (y - y_1)\frac{d^2y}{dx^2} = 0,$$

which reduces the latter to

$$-\frac{dx_1}{dx} - \frac{dy_1}{dx}\cdot\frac{dy}{dx} = 0,\ \text{or}$$

$$\frac{dx}{dy} = -\frac{dy_1}{dx}\cdot\frac{dx}{dx_1} = -\frac{dy_1}{dx_1}.$$

Whence the equation of the normal of the proposed curve at xy, namely,

$$y_1 - y = -\frac{dx}{dy}(x_1 - x),$$

is expressed by

$$y_1 - y = \frac{dy_1}{dx_1}(x_1 - x),\ \text{or}\ y - y_1 = \frac{dy_1}{dx_1}(x - x_1);$$

which is the equation of the tangent to the evolute at $x_1 y_1$, and passing through xy of the proposed curve.

PROP. V.

The difference between any two radii of curvature is equal to the distance of the two centres of curvature estimated on the arc of the evolute.

The equation of the circle of curvature being as before

$$\rho^2 = (x - x_1)^2 + (y - y_1)^2;$$

in which x_1, y_1, ρ, y, are functions of x, if we differentiate with respect to x as the principal variable, we get

$$(x - x_1)\left\{1 - \frac{dx_1}{dx}\right\} + (y - y_1)\left\{\frac{dy}{dx} - \frac{dy_1}{dx}\right\} = \frac{\rho d\rho}{dx};$$

and by the first differential equation of the circle of curvature,

$$x - x_1 + (y - y_1)\frac{dy}{dx} = 0,$$

this is changed to

$$(x - x_1)\frac{dx_1}{dx} + (y - y_1)\frac{dy_1}{dx} = -\frac{\rho d\rho}{dx},$$

$$\text{or } x - x_1 + (y - y_1)\frac{dy_1}{dx_1} = -\frac{\rho d\rho}{dx_1};$$

and hence,

$$(x - x_1)^2 + 2(x - x_1)(y - y_1)\frac{dy_1}{dx_1} + (y - y_1)^2\frac{dy_1^2}{dx_1^2} = \frac{\rho^2 d\rho^2}{dx_1^2}.$$

But since, by the preceding investigation, $y - y_1 = \frac{dy_1}{dx_1}(x - x_1)$, we have

$$(x - x_1)(y - y_1)\frac{dy_1}{dx_1} = (x - x_1)^2\frac{dy_1^2}{dx_1^2} = (y - y_1)^2; \text{ hence,}$$

$$\left\{(x - x_1)^2 + (y - y_1)^2\right\}\left\{1 + \frac{dy_1^2}{dx_1^2}\right\} = \rho^2 . \frac{d\rho^2}{dx^2};$$

or, substituting for ρ^2,

$$= \left\{(x - x_1)^2 + (y - y_1)^2\right\}\frac{d\rho^2}{dx_1^2}.$$

$$\text{Whence, } \frac{d\rho^2}{dx_1^2} = 1 + \frac{dy_1^2}{dx_1^2}, \text{ or } d\rho = \sqrt{dx_1^2 + dy_1^2}.$$

The first side of this equation is the differential of the radius of curvature, and the second that of the arc of the evolute: and hence the radius of curvature differs from the arc of the evolute by only a constant quantity, or

$$\rho = s + c.$$

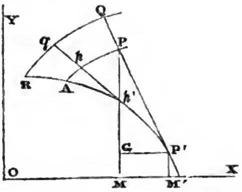
If, therefore, there be two radii, ρ_1, ρ_2, and two corresponding arcs of the evolute s_1, s_2, we have

$$\rho_1 - \rho_2 = (s_1 + c) - (s_2 + c) = s_1 - s_2;$$

which algebraically expresses the enunciated proposition.

Scholium 1.

The terms *involute* and *evolute* have been given in consequence of the property just established of these two connected curves.



If $Ap'P'$ be the evolute, and a string kept stretched pp' or PP' be gradually unwound from $Ap'P'$, then any fixed point in it, p or q, will trace the curve of which $Ap'P'$ is the evolute. For it has been just shown that

$$AP' - Ap' = PP' - pp' = P'Q - p'q, \textit{ etc.}$$

If the tracing point p be the point at which the string commences its evolution, then the constant c of the former solution is 0, and $Ap' = pp'$, $AP' = PP'$, *etc.* If the tracing point be at R, (in the tangent at A,) then

$$c = AR = pq = PQ, \textit{ etc.}$$

The curve $Ap'P$ is thence called the *evolute of* ApP, and ApP is called the *involute of* $Ap'P'$.

Also, PM, GP′ are the *semi-chords of curvature.*

Scholium 2.

The curve $Ap'P'$ being given, the evolute ApP can be formed from the same equations as have before been used.

For,

$$x = OM = OM' - MM' = x_1 - P'G = x_1 - \rho \cos QP'G = x_1 - \frac{\rho dx_1}{\sqrt{dx_1^2 + dy_1^2}},$$

$$y = MP = M'P' + PG = y_1 + PG = y_1 + \rho \sin QP'G = y_1 + \frac{\rho dy_1}{\sqrt{dx_1^2 + dy_1^2}},$$

Whence, ρ being expressed in terms of its co-ordinates, and $x_1 y_1$ being eliminated, we shall have the equation of ApP in terms of xy and constants.

EXAMPLES.

Find the radius and centre of curvature of the *parabola.* Here $y^2 = 4ax$; $\frac{dy}{dx} = \sqrt{\frac{a}{x}}$; and $\frac{d^2y}{dx^2} = -\frac{1}{2}\sqrt{\frac{a}{x^3}}$.

Substitute these, and reduce; then $\rho = \dfrac{2(a+x)^{\frac{3}{2}}}{a^{\frac{1}{2}}} = \dfrac{2r^{\frac{3}{2}}}{\sqrt{a}}$, where r is the radius-vector of the point in reference to the focus.

Again, $y - y_1 = \dfrac{1+p^2}{-q} = y \cdot \dfrac{a^2 + ax}{a^2} = y + \dfrac{yx}{a}$, or $y_1 = -\dfrac{y^3}{4a^2}$, and

$$x - x_1 = -p(y - y_1) = -2(x + a), \text{ or } x_1 = 3x + 2a,$$

which are the co-ordinates of the centre of curvature.

From the two last results we have

$$y = -(4a^2 y_1)^{\frac{1}{3}}, \text{ and } x = \tfrac{1}{3}(x_1 - 2a).$$

These, substituted in the equation of the parabola, give

$$(4a^2 y_1)^{\frac{2}{3}} = 4a\,\tfrac{1}{3}(x_1 - 2a), \text{ or cubing both sides,}$$

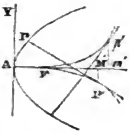
$$27ay_1^2 = 4(x_1 - 2a)^3;$$

or, putting $x_1 - 2a = x$, and dropping the accent from y,

$$27ay^2 = 4x^3.$$

This curve is called the *semi-cubical parabola,* and, as we have written it, has its origin at the focus.

In the annexed figure P′F and p′F are the branches of this curve. PP′ is the radius of curvature at P, and AM′, M′P′, are the co-ordinates x_1, y_1, of the centre of curvature,



2. Show that in *all the conic sections* the radius of curvature is proportional to the cube of the normal.

$$\mathrm{N} = y\sqrt{1 + \frac{dy^2}{dx^2}}\text{; and hence }\sqrt{1 + \frac{dy^2}{dx^2}} = \frac{\mathrm{N}}{y}\text{; and}$$

$$\rho = \frac{\left\{1 + \frac{dy^2}{dx^2}\right\}^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}} = -\frac{\mathrm{N}^3}{y^3.\frac{d^2y}{dx^2}} = -\frac{\mathrm{N}^3}{qy^3}.$$

Referring the curve to the vertex and transverse axis, we have

$$y^2 = 2hx + kx^2;$$

and hence, successively,

$$py = h + kx;\ qy + p^2 = k;\text{ and}$$

$$qy^3 = ky^2 - p^2y^2 = k\,(2hx + kx^2) - (h + kx)^2 = h^2.$$

$$\text{Wherefore, } \rho = -\frac{\mathrm{N}^3}{qy^3} = \frac{\mathrm{N}^3}{h^2}.$$

3. Find the radius, and centre of curvature, and evolute of the *ellipse*, referred to the centre and principal axes.

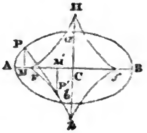
Here $a^2y^2 + b^2x^2 = a^2b^2$; whence, differentiating, we get

$$p = \frac{dy}{dx} = \frac{b}{a}.\frac{x}{\sqrt{a^2 - x^2}}\text{; and } -q = -\frac{d^2y}{dx^2} = \frac{ab}{(a^2 - x^2)^{\frac{3}{2}}}.$$

$$\text{Wherefore, } \rho = \frac{(a^2 - e^2x^2)^{\frac{3}{2}}}{ab}.$$

Again, for the centre of curvature we have, by proceeding as before, the co-ordinates of the required point,

$$y_1 = -\frac{a^2\,e^2\,y^3}{b^4}\text{; and } x_1 = \frac{e^2\,x^3}{a^2}.$$



Lastly, from these we have $x = \sqrt[3]{\frac{a^4\,x_1}{a^2\,e^2}}$, and $y = \sqrt[3]{\frac{b^4\,y_1}{a^2\,e^2}}$; and these, substituted in the equation of the curve, give, after slight reduction,

$$(ax_1)^{\frac{2}{3}} + (by_1)^{\frac{2}{3}} = (a^2\,e^2)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

The figure FHfh represents all the parts of the curve, F and f being the foci of the ellipse.

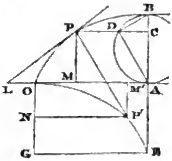
4. In the *hyperbola* referred to the centre, the evolute is expressed by

$$(ax_1)^{\frac{2}{3}} - (by_1)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}.$$

5. The equation of the evolute of the *rectangular hyperbola* referred to its axes, (viz. $xy = 4a^2$,) is

$$(x_1 + y_1)^{\frac{2}{3}} - (x_1 - y_1)^{\frac{2}{3}} = (4a)^{\frac{2}{3}}.$$

6. To find the radius of curvature, the involute and evolute of the *common cycloïd.*



Let A be the middle of the base, AB the diameter of the generating circle, and O, the extremity of the base, be taken as origin; let OA be X, and OY perpendicular to it be Y; let P be a point in the curve, P′ the corresponding point of the evolute; and take AE = AB, and draw the other lines as obviously pointed out in the figure.

Let OM $= x$, MP $= y$, and AB $= 2r$: then (*page* 346)

$$y + r\cos\frac{x + \sqrt{2ry - y^2}}{r} - r = 0$$

$$\frac{dy}{dx} = \frac{\sqrt{2ry - y^2}}{y}; \text{ and } \frac{d^2y}{dx^2} = -\frac{r}{y^2}.$$

These substitutions being made, we get

$$y_1 = -y \text{ and } x_1 = x - 2\sqrt{2ry - y^2},$$

$$\rho = \left(\frac{4y^4}{r}\right)^{\frac{1}{3}}.$$

Inserting the values of x and y from these in the equation of the cycloïd, we get the equation of the evolute,

$$y_1 - r\cos\frac{x_1 - \sqrt{2ry_1 - y_1{}^2}}{r} + r = 0,$$

which is the equation of a cycloïd whose origin is at the vertex O, and the diameter of whose generating circle is $2r$. It follows that the involute of the cycloïd is another cycloïd equal to itself.

When the tracing point has arrived at B, we have $y = 2r$, and hence $\rho = 4r$. But ρ is equal to the arc of the semi-cycloïd OPE, which is equal to OPB; and hence the length of the entire cycloïd is four times the diameter of the generating circle.

7. In the curve whose equation is $a\theta + a\sec^{-1}\frac{r}{a} = \sqrt{r^2 - a^2}$, show that the evolute is a circle to radius a.

8. Find the greatest and least circles of curvature in each of the *conic sections.*

9. In the *catenary* defined by $y = \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}$, we have

$$\frac{dy}{dx} = \tfrac{1}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}; \frac{d^2y}{dx^2} = \frac{y}{a^2}; \text{ and } \rho = -\frac{y^2}{2a}.$$

10. In the *hypo-cycloïd* $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, we have $\rho = -3\,(axy)^{\frac{1}{3}}$.

11. Find the *evolute of the circle*, by the use of the equations.

12. Show that if y and s be made successively the independent variables, the radius of curvature and its reciprocal will be respectively

$$\rho = \frac{\left\{1 + \frac{dx^2}{dy^2}\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^4}}; \text{ and } \frac{1}{\rho} = \left\{\frac{d^2x}{ds^2} + \frac{d^2y}{ds^2}\right\}^{\frac{1}{2}}.$$

13. Show that if x and y be functions of any variable z, then we shall have

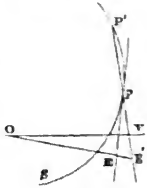
$$\rho = \frac{dy}{dz}$$

CHAPTER IX.

OSCULATION AND CURVATURE, WITH THE INVOLUTES AND EVOLUTES OF CURVES; REFERRED TO POLAR CO-ORDINATES *.

PROP. I.

To find the differential of the angle ω (page 396) in a polar curve represented by an equation between p and r.



Let O be the pole, OV the polar axis, PE the tangent at P, OE the perpendicular; and let P′ be another point in the curve, and PP′ produced to meet OE in E′.

Let OP $= r$, OP′ $= r + h$: then since $p = f(r)$ we have

$$p' - p = \frac{dp}{dr} \cdot \frac{h}{1} + \frac{d^2p}{dr^2} \cdot \frac{h^2}{1\,.\,2} + \frac{d^3p}{dr^3} \cdot \frac{h^3}{1\,.\,2\,.\,3} + \ldots.$$

$$\text{Also, } d\omega = \left[\tan \mathrm{EPE'}\right] = \left[\frac{\mathrm{EE'}}{\mathrm{EP}}\right] = \left[\frac{\mathrm{OE'} - \mathrm{OE}}{\mathrm{EP}}\right] = \frac{dp}{\sqrt{r^2 - p^2}},$$

whence, inserting the limit of the preceding equation in this, we get

$$d\omega = \left\{\frac{dp}{dr} \cdot \frac{h}{1} + \frac{d^2p}{dr^2} \cdot \frac{h^2}{1\,.\,2} + \ldots\right\} \frac{1}{\sqrt{r^2 - p^2}}$$

Def. If there be two curves referred to the same polar co-ordinates

$$p = f(r) \text{ and } p_1 = \phi(r_1),$$

and ω, ω_1 denote the angles made by the respective radii vectors with the tangents, as in the preceding proposition; then in the two developments for the increment h, as

$$d\omega = \left\{\frac{dp}{dr} \cdot \frac{h}{1} + \frac{d^2p}{dr^2} \cdot \frac{h^2}{1\,.\,2} + \frac{d^3p}{dr^3} \cdot \frac{h^3}{1\,.\,2\,.\,3} + \ldots\right\} \frac{1}{\sqrt{r^2 - p^2}},$$

$$d\omega_1 = \left\{\frac{dp_1}{dr_1} \cdot \frac{h}{1} + \frac{d^2p_1}{dr_1^2} \cdot \frac{h^2}{1\,.\,2} + \frac{d^3p_1}{dr_1^3} \cdot \frac{h^3}{1\,.\,2\,.\,3} + \ldots\right\} \frac{1}{\sqrt{r_1^2 - p_1^2}},$$

---

* The method adopted here is the ordinary one, of an equation between r and p. In certain cases it is, perhaps, more simple, and as a method of investigation will be comprised in rather less space than when the equation is given between r and θ. On these accounts it has been used here. However, the equation between r and p is often only a differential one, and hence as a general method of investigation, though convenient in these particular branches of the inquiry, which are conversant mainly with differentials, the system is not generally a favourable one for the discussion of polar curves.

The equations of the tangent and normal at the point $r_1\theta_1$ of a polar curve, given at pages 396—8, will place the whole investigation on the same footing as are rectangular co-ordinates; and though the development of the inquiry on this principle has never been published, I hope to avail myself of an early opportunity of giving such a system. A few applications of it to the conic sections have been given at *pp.* 329—37, of this work; and they are sufficient to show that the system is worthy of further development.

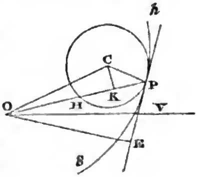
the spirals are said to *meet* if $r_1 = r$; and if

$$r_1 = r, \frac{dr_1}{dp_1} = \frac{dr}{dp}, \frac{d^2r_1}{dp_1^2} = \frac{d^2r}{dp^2}, \ldots \frac{d^{n-1}r_1}{dp_1^{n-1}} = \frac{d^{n-1}r}{dp^{n-1}},$$

they are said to have a contact of the *n*th order *.

PROP. II.

To find the equation between r and p in the polar equation of the circle.



THE general expression for the length is

$$\frac{r_1^2 d\theta_1}{\sqrt{r_1^2 d\theta_1^2 + dr_1^2}},$$ (*p.* 397, *eq.* 5,) and the equation of the circle is

$$\rho^2 = r_1^2 - 2ar \cos \theta_1 + a^2, \text{ or}$$

$$r = a \cos \theta \pm \sqrt{\rho^2 - a^2 \sin^2\theta},$$

where $r_1\theta_1$ are the polar co-ordinates, and a the distance of the pole from the centre, and the centre in the polar axis. From these we readily get,

$$p_1 = \frac{r_1^2 - a^2 + \rho^2}{2\rho};$$

and hence,

$$\frac{dp_1}{dr_1} = \frac{r_1}{\rho}.$$

If now a be the polar distance of the centre of the circle, and we take β the angle † which it makes with the polar axis, (still β will not require to be determined); whilst from the preceding results and the conditions we have

$$r_1 = r, p_1 = p, \frac{dp_1}{dr_1} = \frac{dp}{dr};$$

and hence $\frac{dp}{dr} = \frac{r}{\rho}$, or $\rho = \frac{rdr}{dp}$ = the radius of the circle of curvature.

$$\text{Also } a^2 = r_1^2 - 2p_1\rho + \rho^2 = r^2 + \frac{r^2dr^2}{dp^2} - \frac{2prdr}{dp}, \text{ or}$$

$$a = \sqrt{r^2 - 2pr \cdot \frac{dr}{dp} + r^2 \cdot \frac{dr^2}{dp^2}}.$$

These with the value of r give the three sides of a triangle, one summit of which is the centre of curvature. The value of β will, therefore, be unnecessary, as stated above. The circle of curvature is, hence, completely determined.

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* In comparing this definition with that of orders of contact in respect of rectangular equations, it would seem that the differential co-efficient of the polar curve is one degree lower for the contact of the *n*th order than it is in rectangular co-ordinates. This, however, is not the case: for, in fact, the equation between p and r is itself a differential equation of the first order, and hence $\frac{d^{n-1}p}{dr^{n-1}}$ is of the *n*th order.

† Should, for any subsequent object, β be required, it is easily found; viz.

$$\theta_1 + \cos^{-1} \frac{r_1^2 + a^2 - \rho^2}{2r_1 a};$$

where $r_1\ \theta_1$ are the co-ordinates of the point at which the circle of contact is supposed to touch the spiral.

Corollary 1.

If C be the centre of curvature, and CK be drawn perpendicular to the radius vector OP, and OE perpendicular to the tangent at P; then the *polar chord of curvature* PH is easily found.

For, the triangles CPK, POE, are similar; and hence,

$$2PK = 2CP.\frac{OE}{OP} = \frac{2rdr}{dp}.\frac{p}{r} = \frac{2pdr}{dp}.$$

Corollary 2.

Also from (*equation* 5, *p*. 397), $p = \frac{r^2}{\sqrt{r^2 + \frac{dr^2}{d\theta^2}}}$, we have by differentiating,

$$\frac{dp}{dr} = \frac{r^2 + 2r\frac{dr^2}{d\theta^2} - r^2\frac{d^2r}{d\theta^2}}{\left\{r^2 + \frac{dr^2}{d\theta^2}\right\}^{\frac{3}{2}}}.$$

The sign of this is the criterion, whether the curve be concave or convex to the radius vector (*p*. 400).

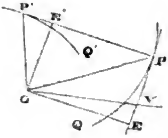
Corollary 3.

From this also $\rho = \frac{rdr}{dp}$ is converted into the following very elegant form:

$$\rho = \frac{\left\{r^2 + \frac{dr^2}{d\theta^2}\right\}^{\frac{3}{2}}}{r^2 - r.\frac{d^2r}{d\theta^2} + 2\frac{dr^2}{d\theta^2}}.$$

PROP. III.

To find the evolute of a curve referred to polar co-ordinates.



Let PE be the tangent at P, OE the perpendicular on it from the origin; and let Q'P' be the evolute, and OE' the perpendicular on the normal PP'.

Let $p = f(r)$ be the equation of PQ, and $p_1 = \phi(r_1)$ be that of the evolute P'Q'; where $OP = r$, $OP' = r_1$, $OE = p$, and $OE' = p_1$. Then $p_1 = PE = \sqrt{r^2 - p^2}$; and likewise

$$r_1 = OP' = \sqrt{OE'^2 + EP'^2} = \sqrt{r^2 - p^2 + \left(\frac{rdr}{dp} - p\right)^2},$$

$$= \sqrt{r^2 - 2pr\frac{dr}{dp} + r^2\frac{dr^2}{dp^2}};$$

and if by means of these equations, together with that of the given spiral $p = f(r)$, we can eliminate p, r, the resulting equation in $p_1 r_1$ will be that of the evolute.

Corollary 1.

The equality of the differences of the curves of the evolute, and of the corresponding radii vectores, being independent of the nature of the co-ordinates employed, it may be assumed, in virtue of the demonstration at *page* 410, without a formal demonstration by polar co-ordinates. It would, however, be easy to effect it by this method.

Corollary 2.

In precisely the same way as in rectangular co-ordinates, the equation of the involute may be found; for in this case p_1, r_1, are the polar co-ordinates of the given curve, and we have only to insert their respective values in the equation of the given curve, and the result is the equation of the involute.

Corollary 3.

The equation of the evolute is very simple in reference to the radius vector and perpendicular; thus,

$$r_1^2 = r^2 - 2\rho p + \rho^2 \qquad\qquad p = f(r)$$

$$p_1^2 = r^2 - p^2 \qquad\qquad \rho = \frac{r\,dr}{dp}.$$

EXAMPLES.

1. In the *lemniscate of Bernoulli*, $r^2 = a^2 \cos 2\theta$, we have $\frac{dr}{d\theta} = -\frac{\sqrt{a^4 - r^4}}{r}$, and $\frac{d^2r}{d\theta^2} = -\frac{a^4}{r^3} - r$; which inserted in the expression for the radius of curvature, give $\rho = \frac{a^2}{3r}$.

2. In the *cardioïd*, $r = a\,(1 - \cos\theta)$ we have $\rho = 3\sqrt{8ar}$.

3. In the *spiral of Archimedes*, $r = a\theta$, we have $\rho = \frac{(a^2 + r^2)^{\frac{3}{2}}}{2a^2 + r^2}$.

4. In the *hyperbolic spiral*, $r = \frac{a}{\theta}$, we have $\rho = \frac{r\,(a^2 + r^2)^{\frac{3}{2}}}{a^3}$.

5. In the *lituus* $r^2 = \frac{a^2}{\theta}$, the value of ρ is $\rho = \frac{r\,(4a^4 + r^4)^{\frac{3}{2}}}{2a^2\,(4a^4 - r^4)}$.

6. In the *epicycloïd*, $p^2 = \frac{m^2\,(r^2 - a^2)}{(m^2 - 1)\,a^2}$, $\rho = \frac{\sqrt{(m^2 - 1)\,(r^2 - a^2)}}{m}$.

7. In the *involute of the circle*, $p^2 = r^2 - a^2$, and $\rho = p$.

8. In *Cotes's spirals*, $p = \frac{br}{\sqrt{a^2 + r^2}}$, and $\rho = \frac{r\,(a^2 + r^2)^{\frac{3}{2}}}{a^2 b}$.

9. In the *logarithmic spiral*, $p = mr$, and $\rho = \frac{r}{m} = \frac{p}{m^2}$.

10. Find the *evolute* of the curve $a\theta + \cos^{-1}\frac{a}{r} = \sqrt{r^2 - a^2}$

$p^2 = r^2 - a^2$, $\rho = p$, $r_1^2 = r^2 + p^2 - 2p^2 = r^2 - p^2 = a^2$, and $p_1^2 = r^2 - p^2 = a^2$; whence r_1 and p_1 are constant, and the evolute is a circle.

11. The *logarithmic spiral*, in which $p = mr$, gives

$\rho = \frac{r}{m}$, $p_1^2 = r^2\,(1 - m^2)$, $r_1^2 = r^2 + \frac{r^2}{m^2} - 2r^2 = \frac{r^2\,(1 - m^2)}{m^2}$, and $p_1 = mr_1$,

which is the equation of a similar logarithmic spiral. The log spiral is hence both the involute and evolute of the log spiral: and therefore analogous in these respects to the common cycloïd.

CHAPTER X.

CONSECUTIVE LINES AND THEIR ENVELOPES.

We have seen already that one absolutely indeterminate quantity entering amongst the conditions of a geometrical problem, converts the final equation of solution into that of a curve line; all the points of which equally satisfy the proposed conditions. Should there also be a second indeterminate, one of these two circumstances will be readily seen to take place.

(1). If the conditions of the problem allow of our considering a point which is not situated in the plane XY as belonging to the solution, this second indeterminate quantity will represent a line drawn parallel to a given fixed axis not coinciding with the plane XY, but intercepted between a point in a curve surface represented by that final equation and the plane XY itself. Under this aspect, the inquiry is of the same general character with that which has preceded; but it leads to expressions of greater complexity, and inquiries of greater difficulty. It constitutes *the geometry of co-ordinates of three dimensions*—a subject into which in this volume we do not propose to enter, though, should our publishers allow us another volume, it will be amply discussed in that place. It has the same relation to the geometry of co-ordinates of two dimensions, that the ordinary geometry has to that which in our first volume is treated under the title of "Planes and Solids," *pp.* 347—70.

(2). If the conditions require us to confine our consideration wholly to one plane, XY, then there will not only be one curve, every point of which will fulfil the given conditions, but innumerable curves, every point of each of which will also fulfil the given conditions. It is clear that such an equation may, by giving suitable values to the indeterminate quantities, be made to represent any, and therefore every, point in that plane. It is to particular results of such a system of conditions that this brief chapter is devoted.

We shall suppose a curve denoted by $f(x, y, v) = 0$, where x and y are the current co-ordinates on X and Y of any one of the innumerable curves denoted by the equation, and each of which is particularized by the value assigned to the arbitrary quantity v.

Let, then, two values, a and $a + h$, be given to v; in consequence of which there will be two distinct and perfect loci indicated. These loci will in general intersect in one or more points, and their common co-ordinates at the points of intersection are, theoretically, determinable. If $h = 0$, then the curves become coincident, and the co-ordinates are common throughout the entire course; but h being made continually to diminish, the curves only approach to coincidence, whilst their points of intersection are continually varied. A little consideration of the circumstances will, however, render it obvious, that there is a certain set of points towards which the general points of intersection tend in their progress, which they never reach for any finite value of h, which they only pass by giving negative values to h, and from which they recede more and more as the negative value of h is increased.

Again, the curve dependent on the value $a + h$ has similar points in reference to that dependent on the value $a + 2h$; and this again has similar points in reference to that dependent on $a + 3h$; and so on.

When h (or $v = a + h$) is supposed to increase *continuously*, each curve and its differentially adjacent ones are called *consecutive curves;* and the assemblage of their points of successive ultimate intersection is called the *envelope* of the system of curves represented by the equation $f(x, y, v) = 0$. The variable magnitude v, upon which the particular curve depends, is called its *variable parameter.*

Let, v being the variable parameter, the equation be

$$f(x, y, v) = 0.$$

Then also will the equation hold good when $v + h$ is put for v,

$$\text{or } f(x, y, v + h) = 0.$$

Now at the points of intersection x and y become coincident in the two curves, and hence, in respect to any two such simultaneous curves, they may be considered constant. Whence, expanding the latter equation in respect of v as the variable, we get

$$f(x, y, v + h) = f(x, y, v) + \frac{df(x, y, v)}{dv} \cdot \frac{h}{1} + \frac{df^2(x, y, v)}{dv^2} \cdot \frac{h^2}{1.2} + \dots .$$

But $f(x, y, v + h)$ and $f(x, y, v)$ are each zero; making which substitution in the equation, and dividing by h, we get

$$0 = \frac{df(x, y, v)}{dv} + \frac{d^2f(x, y, v)}{dv^2} \cdot \frac{h}{1.2} + \dots .$$

whence, diminishing the value of h, as in former cases, we get for the condition required to be annexed to the given general equation,

$$\frac{df(x, y, v)}{dv} = 0.$$

Had there been more variable parameters, t, w, ... the conditions would have been, with respect to each, of the same kind as the preceding one with respect to v.

The rule, then, is, to differentiate the given equation with respect to the variable parameter, and between this differential and the given equation eliminate that parameter.

The result is the equation of the *envelope* *, or of that curve which is independent of the parameter.

EXAMPLES.

1. A line of given length, a, moves with its extremities always situated on two lines at right angles to one another: find its envelope.

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* This name is given from the circumstance that it *touches* every curve dependent on v, as may be thus shown.

Take $u = f(x, y, v) = 0$, the general equation of the family of curves; then to find the tangent for v constant, and x, y, variable, we have

$$du = \frac{du}{dx} \cdot dx + \frac{du}{dy} \cdot dy = 0.$$

Also, differentiating on the hypothesis of x, y, v, variable, we have

$$du = \frac{du}{dx} \cdot dx + \frac{du}{dy} \cdot dy + \frac{du}{dv} \cdot dv = 0.$$

But at the points of consecutive intersection we have just seen that $\frac{du}{dv} = 0$, and hence the inclination of the tangent denoted by $\frac{dy}{dx}$ is the same in both curves. The envelope, therefore, *touches* all the curves $f(x, y, v) = 0$, and hence either envelopes them all, or is enveloped by them.

Let the moving line in any one of its positions cut from the given lines taken as axes of co-ordinates the segments $x_1\ y_1$: then we have

$$u = x_1 y + y_1 x - x_1 y_1 = 0, \text{ and } x_1^2 + y_1^2 = a^2.$$

Differentiate these as functions of x_1, considering x and y constants,

$$y - y_1 + (x - x_1)\frac{dy_1}{dx_1} = 0, \text{ and } x_1 + y_1\frac{dy_1}{dx_1} = 0.$$

From these four equations eliminate x_1, y_1 and $\frac{dy_1}{dx_1}$ and the result is readily found to be

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

This curve possesses a remarkable property of the tangent: viz. of being the shortest line that can be drawn through the point of contact. It is, also, though in another form, identical with *ex.* 7. *p.* 383.

But a much more general one, in reference to curve surfaces, may be seen in the *Math. Repos. vol. vi.* p. 37, by the editor of this work.

2. Find the envelope of all the ellipses expressed by the following equation, in which x_1 is the variable parameter:

$$\frac{x^2}{x_1^2} + \frac{y^2}{(a - x_1)^2} = 1.$$

In this case $\frac{du}{dx_1} = 0$ gives $\frac{x^2}{x_1^3} - \frac{y^2}{(a - x_1)^3} = 0$, or $x_1 = \frac{ax^{\frac{2}{3}}}{x^{\frac{2}{3}} + y^{\frac{2}{3}}}$.

Substitute this in the given equation, and we obtain

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}};$$

which is, again, the same equation as that of the preceding example.

3. A line so moves with its extremities upon two given lines, as to cut off a triangle of given area: what is the envelope?

Let the given lines be taken as axes of co-ordinates, the given arc a^2 the constant area of the triangle; also let $x_1 y_1$ be the segments of the axes intercepted from the vertex by the line in any one of its variable positions.

Then, the two equations of the line and its conditions are

$$y_1 x + x_1 y = x_1 y_1, \text{ and } x_1 y_1 \sin a = 2a^2.$$

By differentiating, we have

$$y - y_1 + (x - x_1)\frac{dy_1}{dx_1} = 0, \text{ and } y_1 + x_1\frac{dy_1}{dx_1} = 0.$$

Eliminate x_1, y_1, and $\frac{dy_1}{dx_1}$ from the two last and the first equations; then we get the equation of the envelope, viz.: that of an hyperbola referred to its asymptotes.

$$xy = \frac{a^2}{2 \sin a}$$

4. If from every point in a given curve of the second order, tangents be drawn to another curve of the second order; show that the chords of contact which constantly touch, will by their consecutive intersections generate a third curve of the second order.

Let the variable pole or point from which the tangents emanate move in

$$\alpha y_1^2 + \beta x_1 y_1 + \gamma x_1^2 + \delta y_1 + \epsilon x_1 + \phi = 0 \ \ldots\ldots\ldots\ldots \ (1),$$

and let the tangents be drawn to the curve

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0 \ \ldots\ldots\ldots\ldots \ (2).$$

Then (*p.* 306) the equation of the variable chord of contact or corresponding polar with respect to (2) will be

$$y_1(2Ay + Bx + D) + x_1(2Cx + By + E) + Dy + Ex + 2F = 0 \ldots. (3).$$

Differentiate (1, 3); and between the differentials and (1, 3) themselves, eliminate x_1, y_1, $\frac{dy_1}{dx_1}$: the result is an equation of the second degree in xy. The envelope is therefore another conic section.

5. Find the envelope of all the consecutive normals to a given curve.

Let $y_1 = f(x_1)$ be the equation of the given curve, and xy the co-ordinates of a point in the normal through x_1y_1: then

$$y - y_1 = -\frac{dx_1}{ay_1}(x - x_1), \text{ or } x - x_1 + (y - y_1)\frac{dy_1}{dx_1} = 0.$$

Now, at the point of intersection of two of these consecutive lines, x and y are constant, whilst x_1, y_1, and $\frac{dy_1}{dx_1}$ vary; whence differentiating again on this hypothesis we get

$$1 + \frac{dy_1^2}{dx_1^4} + (y_1 - y)\frac{d^2y_1}{dx_1^2} = 0.$$

By comparing these results with those found at page 409, we find that the envelope of the normals is identical with the evolute of the given curve.

6. To determine the curve which envelopes all the circles described with radius r, and having their centres in the circumference of another circle whose radius is R.

Ans. Circles concentric with the given one, having their radii equal to $R + r$ $R \sim r$.

7. The centres of an infinite number of equal circles are situated on a given line; assign their envelope.

Ans. Two lines parallel to the line of centres.

8. From every point in a parabola lines are drawn, making the same angle with the tangent that the tangent makes with the diameter; required the envelope.

Ans. A point, the focus.

9. The lines $y = vx + \frac{m}{v}$, and $y = vx + r\sqrt{1 + v^2}$, where v is the variable parameter, are enveloped respectively by a parabola ($y^2 = 4mx$) and a circle ($x^2 + y^2 = r^2$).

10. Show that the envelope of a series of parabolas determined by the equation $y = vx - (1 + v^2)\frac{x^2}{4c}$ is another parabola denoted by $x^2 = 4c(c - y)$, v being the variable parameter[*].

11. Let there be a parabola $y_1^2 = 4kx_1$, and a series of circles $(x - x_1)^2 + y^2 = y_1^2$; show that the envelope of the circle is another parabola equal to the former, having with it a coincident principal axis, and its vertex at the distance $-m$ from the origin of the co-ordinates of the circle.

12. The envelope of the parabolas $y^2 = v(x - v)$ is the system of two straight lines making angles of $\pm$ 30° with the axis of x.

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[*] This is the first problem of this nature ever resolved: it arises from the inquiry, as to what curve is the locus of the intersection of all the projectiles discharged from a fixed point with a constant given velocity.—*Gregory's Examples*, *p.* 228.

CHAPTER XI.

THE SINGULAR POINTS OF CURVE LINES.

Notwithstanding that the general equation of a curve represents the relation between the absciss and ordinate of every point in that curve, there are still points of the curve which are marked by certain peculiarities, that can in general only be discovered by means of the successive differential co-efficients. Whatever that peculiarity may be, it is a peculiarity from the points situated on each side, at the smallest theoretical distance. They depend upon a value being given to x in the equation $y = f(x)$ which renders one or more of the differential co-efficients of the forms 0, $\frac{1}{0}$, or $\frac{0}{0}$. These are called *singular points.*

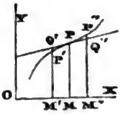
I. MAXIMUM AND MINIMUM ORDINATES.

These are indicated by $\frac{dy}{dx} = 0$, or $\frac{dx}{dy} = 0$, whilst $\frac{d^2y}{dx^2}$, or $\frac{d^2x}{dy^2}$ have respectively real and finite values. In the former hypothesis x is treated as the abscissa, and in the latter y and x are interchanged in the construction, as in the expressions of the curve.

Without regard to the value of the second differential co-efficients, the first equated to zero merely indicate the parallelism of the tangent to X or Y respectively.

These subjects have been already noticed at *page* 390.

II. POINTS OF INFLEXION, OR OF CONTRARY FLEXURE.



These are points at which the concavity being turned towards X at points adjacent on one side, are turned from it at points on the other; or in which the branches of the curve on different sides of the point are situated on different sides of the tangent.

The annexed figure, in connection with our usual notation, sufficiently explains the character of the point P of the curve P′PP″; the line Q′PQ″ being a tangent at P to both the concave and convex branches P′P and PP″.

The analytical expression of this circumstance is, $\frac{d^2y}{dx^2} = 0$ at the point P, whilst on adjacent sides of it the values of this co-efficient have different signs; and $\frac{d^3y}{dx^3}$ not rendered zero by the same value of $x =$ MP.

For let generally $\frac{d^2y}{dx^2} = \phi(x)$: then we have

$$\phi\,(x \pm h) = \phi\,(x) \pm \frac{d\phi\,(x)}{dx} \cdot \frac{h}{1} + \frac{d^2\phi\,(x)}{dx^2} \cdot \frac{h^2}{1\,.\,2} \pm \frac{d^3\phi\,(x)}{dx^3} \cdot \frac{h^3}{1\,.\,2\,.\,3} + \ldots.$$

or, since $\phi\,(x) = \frac{d^2y}{dx^2}$, $\frac{d\phi\,(x)}{dx} = \frac{d^3y}{dx^3}$, etc., this becomes

$$\phi\,(x + h) = \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} \cdot \frac{h}{1} + \frac{d^4y}{dx^4} \cdot \frac{h^2}{1\,.\,2} + \ldots.$$

$$\phi\,(x - h) = \frac{d^2y}{dx^2} - \frac{d^3y}{dx^3} \cdot \frac{h}{1} + \frac{d^4y}{dx^4} \cdot \frac{h^2}{1\,.\,2} - \ldots.$$

But at P, the point of inflexion, $\frac{d^2y}{dx^2} = 0$, and hence the deflexion from the tangent on the two sides of that point, at P′ and P″, are

$$\frac{d^3y}{dx^3} \cdot \frac{h}{1} + \frac{d^4y}{dx^4} \cdot \frac{h^2}{1\,.\,2} + \ldots, \text{ and } -\frac{d^3y}{dx^3} \frac{h}{1} + \frac{d^4y}{dx^4} \cdot \frac{h^2}{1\,.\,2} - \ldots.$$

But if $\frac{d^3y}{dx^3} = 0$, and $\frac{d^4y}{dx^4}$ do not vanish also, the curve will have the same sign before and after passing through P; and P will not in this case be a point of inflexion: the branches being either both concave or both convex to the axis and to the tangent at P.

Proceeding to reason in the same manner, we shall see that there will be a point of contrary flexure when the first differential co-efficient that does not vanish is of an odd order, as the 3rd., 5th., 7th., *etc.*

Scholium 1.

This rule will not be applicable when the tangent is parallel to the axis of y, since $\frac{dy}{dx} = \frac{1}{0}$, and all that follow it are also infinite. Neither will it be applicable when Taylor's series "fails" after the third term, since the co-efficients then become infinite, or zero, or some of both. Our criteria cannot, therefore, in either case be applied.

Sometimes, however, this difficulty is evaded by changing the principal variable, and consequently the axes of reference X and Y.

The surest method of proceeding in such cases, however, is very similar to that employed in doubtful or difficult cases of the maxima and minima: to examine the values of the ordinates of the curve and its tangent before and after passing through P.

Scholium 2.

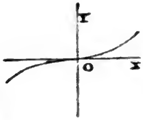
In spirals, or polar curves, the same principle operates. For if we consider the curve given by an equation between r and p, we shall have this corresponding to a first differential of the rectangular equation; the first differential of this, to the second of the rectangular, *etc.*

We hence infer at once that a point of inflexion, $\frac{dp}{dr} = 0$, and the curve changes its sign in passing through that value.

EXAMPLES.

1. The *cubical parabola* $a^2y = x^3$.

$$y = \frac{x^3}{a^2};\ \frac{dy}{dx} = \frac{3x^2}{a^2};\ \frac{d^2y}{dx^2} = \frac{6x}{a^2}.$$



If $x = 0, \frac{d^2y}{dx^2} = 0$; whence if x be positive or negative, $\frac{d^2y}{dx^2}$ is positive or negative, respectively; and hence the curve is always convex to X.

Apply the criterion of $x_1 = 0 + h$ and $x_2 = 0 - h$: then $\frac{d^3y}{dx^3} = \pm \frac{6h}{a^2}$: and the origin is a point of inflexion. Also, since at this point $\frac{dy}{dx} = 0$, the axis X is a tangent to the curve.

2. The *Witch*, $xy^2 = 4a^2\ (2a - x)$; and $\frac{d^2y}{dx^2} = \frac{2a^2\,(3a - 2x)}{x\,(2ax - x^2)^{\frac{3}{2}}}$. Putting $\frac{d^2y}{dx^2} = 0$, we have $x = \frac{3a}{2}$ and $y = \pm \frac{2a}{\sqrt{3}}$. Also, as $\frac{3a}{2} + h$ and $\frac{3a}{2} - h$ give $\frac{d^2y}{dx^2}$ different signs, the points $\left(\frac{3a}{2}, \pm \frac{2a}{\sqrt{3}}\right)$ are points of inflexion.

3. In the curve $x^3 - 3bx^2 + a^2y = 0$, the point $x = b, y = \frac{2b^3}{a^2}$ is a point of inflexion.

4. $x = 0, y = c\ \left(\frac{c}{b}\right)^{\frac{1}{3}}$ is a point of inflexion of the curve $ax^3 + by^3 = c^4$.

5. In the *harmonic curve*, $y = a\ \sin\frac{x}{r}$, every point of intersection of the curve with X is a point of inflexion.

6. In the *lemniscate of Bernoulli*, $r^2 = a^2\cos 2\theta$, the centre is a point of inflexion for both branches of the curve.

7. Determine whether, and if so, under what circumstances of construction, the *conchoïd* can have points of inflexion.

8. Have the *conic sections* points of inflexion?

9. In the *lituus* $r = a\theta^{-1}$, there is an inflexion at $\theta = \frac{1}{2}, r = \pm a\sqrt{2}$.

10. In $r = a\theta^n$, there are generally points of inflexion at the points determined by $r^{-\frac{1}{n}}\{nr^{\frac{2}{n}} + n^2\,(n+1)\,a^{\frac{2}{n}}\} = 0$.

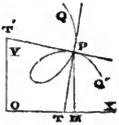
11. In the *trochoïd* $x = a\,(\omega - m\sin x)$, $y = a\,(1 - m\cos\omega)$, there is a point of inflexion at $\cos\omega = m$ and $y = a\,(1 - m^2)$.

12. Show that at a point of inflexion the *radius of curvature is infinite*.

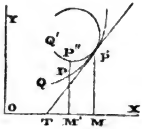
III. MULTIPLE POINTS OF THE FIRST SPECIES.

Some instances of specific curves which have two branches passing through the same point have already been given: as, for instance, the *lemniscate of Bernoulli*, the curve of *Watt's parallel motion*, and one variety of the *conchoïd*—the inferior one when b is greater than a. There are curves, however, in which three, four, or indeed any number of branches pass through the same point—of which the *corolla* is an instance. The points in which two or more branches intersect are called *multiple points;* and are specially called *double, triple,* etc., according as *two, three,* etc., branches pass through the point.

These multiple points are of the *first* or *second species,* according as the branches *cut* each other or *touch* each other in the multiple point. In the first place, let P be a double point of the *first species*: then, since the branches QP, Q'P *cut* in P, there will be two different values of $\frac{dy}{dx}$, or of the inclination of the tangent at P to the axis X.



Again, for the *second species,* the curves touching at P, the value of $\frac{dy}{dx}$ will be the same for both branches: but the deflexion from the tangent, expressed by $\frac{d^2y}{dx^2}$ will have two different values.



These circumstances, supposing any number of branches to pass through P, and modifying the statement accordingly, are strictly and fully characteristic of the two species of multiple points.

Let $u = f(x, y) = 0$ be the equation of the curve, freed from radical expressions: then at a multiple point, $\frac{dy}{dx} = \frac{0}{0}$. For the differentiated equation takes the form $\frac{du}{dy} \cdot \frac{dy}{dx} + \frac{du}{dx}$; and which we shall write $\mathrm{M}.\frac{dy}{dx} + \mathrm{N} = 0$, in which $\mathrm{M} = \frac{du}{dy}$, and $\mathrm{N} = \frac{du}{dx}$.

Now, since the branches intersect, $\frac{dy}{dx}$ will, as before explained, have as many values as there are branches passing through P: but which, for simplicity, we shall in the first place reason upon as two. Also, evidently, M and N will be the same in both. Let α, β be the two values of $\frac{dy}{dx}$: then we shall have

$$\mathrm{M}\alpha + \mathrm{N} = 0, \text{ and } \mathrm{M}\beta + \mathrm{N} = 0; \text{ or } \mathrm{M}(\alpha - \beta) = 0.$$

Now, since α, β are by hypothesis different, this equation is only fulfilled by $\mathrm{M} = 0$; and hence, also, $\mathrm{N} = 0$. Wherefore,

$$\frac{dy}{dx} = -\frac{\mathrm{N}}{\mathrm{M}} = \frac{0}{0}.$$

which establishes the proposition.

Again, as at *page* 423, we may seek the value of $p = \frac{dy}{dx}$. Thus, let x_1, y_1 be the values of x and y at the multiple point: then differentiating the numerator and denominator, we get

$$p = -\frac{\mathrm{N}}{\mathrm{M}} = -\frac{\frac{d\mathrm{N}}{dy_1}p + \mathrm{N}_2}{\frac{d\mathrm{N}}{dy_1} + \mathrm{M}_2 p} = -\frac{\mathrm{M}_1 p + \mathrm{N}_2}{\mathrm{M}_1 + \mathrm{M}_2 p}\left(\text{putting } \mathrm{M}_1 = \frac{d\mathrm{N}}{dy_1} = \frac{d\mathrm{M}}{dx_1}\right),$$

$$\text{or by reduction, } \mathrm{M}_2 p^2 + 2\mathrm{M}_1 p + \mathrm{N}_2 = 0;$$

from which the two values of p belonging to the double point may be found.

If, however, for the values x_1 y_1 of xy, this expression should become illusory (which will be the case when $M_1 = N_1 = M_2 = 0$) we may proceed to another differentiation. We shall thus have, putting q for $\frac{d^2y}{dx^2}$, and similar notation to that above for $\frac{dN_1}{dy_1}$, *etc.*

$$p = -\frac{M_1 q + N_2 p^2 + N_3 p + N_4}{M_2 q + M_3 p^2 + M_4 p + M_5} = -\frac{N_2 p^2 + N_3 p + N_4}{M_3 p^2 + M_4 p + M_5}, \text{ or}$$

$$M_3 p^3 + (M_4 + N_2) p^2 + (M_5 + N_3) p + N_4 = 0.$$

From this cubic (if all the roots be real) we can determine the values of p, at the *triple point* P.

We may proceed, obviously, in this way till we get a final result for the value of p; and there will be as many branches of the curve passing through P as there are *real roots* in the final equation. We should remark here, however, that should there be only one real value of p, the curve has not, in the sense we have used the term, a multiple point at P. The peculiarity of the case, where there are either none or only one real value of p, will be noticed presently, under the head of *conjugate points.*

EXAMPLES.

1. To find whether the curve $ay^2 = x^3 + bx^2$ has multiple points.

$$\text{Here } \frac{dy}{dx} = \frac{3x^2 + 2bx}{2ay} = \frac{0}{0} \text{ (when } x = 0 \text{ and } y = 0) = \frac{6x + 2b}{2a\frac{dy}{dx}};$$

$$\text{or, } \left(\frac{dy}{dx}\right)^2 = \frac{6x + 2b}{2a} = \frac{b}{a} \text{ (when } x = 0,\ y = 0); \text{ and hence, } \frac{dy}{dx} = \pm\sqrt{\frac{b}{a}}.$$

This indicates that the tangents of the curve at the origin intersect the axis X under angles whose trigonometrical tangents are $\left(\frac{b}{a}\right)^{\frac{1}{2}}$ and $-\left(\frac{b}{a}\right)^{\frac{1}{2}}$.

Or thus: $y = \pm\frac{x}{\sqrt{a}} \cdot \sqrt{x + b}$; $\frac{dy}{dx} = \pm\sqrt{\frac{x + b}{a}} + \frac{x}{2\sqrt{a\,(x + b)}}$. Make $x = 0$, then $y = 0$, and $\frac{dy}{dx} = \pm\sqrt{\frac{b}{a}}$, the same result as before.

2. *The lemniscate of Bernoulli,* $(x^2 + y^2)^2 = a^2(x^2 - y^2)$. At the origin, $x = 0$, $y = 0$, gives, putting $\frac{dy}{dx} = p$,

$$p = \frac{a^2x - 2x(x^2 + y^2)}{a^2y + 2y(x^2 + y^2)} = \frac{0}{0} = \frac{a^2 - 2(x^2 + y^2) - 2x(2x + 2py)}{a^2p + 2p(x^2 + y^2) + 2y(x + 2py)} = \frac{1}{p},$$

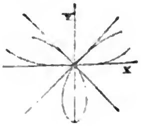
$$\text{or } p^2 = 1, \text{ and } \frac{dy}{dx} = \pm 1.$$

The tangents to the curve, therefore, make angles ± 45° with X.

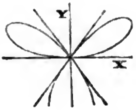
3. The annexed curve, having a triple point at the origin, is represented by the equation

$$x^4 - 2ax^2y - 2x^2y^2 + ay^3 + y^4 = 0.$$

The values of p are $\pm\frac{1}{2}$ and 0; showing that two of the tangents make ± 45° with X, and that the third tangent is X itself.



4. The following curve has a triple point at the origin:



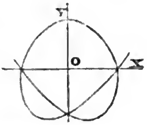
$$x^4 - ayx^2 + by^3 = 0,$$

and the values of p are 0 and $\pm\left(\frac{a}{b}\right)^{\frac{1}{2}}$.

5. To find the multiple points of the curve

$$u = x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0.$$

$$\frac{du}{dx} = 4\,(x^3 - a^2x),\ \frac{du}{dy} = -6\,(ay^2 + a^2y).$$



These vanish for the values of x and y, $(a, 0)$, $(-a, 0)$, and $(0 - a)$; and the corresponding values of $\frac{dy}{dx}$ are

$$\pm\left(\frac{4}{3}\right)^{\frac{1}{2}},\ \pm\left(\frac{4}{3}\right)^{\frac{1}{2}},\ \text{and}\ \pm\left(\frac{2}{3}\right)^{\frac{1}{2}}.$$

Its general form is annexed in the margin, but without the tangents drawn.

5. The curve $y^3 - 3axy + x^3 = 0$ has a double point at the origin, and its tangents are the axes X and Y.

6. The following curves have triple points:

(1). $ay^3 - bx^2y + x^4 = 0$.
(2). $(y^2 + ax)^2 = x^2\,(a^2 + 2ax - x^2)$.
(3). $(y^2 - x^2)^2 = ax^3 - 2bxy^2$.
(4). $y^4 - 2x^2y^2 + x^4 + 2axy^2 - 5ax^3 = 0$.
(5). $y^4 + x^4 - 4ay^3 + 2ay^2x + 2ax^3 + 8a^2y^2 + 4a^2xy - 8a^3y + 2a^4 = 0$.
(6). $x^4 - ay^3 + 2ax^2y + 4ax^3 + 3a^2y^2 + 4a^2xy + 4a^2x^2 - a^3y = 0$.

7. The following one has a quadruple point:

$$(x^2 + y^2)^3 = a^2\,(x^2 - y^2)^2.$$

IV. MULTIPLE POINTS OF THE SECOND SPECIES.

These, as explained in the last section, are such that the branches have a common tangent. They are hence often called *osculating branches*, and the points *points of osculation*.

This class of points are also expressed by $\frac{dy}{dx} = \frac{0}{0}$, and the values of this fraction the same for two or more of the branches. We shall consider, for simplicity, that it has but one value.

For let the order of osculation (*page* 406) be the nth between two branches at the point x_1y_1 of the curve. Then the differential co-efficients, as far as the $(n + 1)$th will be the same for both branches, and all the higher orders different. Let, as in the former case, the first differential equation be

$$\mathrm{M}\,.\,\frac{dy_1}{dx_1} + \mathrm{N} = 0.$$

Then the $(n + 1)$th will be N_1, being composed of differentials of a lower order, and functions of x_1 and y_1, and M the same throughout,

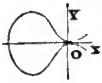
$$\mathrm{M}\,\frac{d^{n+1}\,y_1}{dx_1^{\,n+1}} + \mathrm{N}_1 = 0.$$

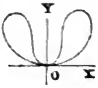
But $\frac{d^{n+1}\,y_1}{dx_1^{\,n+1}}$ has two values, α, β, while M and N_1 are constant; wherefore, as before, $\frac{dy}{dx} = \frac{0}{0}$.

As far as *double points* of this species are concerned, they depend upon $\frac{dy}{dx}$ having only one value, whilst $\frac{d^2y}{dx^2}$ has two: *triple points* upon $\frac{dy}{dx}$ having only one value, whilst $\frac{d^2y}{dx^2}$ has three; and so on.

EXAMPLES.

The curves $x^5 + bx^4 - a^3y^2 = 0$, and $x^4 + x^2y^2 - 6ax^2y + a^2y^2 = 0$ have osculating points at the origin, as represented in the figures.

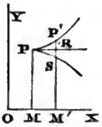


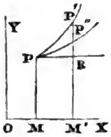


V. POINTS OF REGRESSION, OR CUSPS.

These are, in fact, a variety of the multiple points of the second species. They are distinguished by this:

The branch after arriving at P, instead of proceeding beyond it on the opposite side, they abruptly return, forming a new branch which has the same tangent at P that the former branch had.





When the branches are on opposite sides of the tangent, as in the first figure, the cusp is said to be of the *first species;* and when on the same side, as in the second figure, it is said to be of the *second species.*

The peculiar algebraical character of these points is, that they are osculating points, or $\frac{dy_1}{dx_1} = \frac{0}{0}$; and that, whilst for a slight increase of the value of x_1, $\frac{d^2y_1}{dx_1^2}$ will have two *real values*, and for a slight decrease of x_1, $\frac{d^2y_1}{dx_1^2}$ will have *two imaginary* values. Moreover, the cusp will be of the first or second species, according as these values have different or the same signs.

EXAMPLES.

1. The curve $(by - cx)^2 = (x - a)^4$ has a cusp of the first species when $x = a$, and the common tangent is parallel to the axis of x, similar to the first figure above.

2. The curve $x^4 - ax^2y - axy^2 + a^2y^2 = 0$ has a cusp of the second species at the origin, the axis of x being the common tangent. It would be represented by *fig.* 2 if the origin of co-ordinates were at P.

3. The curve $y^4 - axy^2 + x^4 = 0$ has at the origin a cusp of the first kind touching X, and a branch also touching Y.

4. The *semi-cubical parabola*, $ay^2 = x^3$, has a cusp of the first kind at the origin, both branches touching X.

5. At $x_1 = a$, the curve $y = b + cx^2 + (x - a)^{\frac{5}{2}}$ has a cusp of the second kind, and the inclination of its tangent to the axis of x is expressed by $\frac{dy}{dx} = 2ac$.

VI. CONJUGATE OR ISOLATED POINTS.

These are points which fulfil the *equation of the curve*, but do not belong to the geometrical genesis of it; and this being the case, though their existence may be detected as the result of the equation, they may be said to have no real existence, and no branches pass from them or through them *.

It will follow from the description we have given of this class of points, that if $x_1 y_1$ fulfil the equation, then $x = x_1 - h$ and $x = x_1 + h$ will render y and all the differential co-efficients imaginary.

At a conjugate point, $\frac{dy}{dx} = \frac{0}{0}$, as in the other singular points.

For let $u = f(x, y) = 0$; then, as before, $\mathrm{M} \, . \, \frac{dy}{dx} + n = 0$; but by definition, $\frac{du}{dx} = \alpha + \beta \sqrt{-1}$, or imaginary; whence $\mathrm{M}\alpha + \mathrm{N} + \mathrm{M}\beta\sqrt{-1} = 0$. From this we get $\mathrm{M}\alpha + \mathrm{N} = 0$, and $\mathrm{M} = 0$ (*vol.* i. *p.* 150); and hence $\mathrm{N} = 0$. Wherefore also, $\frac{dy}{dx} = -\frac{\mathrm{N}}{\mathrm{M}} = \frac{0}{0}$.

EXAMPLES.

1. Has the curve $ay^2 - x^3 + bx^2 = 0$ a conjugate point?

Here $y = x\sqrt{\frac{x-b}{a}}$; and $\frac{dy}{dx} = \sqrt{\frac{x-b}{a}} + \frac{x}{2\sqrt{a(x-b)}}$.

When $x = 0$, $y = 0$, by the equation of the curve; and $\frac{dy}{dx} = \sqrt{-\frac{b}{a}}$.

The origin is therefore a conjugate point; and it may be seen from the equation itself that all values of x algebraically less than $+ b$ will give y imaginary.

2. The logarithmic curve has an infinite number of conjugate points, situated in another logarithmic curve which is symmetrical with the given one in respect of the axis X.

3. The logarithmic spiral has, similarly, an infinite number of conjugate points, situated on another logarithmic spiral similar to itself.

4. The curve $a(y^2 - b^2) = (x^2 - c^2)\sqrt{c^2 - 2x^2}$ has four conjugate points determined by $x = \pm c$, and $y = \pm b$.

5. Establish that in the following curves, there are conjugate points as stated:

Curves.	*Conjugate points at*
(1). $y = (x + 2)\sqrt{x + 1 + 3}$.	$x = \quad -2, y = 3$.
(2). $y + a = (a - b + x)\sqrt{x - b}$.	$x = b - a, y = a$.
(3). $y + a = (x - b)\sqrt{x - 2b}$.	$x = \quad b, y = a$.
(4). $a^2(y - b) = (2cx - x^2)(c + x)^2$.	$x = \quad -c, y = b$.
(5). $y^2 - 1 = (x + 1)\sqrt{x - 1}$.	$x = \quad -1, y = \pm 1$.
(6). $(y - 3)^6 = (x - 2)^2(1 - x)^3$.	$x = \quad 1, y = 3$.

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* A theory has for some years been gaining ground, that imaginary expressions in geometry represent points or lines situated out of the assumed limits of the ordinates; and as applied to this case, that they represent the intersection of the plane XY with curves situated out of that plane. When we consider the distinguished names that are ranged in support of this view, though we think the doctrine unsound, it would be most unbecoming to treat it with other than the respect to which the high character of Dr. Peacock and Mr. Gregory, its great supporters, entitles everything that comes from their hands. It may possibly, however, hereafter become the subject of a dissertation by the editor of this work.

CHAPTER XII.

ON TRACING CURVES BY MEANS OF THEIR EQUATIONS.

This consists in finding the course of the curve, assigning the character of the several branches and single points. It becomes, in fact, the application of the several subjects already discussed in reference to peculiar and given curves; and as each of the steps has been illustrated by reference to the examples furnished by particular curves, it will be unnecessary to add more here than a general synopsis of the process, analogous to that given for the solution of equations in *vol. i. p.* 234. We shall annex a single example by way of illustration.

1. If the geometrical genesis of the curve be given, find its equation, either in rectilinear or polar co-ordinates as may be found most convenient.

2. Supposing the equation either found or given in any systen of co-ordinates, examine whether it can be simplified by the transformation of co-ordinates, to either of the other systems; and try also whether it can be simplified by the transformation of origin and direction.

3. Resolve, if possible, the equation of the curve with respect to one of the variables; as, generally, in this form, both the calculations and constructions are more easily performed.

Set off in their identical direction, according to the system of co-ordinates employed, any successive values of the other variable; and then the corresponding computed value of the one for which the equation has been solved. Any number of points may thus be found, and the curve traced with a greater or less degree of accuracy, according to the number of points laid down.

It is, however, more frequently our object to discover merely the general features of the curve, than to make a figure of great exactness; and in this case a comparatively small number of points will be requisite, and even these may be, mainly, deferred to a later part of the inquiry.

4. When the equation does not admit of symbolical resolution in the way directed in the last case, we may assume successive *numerical* values for one of the variables, and compute the values of the other which correspond to each of these. Then each of these values will give one point in each, or in certain branches of the curve: viz., those values which are real give points in the curve, whilst those that are imaginary, express that the lines upon which the values are set off, do not meet that line.

This, like the preceding, except for actual tracing, may be deferred till a later period of the investigation.

5. Find when the curve cuts the axes of co-ordinates by making successively y and x equal to zero.

6. Find whether the curve has symptotes, and assign their position if any such exist.

7. Find for what values of the variables the ordinates are maxima or minima; and determine which each of them is, either by means of the sign of $\frac{d^2y}{dx^2}$, or by the signs of $\frac{dy}{dx}$ for quantities less or greater than that now found.

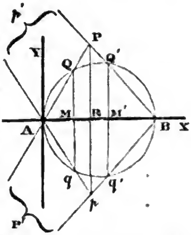
8. Examine which way the concavity of the curve lies with respect to the axis

in each region of the curve; and the points of inflexion, by means of the sign of $\frac{d^2y}{dx^2}$.

9. Discuss the equation* $\frac{dy}{dx} = \frac{0}{0}$, arising from putting both numerator and denominator equal to zero, and resolving the resulting equations. If the values of x and y resulting from their solution be real, the curve has singular points—either ***multiple intersections, multiple osculations, cusps,*** or ***conjugate points.*** In all cases the complete assignment of each of these where they exist, or the proof that none do exist, is essential to the entire discussion of the curve.

10. There are very few cases in which any curve possesses all these peculiarities; but the discussion is requisite, nevertheless, as it is by direct investigation only, that we can be assured of the non-existence of any one of them in a given curve.

EXAMPLE.



On the diameter AB of a given circle, take BM′=2AM and draw the ordinates MQ, M′Q′ to the circle; then draw AQ, BQ′, to meet in P: find the curve traced by P during the passage of the position of M from A to B.

(1). Let the given diameter AB be taken as axis of x, and the tangent at A as the axis of y. Let $AB = 2a$, and denote the co-ordinates of P, Q, and Q′ by xy, x_1y_1, x_2y_2. Then the general equation of the circle being $y^2 = 2ax - x^2$, we have

$$y_1^2 = 2ax_1 - x_1^2 \;\ldots\ldots\; (1) \quad | \quad y_2^2 = 2ax_2 - x_2^2 \;\ldots\ldots\ldots\ldots\; (2).$$

But by the conditions of the question $BM' = 2AM = 2x_1$; and hence

$$x_2 = 2(a - x_1) \;\ldots\ldots\ldots\ldots\ldots\ldots\; (3).$$

Also the equations of AQ, BQ′, are respectively

$$xy_1 = x_1y \;\ldots\ldots\; (4) \quad | \quad y_2(2a - x) = y(2a - x_2) \;\ldots\ldots\; (5).$$

Insert (3) in (2) and (5): then we get

$$y_2^2 = 4x_1(a - x_1) \;\ldots\; (6) \quad | \quad y_2(2a - x) = 2x_1y \;\ldots\ldots\ldots\; (7).$$

Eliminating y_2 from (6, 7), and resolving for x_1 we have

$$x_1 = \frac{a(2a - x)^2}{y^2 + (2a - x)^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (8).$$

From (1, 4) we also get $x_1 = \frac{2ax^2}{x^2 + y^2}$ .... (9),

and from (8, 9) eliminating x_1, we get the equation sought,

$$(2a - x)^2 (y^2 - x^2) = 2x^2y^2 \;\ldots\ldots\ldots\ldots\; (10)\dagger.$$

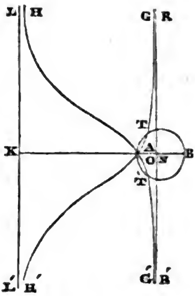
---

[*] That is, supposing both numerator and denominator to contain x and y, or the differentiation to have been performed without previous resolution of the equation of the curve for either of the variables. The fulfilment of the equation of the curve by the values of the variables is in this case (even supposing x, y given by the differential equations, to be real) the test of the accordance of all the conditions. If, however, the equation be $y = f(x)$, then the possibility of $\frac{dy}{dx} = \frac{0}{0}$ depends on x having the same values in the numerator and denominator.

[†] This equation might, however, but for the better exercise in the method employed in the text, have been obtained more briefly. For

$$\frac{y^2}{x^2} = \frac{QM^2}{AM^2} = \frac{2ax_1 - x_1^2}{x_1^2} = \frac{2a}{x_1} - 1, \text{ and } \frac{y^2}{(2a - x)^2} = \frac{M'Q'^2}{M'B^2} = \frac{2x_1(2a - 2x_1)}{(2x_1)^2} = \frac{a}{x_1} - 1.$$

Eliminating x_1 from these, we get the equation above.



Without paying further attention to the geometrical genesis of the curve, let us examine its equation (10). The equation is simplest when resolved for y, giving,

$$y = \pm \frac{(2a - x)\,x}{\sqrt{4a^2 - 4ax - x^2}}.$$

(2). For each possible value of x, y has two equal values with opposite signs; and hence throughout its entire course, there are two equal and symmetrical branches considered with respect to the axis X.

(3). That a point may belong to the curve, the denominator must be real; and hence the possible values of x must be limited by the roots of the equation $4a^2 - 4ax - x^2 = 0$, viz., between $2a(\sqrt{2} - 1)$ and $-2a(\sqrt{2} + 1)$. Wherefore, if we make AK in the negative direction of X, equal to $2a(\sqrt{2} + 1)$, and AN in the positive direction of X, equal to $2a(\sqrt{2} - 1)$: then the curve lies wholly within the parallels KL NR to Y drawn through K and N. For all values of x beyond these limits give y imaginary.

(4). To find where the curve cuts the axis of x, put $y = 0$, then $(2a - x)\,x = 0$. The values of x in this are $2a$ and 0: the former of which rendering the value of y imaginary, shows that the curve does not cut the axis in the corresponding point; and the latter shows that it passes through the origin.

Again, to find where the curve meets Y, put $x = 0$: then the equation is reduced to $4a^2y^2 = 0$, or the curve does not meet the axis of y in any point besides the origin.

(5). The lines KL and NR are asymptotes to the curve. This is obvious from the consideration that as x approaches to the values $2a(\sqrt{2} - 1)$ and $-2a(\sqrt{2} + 1)$, the denominator approaches zero whilst the numerator is of finite value; and hence for a finite value of x, that of y becomes, at these extremes, infinite. This is characteristic of the asymptote. We shall, however, examine it by the differential criterion in (7).

(6). By differentiating the equation of the curve, we get

$$\frac{dy}{dx} = \pm \frac{8a^3 - 12a^2x + 6ax^2 + x^3}{(4a^2 - 4ax - x^2)^{\frac{3}{2}}};\ \text{and,}$$

$$\frac{d^2y}{dx^2} = \pm \frac{24a^2x\,(2a - x)}{(4a^2 - 4ax - x^2)^{\frac{5}{2}}} = \frac{24a^2y}{(4a^2 - 4ax - x^2)^2}.$$

(7). To find where the tangents are parallel to the axes of y and x, put $\frac{dx}{dy} = 0$, and $\frac{dy}{dx} = 0$ respectively.

The former gives $4a^2 - 4ax - x^2 = 0$; which, by what has been shown in (3), proves that KL and NR are tangents to the curve. Also, y being in this case infinite, the criterion of these lines being asymptotes is fulfilled. Also, these are all the roots of that hypothesis, and hence there are no other tangents parallel to Y.

The latter gives

$$x^3 + 6ax^2 - 12a^2x + 8a^3 = 0.$$

In this equation two positive and one negative root are indicated by the signs; and the application of Budan's criterion (*vol. i. p.* 226), with the transformations in a and a^{-1}, show that the two positive roots are imaginary.

By changing the alternate signs to convert the negative into positive roots, and finding the first approximation, we find that the negative root lies in the interval between $-7a$ and $-8a$: to which, by (3), no real value of y corresponds. Wherefore, there is no tangent parallel to the axis of x.

(8). At the origin we have $x = 0$, and hence $\frac{dy}{dx} = \pm \frac{8a^3}{(4a^2)^{\frac{3}{2}}} = \pm 1$; and hence the origin is a double point in the curve; and the branches (that is their tangents) intersect X under angles of ± 45°, and at right angles to each other.

(9). The point A, or origin, is a point of inflexion of both branches of the curve. For $\frac{d^2y}{dx^2} = \frac{24a^2y}{(4a^2 - 4ax - x^2)^2}$; and hence, $\frac{d^2y}{dx^2}$ has always the same sign as y, all the other terms being essentially positive in this expression. Wherefore, as in passing through A, y changes its sign, either from + to −, or from − to +; the second differential co-efficient also changes its sign at that point; which is the criterion of a point of inflexion.

Moreover, y, and therefore $\frac{dy^2}{dx^2}$, does not change its sign at any other point (for no change can take place but by its passing through 0 or infinity); and hence there is no other point of inflexion in either branch.

(10). For further investigation, take the *rational equation of the curve*,

$$(2a - x)^2 (y^2 - x^2) = 2x^2y^2 \text{ ; then}$$

$$\frac{dy}{dx} = \frac{\pm y^3 (8a^3 - 12a^2x + 6ax^2 + x^3)}{x^3 (2a - x)^3}.$$

And we have seen that the curve passes through $x = 0$, $y = 0$, and hence at this point, $\frac{dy}{dx} = \frac{0}{0}$, which is the condition of a singular point; either multiple, osculatory, cusped, or conjugate, or one or more of these together. We have, indeed, seen by (9) that it is a point of inflexion, by means of the resolved or explicit equation; but we may also learn it from this, the implicit equation of the curve. For the criterion is fulfilled by $\frac{dy}{dx} = \frac{0}{0}$ and $\frac{d^2y}{dx^2}$ changing signs at the point (0, 0): whilst taking the third differential co-efficient of the explicit equation (as the more convenient one) we get

$$\frac{d^3y}{dx^3} = \pm \frac{24a^2 (8a^3 + 4a^2x - 10ax^2 - 3x^3)}{(4a^2 - 4ax - x^2)^{\frac{7}{2}}}.$$

This does not become zero when $x = 0$; and hence the origin is a multiple point. Also, as there are only two simultaneous values of y for each value of x, there cannot be more than two branches in the curve, and hence the point can only be a double point.

(11). Since there is but one singular point (the origin), and the character of that point for all the branches has been found, it would be useless to inquire respecting the possibility of osculating points or cusps. It may, however, as an example, be investigated in the present case.

Osculation requires that $\frac{dy}{dx}$ shall have but one value, whilst $\frac{d^2y}{dx^2}$ shall have two. Now we have seen in the present case that $\frac{dy}{dx}$ has two values, as well as $\frac{d^2y}{dx^3}$; and hence the criterion of osculation is not fulfilled, and the curve has no osculating point.

Cusps amongst their conditions involve the same criterion, and hence there is no cusp.

(12). Returning to the general equation $(2a - x)^2 (y^2 - x^2) = 2x^2y^2$, we see that if $2a - x = 0$, then also $y = 0$; and this gives

$$\frac{dy}{dx} = \frac{(2a - x)^3 - 2x^3}{\{(2a - x)^2 - 2x^2\}^{\frac{3}{2}}} = \frac{-(2.2a)^3}{(-2.2a^2)^{\frac{3}{2}}} = \pm \tfrac{1}{2}\sqrt{-2};$$

which, being imaginary, shows that B is a conjugate point.

(13). It only remains to connect the algebraic expression with the genesis of the curve itself.

The two branches lying to the *right* from A form the path of P when y_1 and y_2 have the same sign, either both + or both —; as MQ, M′Q′, or Mq, M′q', in the figure of the genesis. The upper branch, AG, of the curve in the general figure is traced by P, and the lower one, AG′, by p.

The two branches, AH, AH′, to the *left* from A, are traced by the intersections of Aq, BQ′, and those of Bq', AQ, respectively: the former pair meeting in p', and the latter in P′.

THE INTEGRAL CALCULUS.

CHAPTER I.

DEFINITIONS, PRINCIPLES, AND NOTATION.

1. In the differential calculus the function was given to find its differential, or differential co-efficient: in the integral calculus, the differential or differential co-efficient is given to find the original function. The original function from which differentiation would have furnished the given differential, is called the *integral* of that differential expression; and the system of inquiries devoted to the discovery of methods by which the integrals of different classes of expressions can be found, constitutes the *integral calculus*. The operation itself is called *integration*.

2. As the prefix d was used to signify the operation of differentiating the expression to which it was prefixed, so also $\int$ is used to express the operation of integrating the function to which it is prefixed. Thus, the differential of $\sqrt{a^2 - x^2}$ is $d\sqrt{a^2 - x^2}$, or $\dfrac{-xdx}{\sqrt{a^2 - x^2}}$; and $\displaystyle\int\frac{-xdx}{\sqrt{a^2 - x^2}}$, or $\int d\sqrt{a^2 - x^2}$, or $\sqrt{a^2 - x^2}$, expresses the integral of $\dfrac{-xdx}{\sqrt{a^2 - x^2}}$, or of $d\sqrt{a^2 - x^2}$.

3. It follows that the defined senses of the symbols $\int$ and d are those which neutralize each other. So that if u be any function, we have $\int du = u$: and if $\phi_1(x)dx$ be the first differential of the function $\phi(x)$, then $\int \phi_1(x)dx = \phi(x)$, and $d\int \phi_1(x)dx = \phi_1(x)dx$.

4. The results of all *direct processes* can be assigned, as those of multiplication, involution, and differentiation: for the denoted operations can all be performed in an assigned order, and the result will be exact and definite, instead of approximative, as is most frequently the case in the inverse processes of division,

evolution, and (as will presently appear) integration. If we divide 10 by 3, or $a^2 + x^2$ by $a + x$, the operation never terminates; if we extract the square root of 5, or evolve $(a - x)^{\frac{3}{4}}$, we get an interminable series in both cases. In fact, the inverse operation indicated is often one which has no corresponding direct one by which the expressions themselves could have been produced in finite terms: thus, 5 could not have been produced by squaring any finite numerical expression, nor could $a - x$ have been produced by raising any finite function of a and x to the fourth power, and extracting the cube root of that fourth power. So also, the great majority of differential expressions, whether written at hazard or deduced from the expression of the conditions of a problem which involves differential considerations, are of a kind that do not admit of general integration in a finite number of terms. Some, indeed a considerable number, of the most useful ones, will, however, admit of integration in either exponential, logarithmic, circular, or elliptic functions: and this confers facility in calculating their numerical values (a circumstance of great importance in the application of the science) when specific values are given to the variable in terms of which the integral is expressed. This will be better seen as we proceed, as it is only mentioned here that the student may be prepared to comprehend the causes of the impossibility of general integration in a finite number of terms.

5. Those integrals which constitute a class of functions, from which, by *direct differentiation*, differentials composed of single terms, or a definite number of terms arise, are called *elementary integrals*. We obtain these by inspection only, from having observed that the differentials of these expressions are the differential expressions propounded for integration. The great art of general integration consists in transforming a given differential expression (either at once or gradually) into a series of differentials of elementary integrals; and that integration is the most elegant which makes that series of terms of the smallest number and most simple character.

6. We shall close this introductory chapter with the following propositions, which are merely inversions of the corollaries to *prop. i. Differential Calculus, page* 356, 7.

(1). Any constant factor which appears under the sign of integration may be placed without that sign.

For, as $ad\phi(x) = da\phi(x)$, so $a\int d\phi(x) = \int da\phi(x)$.

(2). As the absolute term or constant of any expression proposed for differentiation, disappears by differentiating, so it must re-appear by integrating. The special value of it, however, cannot be found, except we know the exact expression from which the differential was derived, or some circumstance in the problem from which the differential arose, by which its value can be fixed. We shall see instances, hereafter, of the determination of this value in the applications of the calculus; but for the present it will be denoted by $+ c$ or $+ C$, annexed to the integral itself as an addend. These corrections are called *arbitrary constants*. Thus, if $\int nx^{n-1}\,dx$ be supposed to be x^n; then we should write $\int nx^{n-1}dx = x^n + c$ or $x^n + C$.

(3). The integral of any number of differentials connected by addition or subtraction, is the same with the sum or difference (as indicated by their signs) of the integrals of the several terms of the differential.

(4). All radical expressions, as in the differential calculus, must be expressed by fractional indices; and, for the most part, it will be better to bring denominators into the numerator with negative indices.

(5). It is often convenient to express the integral by a single letter, as in the differential calculus, and put its differential equal to the differential expression given for consideration. Thus, $dy = (2ax - x^2)^{\frac{3}{2}} (a - x)\, dx$; in which y is obviously the value of the integral, or expresses what function of x the integral is.

Some writers give only the differential co-efficients for integration. Thus they propose the question to integrate the differential co-efficient $(2ax - x^2)^{\frac{3}{2}} (a - x)$, by which they simply mean to integrate the equation $\frac{dy}{dx} = (2ax - x^2)^{\frac{3}{2}}(a - x)$, which is obviously only the same thing as $dy = (2ax - x^2)^{\frac{3}{2}} (a - x)\, dx$ given above.

CHAPTER II.

ELEMENTARY INTEGRALS.

PROP. I.

To integrate functions of the general form $az^n dz$,

SINCE in differentiation of such functions the index of z in the integral is a unit higher than in its differential, put
$a\int z^n dz = bz^{n+1}$: then $az^n dz = (n + 1)\, bz^n dz$, and hence $(n + 1)\, b = a$, and the integral is

$$a\int z^n dz = \frac{a}{n+1} \cdot z^{n+1} + c \quad \ldots\ldots\ldots\ldots\ldots\ldots \int(1).$$

Corollary.

If $z = \phi x$, still the same conclusion holds good whatever form ϕ may have; namely

$$\int (\phi x)^n d\phi x = \frac{(\phi x)^{n+1}}{n+1} + c.$$

EXAMPLES.

1. Integrate $3ax^5 dx$. Here effacing dx, increasing the index by 1, and dividing by the index so increased, we get

$$\int 3ax^5 dx = \tfrac{1}{2}ax^6 + c.$$

2. Find the integrals of $15x^{\frac{2}{3}}\, dx$ and $-9\, x^{-\frac{1}{3}}\, dx$.

Proceeding as in the last example, we get

$$\int 15x^{\frac{2}{3}}\, dx = 15\int x^{\frac{2}{3}}\, dx = \frac{3}{5} \cdot 15x^{\frac{5}{3}} + C = 9x^{\frac{5}{3}} + C$$

$$\int -9x^{-\frac{1}{3}}\, dx = -9\int x^{-\frac{1}{3}}\, dx = -\frac{5}{4} \cdot 9x^{\frac{4}{3}} + C = -\frac{45}{4}\, x^{\frac{4}{3}} + C.$$

3. Integrate $(a - a_1x + a_2x^2 - \ldots. \pm a_nx^n)\,dx$: that is

$$adx - a_1xdx + a_2x^2dx - \ldots. \pm a_nx^ndx.$$

Integrating each term as before, and collecting the results, we get

$$C + \frac{ax}{1} - \frac{a_1x^2}{2} + \frac{a_2x^3}{3} - \ldots. \pm \frac{a_nx^{n+1}}{n+1},$$

the arbitrary constant being prefixed.

4. To integrate $\sqrt{a^2 + x^2}\,xdx$, or which is the same thing, $(a^2 + x^2)^{\frac{1}{2}}\,xdx$. The part xdx without the vinculum is, with the exception of the factor 2, the differential of that within; and hence, multiplying and dividing by 2, we get

$$\frac{1}{2}\int (a^2 + x^2)^{\frac{1}{2}}.\,2xdx = \frac{1}{2}\int (a^2 + x^2)^{\frac{1}{2}}\,d\,(a^2 + x^2) = \frac{1}{3}\,(a^2 + x^2)^{\frac{3}{2}} + C.$$

5. Let there be given for integration $(b + cx^n)^m\,ax^{n-1}\,dx$.

Multiply and divide by $\frac{a}{nc}$: then we have the requisite form

$$\frac{a}{nc}\int (b + cx^n)^m.\,nc\,x^{n-1}\,dx = \frac{a}{nc\,(m+1)}.\,(b + cx^n)^{m+1} + C.$$

6. Let $(b - cx)^{-n}\,adx$, or $\frac{adx}{(b - cx)^n}$ be proposed for integration.

Multiply and divide by $-\frac{a}{c}$: then we have

$$-\frac{a}{c}\int (b-cx)^{-n} \times -cdx = -\frac{a}{-(n-1)c}\,(b-cx)^{-(n-1)} + C = \frac{a\,(b-cx)^{1-n}}{(n-1)\,c} + C.$$

7. $\int (ax + bx^2 + cx^3)^m\,(a + 2bx + 3cx^2)\,dx = \frac{(ax + bx^2 + cx^3)^{m+1}}{m+1} + C.$

8. Given $dy = adx - \frac{bdx}{x^3} + x^{\frac{3}{2}}dx$ to find y in terms of x.

Ans. $y = ax + \frac{b}{2x^2} + \frac{2}{5}\,x^{\frac{5}{2}} + C.$

10. If dy* $= (2ax - x^2)^{\frac{5}{2}}\,(a - x)\,dx$, $y = \frac{1}{7}\,(2ax - x^2)^{\frac{7}{2}} + C.$

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[*] As it is not always easy to discover by inspection whether the factor which is without the vinculum be the differential of that within, it will be desirable to employ in such cases the following method of investigation. Thus, in the case in the text, let us assume that the integral is $A\,(2ax - x^2)^{\frac{7}{2}}$. Whence we have

$$(2ax - x^2)^{\frac{5}{2}}\,(a - x)\,dx = \frac{7}{2}\,A\,(2ax - x^2)^{\frac{5}{2}}\,(2a - 2x)\,dx;$$

and $a - x = \frac{7}{2}\,A\,(2a - 2x) = 7A\,(a - x)$. This must be fulfilled for both terms (in the manner and on the principle of indeterminate co-efficients) giving $a = 7Aa$, and $-1 = -7A$, or $A = \frac{1}{7}$. This value of A fulfils *both* equations; and the differential is of the appropriate form.

Had the differential function been given $(2ax - x^2)^{\frac{5}{2}}\,(3a - 2x)\,dx$: then $3a = 7Aa$ and $-2 = -\frac{7}{2}\,A$; which give different values of A, and hence the differential is not of a form to fall under the rule.

Even now, however, it may be divided into two parts, one of which falls under the present rule. For, it is equal to

$$(2ax - x^2)^{\frac{5}{2}}\,(2a - 2x)\,dx + a\,(2ax - x^2)^{\frac{5}{2}}\,dx;$$

and the former term falls under the rule, whilst the latter will be hereafter discussed.

Sometimes the results obtained by this method, under different aspects may *seem* to be different in fact, whilst they are only so in form. For instance,

11. Let there be given $dy = (a + bx)^2 dx$, for integration.

Making the outside of the vinculum the differential of the inside,

$$y = \frac{1}{b}\int (a + bx)^2 \, bdx = \frac{1}{b}\int (a + bx)^2 \, d\,(a + bx) = \frac{(a + bx)^3}{3b} + \text{C}.$$

Again, expanding $(a + bx)^2 dx$, we have

$$y = \int a^2\,dx + \int 2abxdx + \int b^2x^2dx = a^2x + abx^2 + \frac{1}{3}\,b^2x^3 + \text{C}'$$

The former result expanded is $\frac{a^3}{3b} + a^2x + abx^2 + \frac{1}{3}\,b^2x^3 + \text{C}$; the only difference, then, is, that C in the first result and C′ in the second are different; but putting $\text{C}' = \text{C} + \frac{a^3}{3b}$, the two results become identical. The form of an integral is often capable of much simplification by means of a little management of the form of the arbitrary constant, of which numerous instances will be presently given.

Many expressions really belong to this form which do not at first seem to do so; or, at least, by very slight transformation may be rendered such. For instance,

12. Let $dy = \frac{2adx}{x\sqrt{2ax - x^2}}$ be given.

Divide all by x^2: then we have $dy = (2ax^{-1} - 1)^{-\frac{1}{2}}\,2ax^{-2}dx$, in which $-2ax^{-2}dx$ is the differential of the part within the vinculum; and hence

$$y = -\,(2ax^{-1} - 1)^{\frac{1}{2}} + \text{C} = -\,\frac{\sqrt{2ax - x^2}}{x} + \text{C}.$$

13. Given $dy = xdx\,(2ax - x^2)^{-\frac{3}{2}} = (2ax^{-1} - 1)^{-\frac{3}{2}}\,x^{-2}dx$; then

$$y = \frac{1}{a}\sqrt{\frac{x}{2a - x}}.$$

14. Given $dy = \frac{\sqrt{a^2 + x^2}}{x^4}\,dx = (a^2x^{-2} + 1)^{\frac{1}{2}}x^{-3}$: then $y = -\,\frac{(a^2 + x^2)^{\frac{3}{2}}}{3a^2x^3}$.

15. Integrate $dy = z^{-1}dz$. Here we get

$$y = \frac{z^0}{0} + \text{C} = \frac{1}{0} + \text{C},$$

an expression altogether ambiguous; and therefore in this particular case the rule does not apply. It will be renewed in the next proposition.

PROP. II.

To integrate functions of the general form, $dy = \frac{d\phi x}{\phi x}$.

This is the differential of $\log_e \phi x$ (by *page* 358); and hence

$$y = \log \phi x + \text{C} = \log_e \phi x + \log_e c = \log_e c\,\phi x \;\ldots\ldots\ldots\ldots \int (2).$$

Where $C = \log_e c$, this form of the constant being more elegant and more useful than the addend C.

EXAMPLES.

1. Let $dy = \dfrac{3x^2dx}{x^3 + a^3}$: then $\phi x = x^3 + a^3$ and $d\phi x = 3x^2dx$.

The example therefore comes under this rule, and we have

$$y = \log_e (x^3 + a^3) + C = \log_e c\,(x^3 + a^3).$$

2. If $dy = \dfrac{ax^n dx}{b^n + b_1 x^{n+1}}$; then $y = \dfrac{a \log_e c\,(b^n + b_1 x^{n+1})}{(n+1)\,b_1}$.

3. The integrals of $\dfrac{dx}{a+x}$ and $\dfrac{x^2dx}{1+x^3}$, are $\log_e c\,(a + x)$ and $\log_e c\,(1 + x^3)^{\frac{1}{3}}$.

4. $\displaystyle\int \frac{5x^3dx}{3x^4 + 2} = \log_e c\,(3x^4 + 2)^{\frac{5}{12}}$; and $\displaystyle\int \frac{(2a - 3x)\,dx}{ax - x^2} = \log_e 3cx^2\,(a - x)$.

5. $\displaystyle\int \frac{a}{b^4} \cdot \frac{(x-a)^3\,dx}{x^2} = \frac{a}{b^4}\left\{\tfrac{1}{2}x^2 - 3ax + a^3x^{-1} + 3a^2 \log x\right\} + C.$

PROP. III.

To integrate functions of the forms $a^x dx$, $e^{ax} dx$, etc.

These are at once seen by inspection of the direct process to be,

$$\frac{a^x}{\log_e a},\ etc. \ldots\ldots\ldots\ldots \int (3), \qquad \Big| \qquad \frac{e^{ax}}{a},\ etc. \ \ldots\ldots\ldots\ldots \int (4).$$

It is quite obvious that if for x we substitute ϕx, the method will still apply. The only care required is to see that the part $d\phi x$ in any expression is really the differential of the index.

It will be at once apparent how to treat any differential involving an exponential (as far as this formula unmodified is concerned): viz. to ascertain whether the differential factor of the expression is that of the variable index. If it be so, the rule applies at once: if not, we must have recourse to other methods hereafter to be explained.

EXAMPLES.

1. Integrate $dy = \dfrac{3xdx}{x^2 + b^4} \cdot a^{\log_e (x^2 + b^2)}$.

This is of the form $a^u du$, where $u = \log_e (x^2 + b^2)$; and hence

$$y = \frac{a^{\log_e (x^2 + b^2)}}{\log_e a} + C.$$

2. Integrate $dy = \dfrac{a^{\frac{3}{3}x^{\frac{5}{3}}}\,adx}{x^3}$. Here as before $y = \dfrac{a^{\frac{3}{3}x^{\frac{5}{3}}}}{\log_e a} + C.$

PROP. IV.

The integration of the forms which follow, is inferred from the direct investigations and results at pp. 364, 5.

First form.	*Second form.*	*Third form.*	*Fourth form.*
$\pm \frac{dx}{\sqrt{a^2 - b^2x^2}}$	$\pm \frac{dx}{a^2 + b^2x^2}$	$\pm \frac{dx}{x\sqrt{b^2x^2 - a^2}}$	$\pm \frac{dx}{\sqrt{a^2x - b^2x^2}}$
or $\pm \frac{1}{a} \cdot \frac{rdx}{\sqrt{r^2 - x^2}}$	or $\pm \frac{1}{a^2} \cdot \frac{r^2dx}{r^2 + x^2}$	or $\pm \frac{b}{a^2} \cdot \frac{r^2dx}{x\sqrt{x^2 - r^2}}$	or $\pm \frac{2b}{a^2} \cdot \frac{rdx}{\sqrt{2rx - x^2}}$
where $r = \frac{a}{b}$	where $r = \frac{a}{b}$	where $r = \frac{a}{b}$	where $2r = \frac{a^2}{b^2}$.

In the following table the integrals are put down in their most general form; and they can be accommodated in all cases to the particular inquiry before us by giving the requisite values to the quantities a and b. The arbitrary constant C belonging to each of them is omitted for the sake of room: but it must always be understood to exist, and therefore be supplied in each instance where the formula is used.

Table of inverse trigonometrical integrals.

$+\int \frac{dx}{\sqrt{a^2 - b^2x^2}} = \frac{1}{b} \sin^{-1} \frac{bx}{a}$	$-\int \frac{dx}{\sqrt{a^2 - b^2x^2}} = \frac{1}{b} \cos^{-1} \frac{bx}{a}$
$+\int \frac{dx}{a^2 + b^2x^2} = \frac{1}{ab} \tan^{-1} \frac{bx}{a}$	$-\int \frac{dx}{a^2 + b^2x^2} = \frac{1}{ab} \cot^{-1} \frac{bx}{a}$
$+\int \frac{dx}{x\sqrt{b^2x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{bx}{a}$	$-\int \frac{dx}{x\sqrt{b^2x^2 - a^2}} = \frac{1}{a} \operatorname{cosec}^{-1} \frac{bx}{a}$
$+\int \frac{dx}{\sqrt{a^2x - b^2x^2}} = \frac{1}{b} \operatorname{vers}^{-1} \frac{2b^2x}{a^2}$	$-\int \frac{dx}{\sqrt{a^2x - b^2x^2}} = \frac{1}{b} \operatorname{covers}^{-1} \frac{2b^2x}{a^2}$

……………………… $\int(5)$.

EXAMPLES.

1. Integrate $\frac{xdx}{\sqrt{a - bx^4}}$. Here if we put $x^2 = z$: then

$$\int \frac{xdx}{\sqrt{a - bx^4}} = \int \frac{\frac{1}{2}dz}{\sqrt{a - bz^2}} = \frac{1}{2\sqrt{b}} \cdot \sin^{-1} \sqrt{\frac{b}{a}}\, z + \mathrm{C} = \frac{1}{2\sqrt{b}} \cdot \sin^{-1} \sqrt{\frac{b}{x}}\, x^2 + \mathrm{C}.$$

2. Let $dy = \frac{x^{n-1}\,dx}{\sqrt{a - bx^{2n}}}$: to find the value of y in terms of x.

Put $x^n = z$: then $x^{n-1}\,dx = \frac{1}{n}\,dz$; and therefore we get

$$y = \frac{1}{n}\int \frac{dz}{\sqrt{a - bz^2}} = \frac{1}{n\sqrt{b}} \cdot \sin^{-1} z\sqrt{\frac{b}{a}} + \mathrm{C} = \frac{1}{n\sqrt{b}} \cdot \sin^{-1} x^n \sqrt{\frac{b}{a}} + \mathrm{C}.$$

3. Suppose $dy = \frac{x^{\frac{1}{2}}\,dx}{a + bx^3}$: then putting $z = x^{\frac{3}{2}}$, we shall get

$$y = \frac{2}{3}\int \frac{dz}{a + bz^2} = \frac{2}{3\sqrt{ab}} \cdot \tan^{-1} \sqrt{\frac{bx^3}{a}} + \mathrm{C}.$$

4. If $dy = \frac{dx}{x\sqrt{bx^n - a}}$ were given for integration; we may convert it into

$\dfrac{x^{\frac{n}{2}-1}\,dx}{x^{\frac{n}{2}}(bx^n-a)^{\frac{1}{2}}}$, by multiplying both terms by $x^{\frac{n}{2}-1}$; whence putting $x^{\frac{n}{2}}=z$, we should have as in the former cases,

$$y=\frac{2}{n}\int\frac{dz}{z\sqrt{bz^2-a}}+\mathrm{C}=\frac{2}{n\sqrt{a}}.\sec^{-1}\sqrt{\frac{b}{a}x^n}+\mathrm{C}.$$

5. $\displaystyle\int\frac{dx}{\sqrt{ax+bx^2}}=\frac{2}{\sqrt{b}}.\cos^{-1}\sqrt{\frac{b}{a}}\,x.$

6. $\displaystyle\int\frac{xdx}{1+x^4}=\tfrac{1}{2}\tan^{-1}x^2.$

7. $\displaystyle\int\frac{x^{\frac{1}{2}}dx}{\sqrt{2-4x^3}}=\frac{1}{3}\sin^{-1}\sqrt{2x^3}.$

8. $\displaystyle\int\frac{x^{n-1}\,dx}{1+x^{2n}}=\frac{1}{n}\tan^{-1}x^n.$

9. $\displaystyle\int\frac{xdx}{\sqrt{a^4-x^4}}=\frac{1}{2}\sin^{-1}\frac{x^2}{a^2}.$

10. $\displaystyle\int\frac{xdx}{a^4+x^4}=\frac{1}{2a^2}\tan^{-1}\frac{x^2}{a^2}.$

11. $\displaystyle\int\frac{xdx}{\sqrt{(x^2-a^2)(b^2-x^2)}}=\sin^{-1}\sqrt{\frac{x^2-a^2}{b^2-a^2}}.$

12. $\displaystyle\int\frac{(1-x)dx}{1-x^3}=\frac{2}{\sqrt{3}}.\tan^{-1}\frac{2x+1}{\sqrt{3}}.$

13. $\displaystyle\int\frac{dx}{\sqrt{(1-x)^2+x^2}}=\tan^{-1}(2x-1).$

14. $\displaystyle\int\frac{dx}{\sqrt{(1+x)^2-2x^2}}=\sin^{-1}\frac{x-1}{\sqrt{2}}.$

In the last ten answers the constant is omitted for the sake of room.

PROP. V.

To integrate functions of the forms $\sin m\theta\,.\,d\theta$, $\cos m\theta\,.\,d\theta$, *etc.*

Put $m\theta=x$, then $\theta=\frac{x}{m}$ and $d\theta=\frac{dx}{m}$: and the expressions to be integrated become

$$\frac{1}{m}\sin x\,.\,dx,\ \frac{1}{m}\cos x\,.\,dx,\ etc.$$

The integrals of which are found by inspection from the tables at *page* 364, to be as follows:

Table of integrals depending on direct trigonometrical functions.

$\int\sin x.dx=-\cos x,$	$\int\cos xdx=+\sin x,$
$\int\frac{dx}{\cos^2x}=+\tan x,$	$\int\frac{dx}{\sin^2x}=-\cot x,$
$\int\frac{\sin x\,.\,dx}{\cos^2x}=+\sec x,$	$\int\frac{\cos x\,.\,dx}{\sin^2x}=-\operatorname{cosec} x,$
$\int\sin m\theta\,.\,d\theta=-\frac{1}{m}\cos m\theta,$	$\int\cos m\theta\,.\,d\theta=+\frac{1}{m}\sin m\theta,$
$\int\frac{d\theta}{\cos^2m\theta}=+\frac{1}{m}\tan m\theta.$	$\int\frac{d\theta}{\sin^2m\theta}=-\frac{1}{m}\cot m\theta,$
$\int\frac{\sin m\theta\,.\,d\theta}{\cos^2m\theta}=+\frac{1}{m}\sec m\theta,$	$\int\frac{\cos m\theta\,.\,d\theta}{\sin^2m\theta}=-\frac{1}{m}\operatorname{cosec} m\theta.$

.... $\int(6)$.

In this table, for brevity, the constants are omitted, but must be inserted in all cases where the formulæ are employed.

EXAMPLES.

1. $\int \tan\theta\, d\theta = \int \frac{\sin\theta}{\cos\theta} . d\theta = -\log_e \cos\theta = \log_e \sec\theta.$

2. $\int \cot\theta\, d\theta = \int \frac{\cos\theta}{\sin\theta} . d\theta = \log_e \sin\theta = -\log_e \operatorname{cosec}\theta.$

3. $\int \frac{d\theta}{\cos\theta \sin\theta} = \int \frac{\sec^2\theta}{\tan\theta}\, d\theta = \log_e \tan\theta.$

4. $\int \frac{d\theta}{\sin\theta} = \int \frac{2d\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \log_e \tan\frac{1}{2}\theta.$

5. $\int \frac{d\theta}{\cos\theta} = \int \frac{d\theta}{\sin(\frac{1}{2}\pi + \theta)} = \log_e \tan(\frac{1}{4}\pi + \frac{1}{2}\theta).$

6. $\int \frac{d\theta}{\cos^2\theta \sin^2\theta} = \int \left\{\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right\} d\theta = \tan\theta - \cot\theta = -2\cot 2\theta.$

7. $\int \frac{d\theta}{a(1+\cos\theta)} = \frac{1}{a}\int \frac{d(\frac{1}{2}\theta)}{\cos^2\frac{1}{2}\theta} = \frac{1}{a}\tan\frac{1}{2}\theta.$

8. $\int \frac{\sin\theta\, d\theta}{a + b\cos\theta} = -\frac{1}{b}\log_e (a + b\cos\theta).$

9. $\int \frac{d\theta}{1 + 2\cos\theta} = \frac{1}{\sqrt{3}} . \log_e \frac{\sqrt{3} + \tan\frac{1}{2}\theta}{\sqrt{3} - \tan\frac{1}{2}\theta}.$

10. $\int \frac{d\theta}{a\cos^2\theta + b\sin^2\theta} = \int \frac{\sec^2\theta\, d\theta}{a + b\tan^2\theta} = \frac{1}{\sqrt{ab}} . \tan^{-1} \left\{\sqrt{\frac{b}{a}} . \tan\theta\right\}.$

11. $\int \frac{\tan\theta d\theta}{\sqrt{a + b\tan^2\theta}} = \frac{1}{\sqrt{b-a}} . \cos^{-1} \left\{\sqrt{\frac{b-a}{b}} . \cos\theta\right\}.$

12. $\int \frac{d\theta}{1 + \cos^2\theta} = \frac{1}{\sqrt{2}} . \tan^{-1} \left(\frac{\tan\theta}{\sqrt{2}}\right).$

13. $\int \sin m\theta \cos n\theta\, d\theta = -\frac{1}{2}\left\{\frac{\cos(m+n)\theta}{m+n} + \frac{\cos(m-n)\theta}{m-n}\right\}.$

14. $\int \sin m\theta \sin n\theta\, d\theta = -\frac{1}{2}\left\{\frac{\sin(m+n)\theta}{m+n} - \frac{\sin(m-n)\theta}{m-n}\right\}.$

15. $\int \cos m\theta \cos n\theta\, d\theta = +\frac{1}{2}\left\{\frac{\sin(m+n)\theta}{m+n} + \frac{\sin(m-n)\theta}{m-n}\right\}.$

16. $\int \cos m\theta \sin n\theta\, d\theta = +\frac{1}{2}\left\{\frac{\cos(m+n)\theta}{m+n} - \frac{\cos(m-n)\theta}{m-n}\right\}.$

The first integral (and the others in a corresponding manner) is thus found:

$$\sin m\theta \cos n\theta\, d\theta = \frac{1}{2}\{\sin(m\theta + n\theta) + \sin(m\theta - n\theta)\} d\theta$$

$$= \frac{1}{2}\left\{\frac{\sin(m+n)\theta\, d(m+n)\theta}{m+n} + \frac{\sin(m-n)\theta\, d(m-n)\theta}{m-n}\right\}.$$

Wherefore, integrating each term, we have

$$\int \sin m\,\theta \cos n\theta\, d\theta = -\frac{1}{2}\left\{\frac{\cos(m+n)\theta}{m+n} + \frac{\cos(m-n)\theta}{m-n}\right\}.$$

CHAPTER III.

THE INTEGRATION OF RATIONAL FRACTIONS:

THAT is, such as have the general form, m being a positive integer:—

$$\frac{dy}{dx}=\frac{Ax^{m-1}+Bx^{m-2}+Cx^{m-3}+\dots}{A_1x^m+B_1x^{m-1}+C_1x^{m-2}+\dots}=\frac{U}{V}.$$

It is always possible to reduce every fraction into which no binomial, trinomial, *etc.* factors enter, to this form, together with a series of terms, $ax^n+bx^n+\dots$, by actual division of the numerator by the denominator, when the degree of the former is higher than that of the latter. We shall, therefore, only attend to those cases in which the numerator is at least one degree lower in dimension than the denominator [*].

PROP. I.

The denominator is decomposable into factors of one or both the forms $x-a$, x^2-ax+b, of which any of either or both kinds may be equal.

FOR, put $A_1x^m+B_1x^{m-1}+C_1x^{m-2}+\dots=0$: then (*vol. i. p.* 219) this equation has m roots either real or imaginary, or of both kinds. To every real root there corresponds a factor $x-a$; and to every imaginary pair there corresponds a double factor $\{x-(\alpha+\beta\sqrt{-1})\}\{x-(\alpha-\beta\sqrt{-1})\}$ or $x^2-2\alpha x+(\alpha^2+\beta^2)$. Whence the multiplication of all the factors of the first and second degree so obtained, will produce the denominator; or in other words, the denominator is decomposable into such factors as $x-a$, and x^2-ax+b. It is also clear, that any number of either kind may be equal.

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[*] It will be readily seen, that by very simple and obvious processes a great number of irrational expressions may be rendered rational. The most simple and useful of such transformations will be noticed in the next chapter; but we should notice one particular case here. It is where such expressions as $a_1x^{\frac{\alpha_1}{\beta^1}}+a_2x^{\frac{\alpha_2}{\beta_2}}+\dots$ occur; and the rationalization is effected by reducing the indices to a common denominator, (suppose it be $\beta_1\,\beta_2\dots\beta_n=\delta$,) and assuming $z^{\delta}=x$, we shall have all the indices rendered integer. Suppose, for instance, the fraction $\frac{x^{\frac{1}{2}}-\frac{1}{2}a}{x^{\frac{1}{3}}-x^{\frac{1}{2}}}\,dx$ given for integration; then the common multiple of the denominators of the indices is $2.3=6$, and putting $z^6=x$, we have

$$x^{\frac{1}{2}}=z^3,\ x^{\frac{1}{3}}=z^2,\text{ and }dx=6z^5dz;$$

wherefore the differential proposed is $-\frac{6z^6-2az^3}{z-1}.\,dz$, which may be reduced to the condition required for solution by actual division. The result is finally restored to terms of x after integration by putting $x^{\frac{1}{6}}$ for z.

PROP. II.

Every rational fraction is decomposable into a series of partial fractions of lower degrees, in the following manner:

(1). To each factor (it having no second equal to it) of the forms $x - a$ and $x^2 - ax + b$, correspond the fractions $\frac{M}{x - a}$ and $\frac{Mx + N}{x^2 - ax + b}$; and

(2). To multiple factors $(x - a)^n$ and $(x^2 - ax - b)^n$ correspond respectively the systems of fractions,

$$\frac{M}{(x - a)^n} + \frac{M_1}{(x - a)^{n-1}} + \dots \frac{M_{n-1}}{x - a}, \text{ and}$$

$$\frac{Mx + N}{(x^2 - ax + b)^n} + \frac{M_1 x + N_1}{(x^2 - ax + b)^{n-1}} + \dots \frac{M_{n-1}\, x + N_{n-1}}{x^2 - ax + b}.$$

For in all cases, the reduction of the series of assumed partial fractions will, by reduction to a common denominator, give a result containing the given denominator, and having for a numerator a function one degree lower than the denominator. The product of the fractions so reduced gives also a fraction having the same property. Such a result, therefore, agrees with the composition of the fraction to this extent.

Again, there are as many (m) indeterminate co-efficients, M, M_1, M_2, ... N, N_1, N_2, ... as there are units in the dimensions of x in the denominator V; that is, the same number as there are of co-efficients in the numerator U. Wherefore, equating the homologous co-efficients of the numerator of U with those of the numerator produced by reducing the partial fraction to a common denominator, we shall obtain the several equations which are required for the determination of M, M_1, ..., N, N_1, ... None of these quantities, moreover, rise above the first degree, and hence they are all real, and therefore determinable. The reduction to partial fractions will then be completed; and the given expression prepared for direct integration.

Scholium.

The great *practical* obstacle to the extended use of this method of solution, arises from actual difficulty of reducing the denominator to its component simple and quadratic factors. When the roots are real and integer, the process is very simple; but when real and incommensurate, the labour is immensely increased. When the denominator contains only one pair of imaginary factors, the simple factors being first obtained, the expression is simply composed of the product of this pair; but when more than one pair of such factors exists in the denominator, we are in possession of no method by which these pairs can be separately exhibited. Such cases, however, seldom occur in actual investigations.

These fractions have also several elegant properties which are worthy of attention; but as they do not, in any degree, subserve the purpose under consideration, we shall not dwell upon them here.

PROP. III.

To integrate a rational fraction when the partial fractions whose sum is the given fraction, the factors of the denominator of which are all real and different; such that

$$V = (x - a_1)(x - a_2)(x - a_3) \dots\dots\dots (x - a_n)$$

$$\text{Let } \frac{U}{V} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \frac{A_3}{x - a_3} + \dots \frac{A_n}{x - a_n}.$$

Reduce to a common denominator, and the numerator of the right side will be equal to U, since both fractions and their denominators are equal by hypothesis: then

$$\begin{aligned} U = {} & A_1 (x - a_2)(x - a_3)(x - a_4) \dots\dots (x - a_{n-1})\ .(x - a_n) \\ & + A_2 (x - a_3)(x - a_4)(x - a_5) \dots\dots (x - a_n)\ (x - a_1) \\ & + A_3 (x - a_4)(x - a_5)(x - a_6) \dots\dots (x - a_1)\ (x - a_2) \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + A_n (x - a_1)(x - a_2)(x - a_3) \dots\dots (x - a_{n-2})\ (x - a_{n-1}). \end{aligned}$$

This expression being expanded, and the homologous co-efficients equated with those of U, we shall have the required equations of condition, and thence the values of $A_1, A_2, \dots A_n$.

The following method is, however, often more simple and direct in its application.

Give to x in succession the several values, $a_1, a_2, \dots a_n$, and let $U_1, U_2, \dots U_n$, be the corresponding values of U. Then, since when $x = a_1$, all the terms which contain $x - a_1$ become zero, the values of U_1, U_2, *etc.*, are reduced to one single term composed of binomial factors, but not containing $x - a_1$ or $a_1 - a_1$; and similarly for all the others. We thus have*

$$U_1 = A_1 (a_1 - a_2)(a_1 - a_3) \dots (a_1 - a_n), \text{ or } A_1 = \frac{U_1}{(a_1 - a_2)(a_1 - a_3)..(a_1 - a_n)},$$

$$U_2 = A_2 (a_2 - a_3)(a_2 - a_4) \dots (a_2 - a_1), \text{ or } A_2 = \frac{U_2}{(a_2 - a_3)(a_2 - a_4)..(a_2 - a_1)},$$

etc., etc.

Whence,

$$\int \frac{U dx}{V} = \int \frac{A_1\, dx}{x - a_1} + \int \frac{A_2\, dx}{x - a_2} + \int \frac{A_3\, dx}{x - a_3} + \dots + \int \frac{A_n\, dx}{x - a_n}$$

$$= A_1 \log_e (x - a_1)\ A_2 \log_e (x - a_2) + A_3 \log_e (x - a_3) .. + A_n \log_e (x - a_n)$$

$$= \log_e (x - a_1)^{A_1} (x - a_2)^{A_2} (x - a_3)^{A_3} \dots\dots (x - a_n)^{A_n}.$$

EXAMPLES.

1. Integrate $\dfrac{a dx}{x^2 - a^2}$.

Here the denominator is decomposable into $(x + a)(x - a)$; hence,

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[*] This process is founded on the principle, that as x is perfectly indeterminate in its value in the given fraction, it may take all possible values, and amongst the rest the values $a_1, a_2, \dots a_n$; and the equations to which the decomposition give rise must subsist for these as well as other values, whatever form the result may take. By observation of the expansion itself, we see that it gives the values of A_1, A_2, *etc.*, which are the objects of our inquiry.

$$\frac{a}{x^2-a^2}=\frac{A}{x-a}+\frac{A_1}{x+a}=\frac{A\,(x+a)+A_1\,(x-a)}{x^2-a^2},$$

$$\text{or } a=A\,(x+a)+A_1\,(x-a).$$

Put $x=a$; then, $a=A\,(a+a)$, or $A=\frac{1}{2}$;

$x=-a$; then, $a=A_1\,(-a-a)$, or $A_1=-\frac{1}{2}$.

Whence,

$$\int\frac{adx}{x^2-a^2}=\int\frac{\frac{1}{2}dx}{x-a}-\int\frac{\frac{1}{2}dx}{x+a}=\tfrac{1}{2}\log_e(x-a)-\tfrac{1}{2}\log_e(x+a)+C,$$

$$=\log_e\left(\frac{x-a}{x+a}\right)^{\frac{1}{2}}+C.$$

The same values of A and A_1 would have resulted from the ordinary method of applying the solution of the equations of the indeterminate co-efficients. For we have

$$a=(A+A_1)\,x+(A-A_1)\,a,\text{ giving } A+A_1=0,\ A-A_1=1.$$

2. To integrate $\dfrac{(a^3+bx^2)\,dx}{a^2x-x^3}$.

Here $\dfrac{a^3+bx^2}{a^2x-x^3}=\dfrac{A}{x}+\dfrac{A_1}{a-x}+\dfrac{A_2}{a+x}$

$$=\frac{A\,(a-x)\,(a+x)+A_1x\,(a+x)+A_2\,(a-x)\,x}{x\,(a-x)\,(a+x)}.$$

Put $x=0$: then $a^3=Aa^2$, or $A=a$,

$x=a$: then $a^3+ba^2=2A_1.a^2$, or $A_1=\frac{1}{2}\,(a+b)$,

$x=-a$: then $a^3+ba^2=-2A_2\,a^2$, or $A_2=-\frac{1}{2}\,(a+b)$.

Hence $\displaystyle\int\frac{(a^3+bx^2)\,dx}{a^2x-x^3}=\int\frac{adx}{x}+\int\frac{\frac{1}{2}\,(a+b)\,dx}{a-x}-\int\frac{\frac{1}{2}\,(a+b)\,dx}{a+x},$

$$=a\log_e x-\tfrac{1}{2}(a+b)\log_e(a-x)-\tfrac{1}{2}(a+b)\log_e(a+x)+C$$

$$=a\log_e x-\tfrac{1}{2}(a+b)\log_e(a^2-x^2)+C,$$

$$=\log_e x^a-\log_e(a^2-x^2)^{\frac{1}{2}(a+b)}+\log_e c,$$

$$=\log_e\frac{cx^a}{(a^2-x^2)^{\frac{1}{2}(a+b)}}.$$

The ordinary mode of reducing the equation in A, A_1, A_2, gives

$$a^3+bx^2=Aa^2-Ax^2+A_1ax+A_1x^2+A_2ax-A_2x^2.$$

3. Let the function $\dfrac{(x^2-7x+1)\,dx}{x^3-6x^2+11x-6}$ be given for integration.

In this we have $x^3-6x^2+11x-6=(x-1)\,(x-2)\,(x-3)$, and hence

$$x^2-7x+1=A\,(x-2)\,(x-3)+A_1\,(x-3)\,(x-1)+A_2\,(x-1)\,(x-2).$$

Put $x=1$: then $1-7+1=A\,(1-2)\,(1-3)$, or $-5=2A$, and $A=-\frac{5}{2}$

$x=2$: then $4-14+1=A_1\,(2-3)\,(2-1)$, or $-9=-A_1$, and $A_1=9$,

$x=3$: then $9-21+1=A_2\,(3-1)\,(3-2)$, or $-11=2A_2$, and $A_2=-\frac{11}{2}$

Hence,

$$\int\frac{x^2dx-7xdx+dx}{x^3-6x^2+11x-6}=-\frac{5}{2}\int\frac{dx}{x-1}+9\int\frac{dx}{x-2}-\frac{11}{2}\int\frac{dx}{x-3}$$

$$=-\frac{5}{2}\log_e(x-1)+9\log_e(x-2)-\frac{11}{2}\log_e(x-3)+\log_e$$

$$=\log_e\frac{c.(x-2)^9}{\sqrt{(x-1)^5\,(x-3)^{11}}}.$$

4. Given $dy = \frac{xdx}{x^2 + 4ax - b^2}$ for integration.

Here $x^2 + 4ax - b^2 = \{x + (2a + \sqrt{4a^2 + b^2})\}\{x + (2a - \sqrt{4a^2 + b^2})\}$.

Let $2a + \sqrt{4a^2 + b^2} = \alpha$, and $2a - \sqrt{4a^2 + b^2} = \beta$: then

$$\frac{x}{x^2 + 4ax - b^2} = \frac{A}{x + \alpha} + \frac{A_1}{x + \beta}; \text{ and } A = \frac{\alpha}{\alpha - \beta}, \text{ and } A_1 = -\frac{\beta}{\alpha - \beta}.$$

$$\text{Hence } y = \frac{\alpha}{\alpha - \beta} \log_e (x + \alpha) - \frac{\beta}{\alpha - \beta} \log_e (x + \beta) + \log_e c,$$

$$= \log_e (x + \alpha)^{\frac{\alpha}{\alpha - \beta}} - \log_e (x + \beta)^{\frac{\beta}{\alpha - \beta}} + \log_e c,$$

$$= \log_e c \left\{\frac{(x + \alpha)^{\alpha}}{(x + \beta)^{\beta}}\right\}^{\frac{1}{\alpha - \beta}}.$$

5. $\int \frac{2xdx + 3dx}{x^3 + x^2 - 2x} = \log_e \frac{c\,.\,(x + 1)^{\frac{5}{3}}}{x^{\frac{3}{2}}\,(x + 2)^{\frac{1}{6}}}$.

6. $\int \frac{x - 1)\,dx}{x^2 + 6x + 8} = \log_e c \left\{\frac{(x + 4)^5}{(x + 2)^3}\right\}^{\frac{1}{2}}$.

7. $\int \frac{x^2dx - xdx + 2dx}{x^4 - 5x^2 + 4} = \log_e c \left\{\frac{(x + 1)^2\,(x - 2)}{(x - 1)\,(x + 2)^2}\right\}^{\frac{1}{2}}$.

8. Let r be less than n: and

$(a_1 - a_2)\,(a_1 - a_3) \dots (a_1 - a_n) = b_1$, $(a_2 - a_3)\,(a_2 - a_4) \dots (a_2 - a_1) = b_2$, *etc.* then

$$\int \frac{x^r dx}{(x - a_1)\,(x - a_2) \dots (x - a_n)} = \frac{a_1{}^r \log_e (x - a_1)}{b_1} + \frac{a_2{}^r \log_e (x - a_2)}{b_2} + \dots \frac{a_n{}^r \log_e (x - a_n)}{b^n}.$$

PROP. IV.

To integrate a rational fraction when any number of the binomial factors of the denominator are equal.

The former method fails in this case, since all the values of U_1, U_2, *etc.* become zero. Let, then, in this case,

$$V = (x - a)^m\,Q;$$

in which Q represents the quotient of V by $(x - a)^m$: and let

$$\frac{U}{V} = \frac{A}{(x - a)^m} + \frac{A_1}{(x - a)^{m-1}} + \dots \frac{A_m}{x - a} + \frac{P}{Q}, \text{ or}$$

$$U = AQ + \{A_1\,(x - a) + A_2\,(x - a)^2 + \dots + A_m\,(x - a)^{m-1}\} + P\,(x - a)^m.$$

Let $x = a$, and denote by U_1, Q_1, the values which U and Q then take: then

$U_1 = AQ_1$, or $A = \frac{Q_1}{U_1}$; and therefore

$$U - \frac{U_1}{Q_1} Q = (x - a)\,\{A_1 + A_2\,(x - a) + \dots A_m\,(x - a)^{m-2} + P\,(x - a)^{m-1}\}.$$

Now the second side being divisible by $x - a$, the first is so too; and if the quotient be U^1 we have

$$U^1 = \{A_1 + A_2\,(x - a) + \dots A_m\,(x - a)^{m-2}\} + P\,(x - a)^{m-1}.$$

Again, putting $x = a$, and proceeding in the same manner, we shall, by m successive steps, get $U^M = P$. If this be a constant, the entire reductions will have been effected; but if it be a function of x, we must apply to it the process of the preceding proposition *.

Scholium.

We might, instead of the mode of finding U^{I}, U^{II}, ... U^{M}, as here investigated, have adopted the ordinary mode of finding the values of the indeterminate co-efficients. Indeed, in many instances this will be found the less operose method. We shall illustrate both methods as we proceed.

EXAMPLES.

1. Integrate the function $\frac{2xdx - 5dx}{(x+3)(x+1)^2}$.

Let $\frac{2x-5}{(x+3)(x+1)^2} = \frac{A}{(x+1)^2} + \frac{A_1}{x+1} + \frac{P}{x+3}$; and then we have

$$2x - 5 = A(x+3) + A_1(x+1)(x+3) + P(x+1)^2.$$

Put $x + 1 = 0$: then $-7 = A(3-1)$, or $A = -\frac{7}{2}$: wherefore,

$$2x - 5 + \frac{7}{2}(x+3)\left(= \frac{11}{2}(x+1)\right) = A_1(x+1)(x+3) + P(x+1)^2,$$

$$\text{and } \frac{11}{2} = A_1(x+3) + P(x+1).$$

Again, put $x + 1 = 0$, and this gives $A_1 = \frac{11}{4}$

Also, put $x + 3 = 0$, and it gives $P = -\frac{11}{4}$.

$$\text{Hence,} \int \frac{2xdx - 5dx}{(x+3)(x+1)^2} = -\frac{7}{2}\int \frac{dx}{(x+1)^2} + \frac{11}{4}\int \frac{dx}{x+1} - \frac{11}{4}\int \frac{dx}{x+3}$$

$$= \frac{7}{2(x+1)} + \frac{11}{4}\log_e \frac{(x+1)}{x+3} + C.$$

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* This assumption might have been made as follows:

$\frac{U}{V} = \frac{A_1 x^{m-1} + B_1 x^{m-2} + \ldots + K_1 x + L_1}{(x-a)^m}$ + terms depending on the other factors of the denominator. For by either method the requisite number of indeterminate co-efficients is introduced into the equation. The reduction, however, by this method, will generally be more laborious than by the assumptions in the text.

We may moreover remark, that some of the expressions for the values of A_1, A_2, A_3, *etc.* admit of a very elegant form in terms of the differential co-efficients of $\frac{U}{V_1}$, where

$$\frac{U}{V} = \frac{A_1}{x-a)^m} + \frac{A_2}{(x-a)^{m-1}} + \ldots \frac{U_1}{V_1} + \ldots$$

the latter term containing all the remaining partial fractions. It is this. Insert a for x in the following series of expressions:—

$$A_1 = \frac{U}{V_1}, A_2 = \frac{d\frac{U}{V_1}}{dx}, A_3 = \frac{d^2\frac{U}{V_1}}{dx^2}, \textit{etc.}$$

As, however, for the mere reduction this is equally laborious and sometimes more so, than the method we have employed, and as, moreover, it will not present itself under any useful aspect to us in the present volume; we shall not stop to offer a proof, but leave it to the student's own sagacity to investigate the truth of the formulæ. A slight use, however, will be made of the method further on, and the investigation as far as that use requires it, also given.

The co-efficients found by the common method.

Multiplying out the factors $(x + 1)(x + 3)$ and $(x + 1)^2$, we have

$$(P + A_1)x^2 + (2P + 4A_1 + A - 2)x + (P + 3A_1 + 3A + 5) = 0.$$

Whence, $P + A_1 = 0$, $2P + 4A_1 + A - 2 = 0$, and $P + 3A_1 + 3A + 5 = 0$.

Subtract the second from the sum of the first and third; then we get $A = -\frac{7}{2}$.

Substitute this in the second, then $4P + 8A_1 = 11$; from which, combined with the first, we have $P = -\frac{11}{4}$ and $A_1 = \frac{11}{4}$.

2. Suppose $\int \frac{adx}{(x-1)^2 (x+1)^2}$ be required. Then we have

$$\frac{a}{(x-1)^2 (x+1)^2} = \frac{A}{(x-1)^2} + \frac{A_1}{x-1} + \frac{B}{(x+1)^2} + \frac{B_1}{x+1}, \text{ or}$$

$$a = A(x+1)^2 + A_1 (x+1)^2 (x-1) + B(x-1)^2 + B_1 (x-1)^2 (x+1).$$

Let $x - 1 = 0$: then $a = A \,.\, 2^2$, or $A = \frac{1}{4}a$.

Substitute this in the preceding equation: then,

$$a - \frac{1}{4}a(x+1)^2 = A_1 (x+1)^2 (x-1) + B(x-1)^2 + B_1 (x-1)^2 (x+1).$$

Now $a - \frac{1}{4}a(x+1)^2 = -\frac{1}{4}a(x+3)(x-1)$; hence dividing the preceding by $x - 1$ we have

$$-\frac{1}{4}a(x+3) = A_1 (x+1)^2 + B(x-1) + B_1 (x-1)(x+1).$$

Again, let $x - 1 = 0$: then this gives $-\frac{1}{4}a \,.\, 4 = A_1 \,.\, 2^2$, or $A_1 = -\frac{1}{4}a$; whence again transposing,

$$-\frac{1}{4}a(x+3) + \frac{1}{4}a(x+1)^2 = B(x-1) + B_1 (x-1)(x+1).$$

But $-\frac{1}{4}a(x+3) + \frac{1}{4}a(x+1)^2 = \frac{1}{4}a(x-1)(x+2)$; wherefore dividing all by $x - 1$, we get

$$\frac{1}{4}a(x+2) = B + B_1 (x+1).$$

Let now $x + 1 = 0$: then $B = \frac{1}{4}a(-1+2) = \frac{1}{4}a$; and the equation is changed to

$$\frac{1}{4}a(x+2) - \frac{1}{4}a = B_1 (x+1), \text{ or}$$

$$\frac{1}{4}a(x+1) = B_1 (x+1), \text{ or } B_1 = \frac{1}{4}a.$$

Inserting these values of A, A_1, B, B_1, we get

$$\int \frac{adx}{(x-1)^2 (x+1)^2} = \frac{a}{4}\left\{\int \frac{dx}{(x-1)^2} + \int \frac{dx}{(x+1)^2} - \int \frac{dx}{x-1} + \int \frac{dx}{x+1}\right\}$$

$$= \frac{a}{4}\left\{-\frac{1}{x-1} - \frac{1}{x+1} - \log_e (x-1) + \log_e (x+1)\right\} + C.$$

$$= -\frac{ax}{2(x^2-1)} + \frac{a}{4}\log_e \frac{x+1}{x-1} + C.$$

Had we adopted the ordinary mode of solution of the equations of the indeterminate co-efficients, we should have had

$$\mathrm{U} = (\mathrm{A}_1+\mathrm{B}_1)\,x^3 + (\mathrm{A}+\mathrm{A}_1+\mathrm{B}-\mathrm{B}_1)\,x^2 + (2\mathrm{A}-\mathrm{A}_1-2\mathrm{B}-\mathrm{B}_1)\,x+\mathrm{A}-\mathrm{A}_1+\mathrm{B}+\mathrm{B}_1-a = 0;$$

and hence,

$$\begin{array}{r|r} \mathrm{A}_1 + \mathrm{B}_1 = 0 & 2\mathrm{A} - \mathrm{A}_1 - 2\mathrm{B} - \mathrm{B}_1 = 0 \\ \mathrm{A} + \mathrm{A}_1 + \mathrm{B} - \mathrm{B}_1 = 0 & \mathrm{A} - \mathrm{A}_1 + \mathrm{B} + \mathrm{B}_1 = a. \end{array}$$

The first and third of these give $2\mathrm{A} - 2\mathrm{B} = 0$, or $\mathrm{A} = \mathrm{B}$; and the second and fourth give $2\mathrm{A} + 2\mathrm{B} = a$. These two give $\mathrm{A} = \mathrm{B} = \frac{1}{4}\,a$. Wherefore, also, the fourth becomes $\mathrm{B}_1-\mathrm{A}_1 = \frac{1}{2}\,a$, which, by the first, becomes $\mathrm{A}_1 = -\frac{1}{4}\,a$, and $\mathrm{B}_1 = \frac{1}{4}\,a$.

3. Show that $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log_e \frac{x - a}{x + a} + \mathrm{C}.$

4. Similarly $\displaystyle\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log_e \frac{x + a}{x - a} + \mathrm{C}.$

5. $$\int \frac{dx}{x^3 - 7x^2 + 12x} = \log_e \frac{c\,.\,x^{\frac{1}{12}}\,(x - 4)^{\frac{1}{4}}}{(x - 3)^{\frac{1}{3}}}.$$

6. $$\int \frac{x^2dx}{x^3 - x^2 - x + 1} = \frac{1}{2\,(1 - x)} + \log_e c \left\{(x^2 - 1)\,(x - 1)^2\right\}^{\frac{1}{4}}.$$

7. $$\int \frac{(x^2 - 2)\,dx}{x^3 + 4x^2 + 4x} = \frac{1}{x + 2} + \log_e c \left\{\frac{(x + 2)^3}{x}\right\}^{\frac{1}{2}}.$$

8. $$\int \frac{(x^2 - 3)\,dx}{x^3 - 7x + 6} = \log_e c\,(x - 1)^{\frac{1}{2}}\,(x - 2)^{\frac{1}{5}}\,(x + 3)^{\frac{3}{10}}.$$

9. $$\int \frac{xdx}{(a + bx)^3} = -\frac{a}{b}\int \frac{dx}{(a + bx)^3} + \frac{1}{b}\int \frac{dx}{(a + bx)^2}.$$
$$= \frac{a}{2b^2\,(a + bx)^2} - \frac{1}{b^2\,(a + bx)} + \mathrm{C}.$$
$$= -\frac{a + 2bx}{2b^2\,(a + bx)^2} + \mathrm{C}.$$

PROP. V.

To integrate a rational fraction, the denominator being decomposable into factors, any number of which is of the quadratic form.

The general form of the simple parts of the quadratic factor is $x-(a \pm \beta\sqrt{-1})$, and of that factor itself $(x - a)^2 + \beta^2$.

Then we shall have

$$\frac{\mathrm{U}}{\mathrm{V}} = \frac{(\mathrm{A}_1x + \mathrm{B}_1)}{(x - a_1)^2 + \beta_1{}^2} + \cdots \frac{\mathrm{A}_mx + \mathrm{B}_m}{(x - a_m)^2 + \beta^2{}_m} + \frac{\mathrm{C}_1}{x - a_1} + \cdots\cdot \frac{\mathrm{C}_n}{x - a_n};$$

or reducing to a common denominator,

$$\left.\begin{array}{l} \mathrm{U}=(\mathrm{A}_1x+\mathrm{B}_1)\left\{(x-\alpha_2)^2+\beta_2{}^2\right\}\left\{(x-\alpha_3)^2+\beta_3{}^2\right\}\ldots(x-a_1)\,(x-a_2)\ldots(x-a_n) \\ \quad+(\mathrm{A}_2x+\mathrm{B}_2)\left\{(x-\alpha_3)^2+\beta_3{}^2\right\}\left\{(x-\alpha_4)^2+\beta_4{}^2\right\}\ldots(x-a_1)\,(x-a_2)\ldots(x-a_n) \\ \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \quad+(\mathrm{A}_mx+\mathrm{B}_m)\left\{(x-\alpha_1)^2+\beta_1{}^2\right\}\left\{(x-\alpha_2)^2+\beta_2{}^2\right\}\ldots(x-a_1)\,(x-a_2)\ldots(x-a_n) \end{array}\right\}\ (e)$$

$$\left.\begin{array}{l} +C_1(x-a_2)(x-a_3)\ldots(x-a_n)\{(x-\alpha_1)^2+\beta_1^2\}\ldots\ldots\{(x-\alpha_m)^2+\beta_m^2\} \\ +C_2(x-a_3)(x-a_4)\ldots(x-a_1)\{(x-\alpha_1)^2+\beta_1^2\}\ldots\ldots\{(x-\alpha_m)^2+\beta_m^2\} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ +C_n(x-a_1)(x-a_2)..(x-a_{n-1})\{(x-\alpha_1)^2+\beta_1^2\}\ldots\{(x-\alpha_m)^2+\beta^2{}_m\} \end{array}\right\}\ldots(b).$$

Of the two series of terms composing the value of U, that marked (*a*) contains all the simple factors in each term, together with all the quadratic factors but one; and hence the series vanishes for either of the n values of x, *viz.* $a_1, a_2, \ldots a_n$. The series marked (*b*) contains all the quadratic factors in each term, together with all the simple factors but one; and hence the series vanishes for each of the m values of x deduced from $(x - \alpha_1)^2 + \beta_1^2 = 0$, *etc.*

Again, not only do all the terms composing the series (*a*) vanish when $x - a_p = 0$, but likewise all the terms of the second series (*b*) but one. Let U_p be the value of U when $x = a_p$: then giving to p successive values, 1, 2, ... n, we have

$$\begin{array}{l} U_1 = C_1(a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n) \\ U_2 = C_2(a_2 - a_3)(a_2 - a_4)\ldots(a_2 - a_1) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ U_n = C_n(a_n - a_1)(a_n - a_2)\ldots(a_n - a_{n-1}) \end{array}$$

from which $C_1, C_2, \ldots C_n$ are found, and all the simple partial fractions are therefore known, as in *prop. iii.*

Further, not only do all the terms composing the series (*b*) vanish when $(x - \alpha_p)^2 + \beta_p^2 = 0$, but likewise all those of (*a*) but one. Let γ_p be the value of U when $(x - \alpha_p)^2 + \beta_p^2 = 0$: then giving to p successive values $1, 2, \ldots m$, and writing $\xi_1, \xi_2 \ldots \xi_m$ for

$\alpha_1 \pm \beta_1\sqrt{-1}$, $\alpha_2 \pm \beta_2\sqrt{-1}$, ... $\alpha_m \pm \beta_m\sqrt{-1}$, we get

$$\begin{array}{l} \gamma_1 = (A_1\xi_1+B_1)\{(\xi_1-\alpha_2)^2+\beta_2^2\}\{(\xi_1-\alpha_3)^2+\beta_3^2\}\ldots(\xi_1-a_1)(\xi_1-a_2)\ldots(\xi_1-a_n) \\ \gamma_2 = (A_2\xi_2+B_2)\{(\xi_2-\alpha_3)^2+\beta_3^2\}\{(\xi_2-\alpha_4)^2+\beta_4^2)\}\ldots(\xi_2-a_1)(\xi_2-a_2)\ldots(\xi_2-a_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \gamma_m=(A_m\xi_m+B_m)\{(\xi_m-\alpha_1)^2+\beta_1^2\}\{(\xi_m-\alpha_2)^2+\beta_2^2\}..(\xi_m-a_1)(\xi_m-a_2)..(\xi_m-a_n). \end{array}$$

Now, if for any one ξ_p write its value; and then expand the equation, keeping the real and imaginary parts separate, we shall have an equation of the general form.

$$hA_p + kB_p \pm (h_1A_p \pm k_1B_p)\sqrt{-1} = 0.$$

from which A_p and B_p can be found in virtue of *vol. i. p.* 157, which gives

$$hA_p + kB_p = 0, \text{ and } h_1A_p \pm k_1B_p = 0.$$

All the partial fractions are therefore determinable. Let them be found: then those with simple denominators are integrable by *prop. iv.*; and it remains to show how to integrate those with quadratic denominators.

Such expressions are of the form $\dfrac{(Ax + B)\,dx}{(x - \alpha)^2 + \beta^2}$.

Put $x - \alpha = z$: then $x = z + \alpha$, $dx = dz$, and $(x - \alpha)^2 + \beta^2 = z^2 + \beta^2$; which converts the expression into

$$\int\frac{Azdz + (A\alpha + B)\,dz}{z^2 + \beta^2} = \int\frac{Azdz}{z^2 + \beta^2} + \int\frac{(A\alpha + B)\,dz}{z^2 + \beta^2}$$

$$= A \log_e (z^2 + \beta^2)^{\frac{1}{2}} + \frac{A\alpha + B}{\beta} \tan^{-1}\frac{z}{\beta}.$$

$$= \log_e \{(x-\alpha)^2+\beta^2\}^{\frac{A}{2}} + \frac{A\alpha + B}{\beta} \tan^{-1}\frac{x - \alpha}{\beta}.$$

EXAMPLES.

Let it be required to integrate $\frac{\mathrm{U}dx}{\mathrm{V}} = \frac{(x^3 - 9x^2 + 2x - 6)\,dx}{(x^2 - 2x + 5)\,(x^2 - x + 1)\,(x - 1)}$.

$$\text{Assume } \frac{\mathrm{U}}{\mathrm{V}} = \frac{\mathrm{A}_1 x + \mathrm{B}_1}{x^2 - 2x + 5} + \frac{\mathrm{A}_2 x + \mathrm{B}_2}{x^2 - x + 1} + \frac{\mathrm{A}_3}{x - 1}: \text{ then}$$

$$\mathrm{U} = (\mathrm{A}_1 x + \mathrm{B}_1)\,(x - 1)\,(x^2 - x + 1) + (\mathrm{A}_2 x + \mathrm{B}_2)\,(x - 1)\,(x^2 - 2x + 5)$$
$$+ \mathrm{A}_3\,(x^2 - 2x + 5)\,(x^2 - x + 1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

Make $x = 1$, or $x - 1 = 0$: then the equation reduces to

$$1 - 9 + 2 - 6 = \mathrm{A}_3\,(1 - 2 + 5)\,(1 - 1 + 1), \text{ or } \mathrm{A}_3 = -3.$$

Substitute this value of A_3 in (a), transpose and divide by $x - 1$: then putting $\mathrm{U}^1 = 3x^3 - 5x^2 + 10x - 9$,

$$\mathrm{U}^1 = (\mathrm{A}_1 x + \mathrm{B}_1)(x^2 - x + 1) + (\mathrm{A}_2 x + \mathrm{B}_2)\,(x^2 - 2x + 5)\ldots\ldots\ldots (b).$$

Put $x = 1 \pm 2\sqrt{-1}$ (from $x^2 - 2x + 5 = 0$) in (b): then reducing, we get

$$-7\mathrm{A}_1 - 3\mathrm{B}_1 + 17 \mp (4\mathrm{A}_1 - 2\mathrm{B}_1 - 6)\sqrt{-1} = 0,$$
$$\text{or } 7\mathrm{A}_1 + 3\mathrm{B}_1 = 17, \text{ and } 4\mathrm{A}_1 - 2\mathrm{B}_1 = 6: \text{ from which}$$
$$\mathrm{A}_1 = 2, \text{ and } \mathrm{B}_1 = 1.$$

Substitute these values in (b), transpose and divide by $x^2 - 2x + 5$: then we get $x - 2 = \mathrm{A}_2 x + \mathrm{B}_2$ whence

$$\int \frac{\mathrm{U}dx}{\mathrm{V}} = \int \frac{(2x + 1)dx}{x^2 - 2x + 5} + \int \frac{(x - 2)dx}{x^2 - x + 1} - \int \frac{3dx}{x - 1}.$$

In the first term put $x - 1 = z$; then $x = z + 1$, and $dz = dx$; whence

$$\int \frac{(2x + 1)dx}{x^2 - 2x + 5} = \int \frac{(2z + 2 + 1)\,dz}{z^2 + 4} = \int \frac{2zdz}{z^2 + 4} + \int \frac{3dz}{z^2 + 4}$$
$$= \log_e (z^2 + 4) + \frac{3}{2} \tan^{-1} \frac{z}{2}$$
$$= \log_e (x^2 - 2x + 5) + \frac{3}{2} \tan^{-1} \frac{x - 1}{2}.$$

In the second term put $x - \frac{1}{2} = y$; then $x = y + \frac{1}{2}$, and $dx = dy$; whence

$$\int \frac{(x - 2)\,dx}{x^2 - x + 1} = \int \frac{(y - \frac{3}{2})\,dy}{y^2 + \frac{3}{4}} = \int \frac{ydy}{y^2 + \frac{3}{4}} - \frac{3}{2} \int \frac{dy}{y^2 + \frac{3}{4}}$$
$$= \frac{1}{2} \log_e (y^2 + \tfrac{3}{4}) - \frac{3}{2} \cdot \sqrt{\frac{4}{3}} \tan^{-1} \frac{2y}{\sqrt{3}}$$
$$= \log_e (x^2 - x + 1)^{\frac{1}{2}} - \sqrt{3} \tan^{-1} \frac{2x - 1}{\sqrt{3}}$$

And the third term gives $-3 \int \frac{dx}{x - 1} = -3 \log_e (x - 1) = -\log_e (x - 1)^3$

Whence, collecting these results, we have the integral required; viz.

$$\frac{\mathrm{U}dx}{\mathrm{V}} = \log_e \frac{c\,(x^2 - 2x + 5)\,(x^2 - x + 1)^{\frac{1}{2}}}{(x - 1)^3} + \frac{3}{2} \tan^{-1} \frac{x - 1}{2} - \sqrt{3} \tan^{-1} \frac{2x - 1}{\sqrt{3}};$$

where the constant c is brought into the logarithmic part of the expression.

With a view to the student's decision between the work by this method, and the ordinary one with indeterminate co-efficients, we shall add the equations for the determination of A_1, A_2, A_3, B_1, B_2.

$$\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3 = 0; \quad \mathrm{B}_1 + 5\mathrm{B}_2 - 5\mathrm{A}_3 - 6 = 0,$$
$$2\mathrm{A}_1 + 3\mathrm{A}_2 + 3\mathrm{A}_3 - \mathrm{B}_1 - \mathrm{B}_2 + 1 = 0,$$
$$2\mathrm{A}_1 + 7\mathrm{A}_2 + 8\mathrm{A}_3 - 2\mathrm{B}_1 - 3\mathrm{B}_2 + 9 = 0,$$
$$\mathrm{A}_1 + 5\mathrm{A}_2 + 7\mathrm{A}_3 - 2\mathrm{B}_1 - 7\mathrm{B}_2 + 2 = 0.$$

2. $$\int\frac{(x-3)\,dx}{x^3+1}=-\frac{4}{3}\int\frac{dx}{x+1}+\frac{1}{3}\int\frac{(4x-5)\,dx}{x^2-x+1}$$

$$=\log_e\frac{c\,(x^2-x+1)^{\frac{2}{3}}}{(x+1)^{\frac{4}{3}}}-\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$$

3. Required the integral of $\dfrac{dx}{(x+1)\,(x+2)^2\,(x^2+1)}$.

$$\textit{Ans.}\ \frac{1}{5\,(x+2)}-\frac{1}{50}\tan^{-1}x+\log_e\frac{c\,.\,(x+1)^{\frac{1}{2}}}{(x+2)^{\frac{9}{25}}\,(x^2+1)^{\frac{7}{100}}}.$$

4. $$\int\frac{dx}{x^3-1}=\log_e\frac{c\,(x-1)^{\frac{1}{3}}}{(x^2+x+1)^{\frac{1}{6}}}+\tfrac{1}{3}\sqrt{3}\,.\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

5. Let $\dfrac{dx}{a+bx+cx^2}$ be proposed for integration; then it is

$$\int\frac{dx}{a+bx+cx^2}=\frac{1}{c}\int\frac{dx}{\left(x+\dfrac{b}{2c}\right)^2-\dfrac{b^2-4ac}{4c^2}}=\text{M}\int\frac{dx}{(x+h)^2\pm k^2},$$

according as $-\dfrac{(b^2-4ac)}{4c^2}=\pm k^2$.

Then,

$$\text{M}\int\frac{dx}{(x+h)^2+k^2}=\frac{\text{M}}{k}\tan^{-1}\frac{x+h}{k}=\frac{2}{\sqrt{4ac-b^2}}.\tan^{-1}\frac{2cx+b}{\sqrt{4ac-b^2}}$$

$$\text{M}\int\frac{dx}{(x+h)^2-k^2}=\frac{\text{M}}{2k}\log_e\frac{x+h-k}{x+h+k}=\frac{1}{\sqrt{b^2-4ac}}\log_e\frac{2cx+b-\sqrt{b^2-4ac}}{2cx+b+\sqrt{b^2-4ac}}$$

6. $$\int\frac{dx}{1+x+x^2}=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

7. $$\int\frac{dx}{1-2x+2x^2}=\tan^{-1}(2x-1).$$

8. $$\int\frac{dx}{1+3x+2x^2}=\log_e\frac{2x+1}{2x+2}$$

9. $$\int\frac{dx}{1+3x+2x^2}=\log_e\frac{2x+1}{2x+2}.$$

10. $$\int\frac{dx}{1+2x-2x^2}=\frac{1}{\sqrt{5}}\log_e\frac{2x-1+\sqrt{5}}{2x-1-\sqrt{5}}.$$

11. $$\int\frac{(1-x\cos a)\,dx}{1-2x\cos a+x^2}=\int\frac{\sin^2 a\,dx-\cos a\,(x-\cos a)\,dx}{\sin^2 a+(x-\cos a)^2},$$

$$=\int\frac{\sin^2 a\,dx}{\sin^2 a+(x-\cos a)^2}-\cos a\int\frac{(x-\cos a)\,dx}{\sin^2 a+(x-\cos a)^2},$$

$$=\sin a\tan^{-1}\frac{x-\cos a}{\sin a}-\tfrac{1}{2}\cos a\log_e(1-2x\cos a+x^2).$$

PROP. VI.

To integrate a rational fraction which has two or more quadratic factors in the denominator.

THIS bears the same relation to the preceding that the fourth does to the third problem; and from the remarks already made, its solution is very obvious. As, moreover, it is clear that the single quadratic factors, and the single and multiple simple factors, are determinable by the processes already given, it will only be necessary to attend to the case of the partial factor $\frac{X}{\{(x-a)^2+\beta^2\}^n}$, where X is a function of x of $(2n-1)$ dimensions at the highest. Then the system of partial fractions corresponding to this term will be

$$\frac{A_1x+B_1}{\{(x-a)^2+\beta^2\}^n}+\frac{A_2x+B_2}{\{(x-a)^2+\beta^2\}^{n-1}}+\cdots\frac{A_nx+B_n}{(x-a)^2+\beta^2}.$$

The values of A_1, B_1, A_2, B_2, *etc.*, will be determined by a similar process to either of those employed in *prop. iv.* Wherefore,

$$\int\frac{Xdx}{\{(x-a)^2+\beta^2\}^n}=\int\frac{A_1xdx}{\{(x-a)^2+\beta^2\}^n}+\int\frac{A_2xdx}{\{(x-a)^2+\beta^2\}^{n-1}}+\cdots\int\frac{A_nxdx}{(x-a)^2+\beta^2}$$
$$+\int\frac{B_1dx}{\{(x-a)^2+\beta^2\}^n}+\int\frac{B_2dx}{\{(x-a)^2+\beta^2\}^{n-1}}+\cdots\int\frac{B_ndx}{(x-a)^2+\beta^2}$$

This expression is composed of two distinct series, the first of which is of the general form, where $x-a=z$.

$$\int\frac{Azdz}{(z^2+\beta^2)^m}=A\int zdz\,(z^2+\beta^2)^{-m}=-\frac{A(z^2+\beta^2)^{-m+1}}{2(m-1)}$$
$$=-\frac{A}{2(m-1)(z^2+\beta^2)^{m-1}}.$$

The second series belongs to a class of integrals which has not yet been discussed, but which may be easily obtained. These integrals are so connected that

$\int\frac{dz}{(z^2+\beta^2)^m}$ depends on $\int\frac{dz}{(z^2+\beta^2)^{m-1}}$; and this, again, on $\int\frac{dz}{(z^2+\beta^2)^{m-2}}$, and so on, down to $\int\frac{dz}{z^2+\beta^2}$; which (as we have already seen) $=\frac{1}{\beta}\tan^{-1}\frac{z}{\beta}$.

To complete the resolution, therefore, we must show how these successive integrals may be deduced, viz.:

$$\int\frac{dz}{(z^2+\beta^2)^m}\text{ from }\int\frac{dz}{(z^2+\beta^2)^{m-1}}.$$

Multiply both terms of $\frac{dz}{(z^2+\beta^2)^{m-1}}$ by $z^2+\beta^2$:

$$\text{then, }\frac{dz}{(z^2+\beta^2)^{m-1}}=\frac{z^2dz}{(z^2+\beta^2)^m}+\frac{\beta^2dz}{(z^2+\beta^2)^m}\ \ldots\ldots\ldots\ (a).$$

$$\text{Also, }d\left\{\frac{z}{(z^2+\beta^2)^{m-1}}\right\}=\frac{dz}{(z^2+\beta^2)^{m-1}}-\frac{2(m-1)z^2dz}{(z^2+\beta^2)^m}.$$

By addition and cancelling $\frac{dz}{(z^2+\beta^2)^{m-1}}$ from both sides of the result, and collecting the like terms, we get

$$d\left\{\frac{z}{(z^2+\beta^2)^{m-1}}\right\}=\frac{\beta^2dz}{(z^2+\beta^2)^m}-(2m-3)\cdot\frac{z^2dz}{(z^2+\beta^2)^m};$$

$$\text{or, }\frac{z}{(z^2+\beta^2)^{m-1}}=\beta^2\int\frac{dz}{(z^2+\beta^2)^m}-(2m-3)\cdot\int\frac{z^2dz}{(z^2+\beta^2)^m}\ \ldots\ldots\ (b).$$

Add the integral of (a) to (b), and collect and cancel like terms: then

$$\int\frac{dz}{(z^2+\beta^2)^{m-1}}=\frac{(2m-2)\beta^2}{2m-3}\int\frac{dz}{(z^2+\beta^2)^m}-\frac{z}{(2m-3)(z^2+\beta^2)^{m-1}};$$

$$\text{or, }\int\frac{dz}{(z^2+\beta^2)^m}=\frac{z}{(2m-2)\beta^2(z^2+\beta^2)^{m-1}}+\frac{2m-3}{2(m-1)\beta^2}\int\frac{dz}{(z^2+\beta^2)^{m-1}}.$$

Hence, by giving to m the successive integer values, we can successively reduce the integrals to the case of $\int \frac{dz}{z^2 + \beta^2}$.

In all the reductions by this method, it will be more advantageous to make the numerical reductions *pari passu*; but as the formula to which they lead is both elegant and remarkable, it is annexed. The student will find it useful to deduce it for himself.

$$\begin{aligned}\int \frac{dz}{(z^2 + \beta^2)^m} &= \frac{z}{2\,(m-1)\,\beta^2\,(z^2+\beta^2)^{m-1}} \\ &+ \frac{(2m-3)z}{2^2\,(m-1)\,(m-2)\,\beta^4\,(z^2+\beta^2)^{m-2}} \\ &+ \frac{(2m-3)\,(2m-5)\,z}{2^3.(m-1)\,(m-2)\,(m-3)\,\beta^6\,(z^2+\beta^2)^{m-3}} \\ &\qquad\qquad \textit{etc.} \qquad\qquad \textit{etc.}\end{aligned}$$

and the general form of the r^{th} term is readily discovered to have the form

$$+\frac{(2m-3)\,(2m-5)\ldots.(2m-2r+1)\,z}{2^r\,(m-1)\ldots.\,(m-r)\,\beta^{2r}\,(z^2+\beta^2)^{m-r}}.$$

The series proceeding till $r = m - 1$; in which case we finally arrive at

$$\int \frac{dz}{z^2+\beta^2} = \tan^{-1}\frac{z}{\beta},$$

as already pointed out in the preceding page.

EXAMPLES.

1. Find the integral of $\frac{(x^4 + 2x^3 + 3x^2 + 3)\,dx}{(x^2+1)^3} = \frac{\text{U}dx}{\text{V}}$.

Here, finding the partial fractions as before, we have

$$\begin{aligned}\int\frac{\text{U}dx}{\text{V}} &= -2\int\frac{xdx}{(x^2+1)^3} + 2\int\frac{xdx}{(x^2+1)^2} + \int\frac{dx}{(x^2+1)^3} + \int\frac{dx}{(x^2+1)^2} + \int\frac{dx}{x^2+1} \\ &= \frac{1}{2\,(x^2+1)^2} - \frac{1}{x^2+1} + \frac{x}{4\,(x^2+1)^2} + \frac{7}{4}\int\frac{dx}{(x^2+1)^2} + \int\frac{dx}{x^2+1} \\ &= \frac{1}{2\,(x^2+1)^2} - \frac{1}{x^2+1} + \frac{x}{4\,(x^2+1)^2} + \frac{7x}{8\,(x^2+1)} + \frac{15}{8}\int\frac{dx}{x^2+1} \\ &= \frac{7x^3 - 8x^2 + 9x - 4}{8\,(x^2+1)^2} + \frac{15}{8}\tan^{-1}x + \text{C}.\end{aligned}$$

2. If it be required to integrate $\frac{(2x^5 - x^2)dx}{(x^2+1)^2\,(x^2+x+1)^2}$, the forms are

$$-\int\frac{(1+2x)dx}{(x^2+1)^2} + \int\frac{4dx}{x^2+1} - \int\frac{4dx}{x^2+x+1} + \int\frac{dx}{(x^2+x+1)^2}.$$

3. $\int\frac{x^2dx}{x^3+x^2+x+1} = \log_e\left\{(x+1)\,(x^2+1)^{\frac{1}{2}}\right\}^{\frac{1}{2}} - \frac{1}{2}\tan^{-1}x.$

4. $\int\frac{(3x^2+x-2)dx}{(x-1)^3(x^2+1)} = -\frac{1}{2\,(x-1)^2} - \frac{5}{2\,(x-1)} + \log_e\left\{\frac{x^2+1}{(x-1)^2}\right\}^{\frac{3}{4}} - \tan^{-1}x.$

5. $\int\frac{dx}{(x^2+1)^2(x+1)} = \frac{x+1}{4\,(x^2+1)} + \log_e\left\{\frac{x+1}{(x^2+1)^{\frac{1}{2}}}\right\}^{\frac{1}{4}} + \frac{1}{2}\tan^{-1}x.$

$$6.\ \int \frac{(x^2+3x-2)dx}{(x^2-x+1)^2(x-1)^2} = \frac{25}{3\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}} - \log_e \frac{(x^2-x+1)^{\frac{1}{2}}}{x-1}$$
$$- \frac{5x-7}{3(x^2-x+1)} - \frac{2}{x-1}.$$

$$7.\ \int \frac{(a+bx)dx}{x^3-1} = \frac{a+b}{3} \log_e \frac{x-1}{(x^2+x+1)^{\frac{1}{2}}} + \frac{b-a}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}}.$$

PROP. VII.

To integrate $\dfrac{Udx}{x^{2n} - 2x^n \cos\beta + 1}$, *where n is integer, and U is a function of x of a lower degree than the $2n^{th}$.*

This integral properly belongs to *prop. v.*, but the practical difficulty of resolving V into its component factors is, as has before been remarked, an insuperable difficulty in the actual solution. In the present case (and many others are reducible to the form here proposed) that difficulty does not exist: but we shall only show the method of resolution, leaving the actual integration as a part of the proposition before referred to.

By Euler's theorem (*vol. i. p.* 437), putting a for θ, we get

$\cos a = \frac{1}{2}\{e^{a\sqrt{-1}} + e^{-a\sqrt{-1}}\}$, and $\cos na = \frac{1}{2}\{e^{na\sqrt{-1}} + e^{-na\sqrt{-1}}\}$.

whence, if we put $e^{a\sqrt{-1}} = x$, then $e^{na\sqrt{-1}} = x^n$; and hence,

$$2 \cos a = x + \frac{1}{x}, \text{ or } x^2 - 2x \cos a + 1 = 0,$$

$$2 \cos na = x^n + \frac{1}{x^n}, \text{ or } x^{2n} - 2x^n \cos na + 1 = 0.$$

Now these equations must be simultaneous, being derived from the same fundamental equation by similar processes; and hence the values of x which fulfil the first must also fulfil the second: in other words, the first is a component factor of the second.

Let $a = \frac{\beta}{n}$; and for β write $(2p\pi + \beta)$: then for successive integer values 0, 1, 2, ... $n-1$, of p, we have the following pairs of simultaneous equations.

$$\begin{array}{lll}
(1).\ x^2 - 2x \cos \frac{\beta}{n} + 1 = 0, & & x^{2n} - 2x^n \cos\beta + 1 = 0, \\
(2).\ x^2 - 2x \cos \frac{2\pi+\beta}{n} + 1 = 0, & & x^{2n} - 2x^n \cos(2\pi+\beta) + 1 = 0, \\
(3).\ x^2 - 2x \cos \frac{4\pi+\beta}{n} + 1 = 0, & & x^{2n} - 2x^n \cos(4\pi+\beta) + 1 = 0, \\
\dots\dots\dots\dots\dots\dots & & \dots\dots\dots\dots\dots\dots \\
(^n).\ x^2 - 2x \cos \frac{2(n-1)\pi+\beta}{n} + 1 = 0, & & x^{2n} - 2x^n \{\cos 2(n-1)\pi + \beta\} + 1 = 0.
\end{array}$$

But the several equations of the second vertical column are identical, since (*vol. i. p.* 428), generally, $\cos\beta = \cos(2p\pi + \beta)$; and hence the first of them $x^{2n} - 2x^n \cos\beta + 1 = 0$ is divisible by each of the equations in the first vertical column, the expressions themselves are component factors of that equation. Moreover, as they are n in number and in the second degree there can be no other factors: wherefore the denominator of the given fraction is resolved into its component factors: that is,

$$x^{2n} - 2x^n \cos\beta + 1 = \left\{x^2 - 2x\cos\frac{\beta}{n} + 1\right\}\left\{x^2 - 2x\cos\frac{2\pi+\beta}{n} + 1\right\}\ldots\ldots$$

the factors being those which form the first vertical column. Hence finally,

$$\frac{\mathrm{U}dx}{\mathrm{V}} = \frac{\mathrm{A}_1 x + \mathrm{B}_1}{x^2 - 2x\cos\frac{\beta}{n} + 1} + \frac{\mathrm{A}_2 x + \mathrm{B}_2}{x^2 - 2x\cos\frac{2\pi+\beta}{n} + 1} + \ldots\ldots$$

and the values of A_1, B_1, A_2, B_2, ... A_n, B_n, can be found as before. There are no known means of abbreviating the process of reduction in this general state; but in particular cases of it, this is otherwise. These cases will form the concluding propositions of the chapter.

PROP. VIII.

To integrate $\frac{dx}{x^n - 1}$, *n being any integer, odd or even* [*].

First. To decompose the denominator.

In the component factors of the preceding proposition, make $\beta = 0$; then

$$x^{2n} - 2x^n + 1 =$$

$$= (x^2 - 2x + 1)\ (x^2 - 2x\cos\frac{2\pi}{n} + 1)\ (x^2 - 2x\cos\frac{4\pi}{n} + 1)\ \ldots\ldots$$

$$\times \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\times (x^2 - 2x\cos\frac{2(n-2)\pi}{n} + 1)\ (x^2 - 2x\cos\frac{2(n-1)\pi}{n} + 1)$$

If now we examine the cosines of these factors in a reverse order, we find

$$\cos\frac{2(n-1)\pi}{n} = \cos\left(2\pi - \frac{2\pi}{n}\right) = \cos\frac{2\pi}{n},$$

$$\cos\frac{2(n-2)\pi}{n} = \cos\left(2\pi - \frac{4\pi}{n}\right) = \cos\frac{4\pi}{n},\ \textit{etc.}$$

Hence $x^2 - 2x\cos\frac{2(n-1)\pi}{n} + 1 = x^2 - 2x\cos\frac{2\pi}{n} + 1$, *etc.*;

and we have the above equation converted into

$$x^{2n} - 2x^n + 1 = (x-1)^2\ (x^2 - 2x\cos\frac{2\pi}{n} + 1)^2\ (x^2 - 2x\cos\frac{4\pi}{n} + 1)^2\ \ldots\ldots\ldots\ (a).$$

When n is odd, there will be an odd number of factors; and hence, apart from $x^2 - 2x + 1$ or $(x-1)^2$, there will be an even number; hence each of the factors, beginning with the second, will have one equal to its value beginning

---

[*] If n be even, $x^n - 1 = (x^{\frac{n}{2}} - 1)\ (x^{\frac{n}{2}} + 1)$; and similarly,

... ½n .... $x^n - 1 = (x^{\frac{n}{4}} - 1)\ (x^{\frac{n}{4}} + 1)\ (x^{\frac{n}{2}} + 1)$, and so on.

If we proceed thus, we shall at last have a factor $x - 1$, or two factors $x - 1$ and $x + 1$, together with other factors $x^2 + 1$, $x^4 - 1$, ...; which are particular cases of the preceding proposition, having $\beta = \frac{1}{2}\pi$. The decomposition of these may therefore be effected by taking

$$\mathrm{V} = \frac{\mathrm{A}_1}{x-1} + \frac{\mathrm{A}_2}{x+1} + \frac{\mathrm{A}_3 x + \mathrm{B}}{x^2+1} + \frac{\mathrm{A}_4 x^3 + \mathrm{B}_4 x^2 - \mathrm{C}_4 x + \mathrm{D}_4}{x^4+1} + \ldots\ldots$$

The method in the text is, however, preferable, on account of the elegance and symmetry of the resulting final equations.

from the end. The preceding expression is therefore a square, since each side is composed of factors which are squares. Wherefore, extracting, we get

$$x^n - 1 = (x - 1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right)\ldots\frac{n+1}{2}\text{ factors.} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(a).$$

When n is even, there will be an even number of factors; and hence there will be one besides $x^2 - 2x + 1$, immediately following the $\frac{1}{2}n^{\text{th}}$, which has no corresponding *equal* factor in the first half of the series. This factor is

$$\int x^2 - 2x\cos\frac{2\,.\,\frac{1}{2}n\pi}{n} + 1 = x^2 - 2x\cos\pi + 1 = x^2 + 2x + 1 = (x+1)^2;$$

and hence, also, in this case, the second side is a square, and therefore

$$x^n - 1 = (x + 1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\ldots\,\frac{n+2}{2}\text{ factors} \ \ldots\ldots\ldots\ldots (b).$$

Wherefore, the given denominator is decomposed as was required to be done.

Second. To find the corresponding partial fractions.

Let n be odd. Differentiate $\log_e$ of both sides of equation (a), which corresponds to this case, and multiply the results by x; then we obtain

$$\frac{nx^n dx}{x^n - 1} = \frac{x\,dx}{x - 1} + \left\{\frac{2x^2 - 2x\cos\frac{2\pi}{n}}{x^2 - 2x\cos\frac{2\pi}{n} + 1} + \frac{2x^2 - 2x\cos\frac{4\pi}{n}}{x^2 - 2x\cos\frac{4\pi}{n} + 1} + \ldots\frac{n-1}{2}\text{ terms}\right\}dx.$$

Also, $n = 1 + \{2 + 2 + \ldots\ldots \frac{n-1}{2}\text{ terms}\}$.

Multiply the latter of these equations by dx, and subtract the former from it, term for term, and divide the result by n; then we have

$$\frac{dx}{x^n - 1} = -\frac{2dx}{n}\left\{\frac{-\frac{1}{2}}{x - 1} + \frac{1 - x\cos\frac{2\pi}{n}}{1 - 2x\cos\frac{2\pi}{n} + x^2} + \ldots + \frac{1 - x\cos\frac{(n-1)\pi}{n}}{1 - 2x\cos\frac{(n-1)\pi}{n} + x\right.$$

Let n be even. Take $\log_e$ of both sides, differentiate and multiply by x; then we have, as in the former case,

$$\frac{nx^n dx}{x^n - 1} = \frac{2x^2 dx}{x^2 - 1} + \left\{\frac{2x^2 - 2x\cos\frac{2\pi}{n}}{x^2 - 2x\cos\frac{2\pi}{n} + 1} + \frac{2x^2 - 2x\cos\frac{4\pi}{n}}{x^2 - 2x\cos\frac{4\pi}{n} + 1} + \ldots\,\frac{n-2}{2}\text{ terms}\right\}dx.$$

And, $n = 2 + \{2 + 2 + \ldots\ldots \frac{n-2}{2}\text{ terms}\}$.

Multiply by dx, subtract and divide by n, as in the preceding case; then we get

$$\frac{dx}{x^n - 1} = -\frac{2dx}{n}\left\{\frac{-1}{x^2 - 1} + \frac{1 - x\cos\frac{2\pi}{n}}{1 - 2x\cos\frac{2\pi}{n} + x^2} + \ldots + \frac{1 - x\cos\frac{(n-2)\pi}{n}}{1 - 2x\cos\frac{(n-2)\pi}{n} + x^2}\right\}$$

The general form of the integral which occurs here has been already obtained in *ex.* 11, *p.* 453.

The integration, therefore, is rendered complete as to method; and the integral can be found in a series of single terms. The student will do well to attempt the formation of a simplified expression for the final result.

PROP. IX.

To integrate $\frac{dx}{x^n+1}$, *where n is either an even or an odd integer.*

This differs from the preceding problem only in taking $\beta = \pi$ in *prop. vii.* (instead of $\beta = 0$), which gives

$$x^n + 1 = (x+1)\left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right) \ldots \frac{n+1}{2} \text{ terms, or}$$

$$x^n + 1 = \left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right) \ldots . \frac{n}{2} \text{ factors;}$$

according as *n* is *odd* or *even;* and in the two cases respectively,

$$\frac{dx}{{}^{n}+1} = \frac{2dx}{n}\left\{\frac{\frac{1}{2}}{1+x} + \frac{1-x\cos\frac{\pi}{n}}{1-2x\cos\frac{\pi}{n}+x^2} + \frac{1-x\cos\frac{3\pi}{n}}{1-2x\cos\frac{3\pi}{n}+x^2} + \ldots . + \frac{1-x\cos\frac{(n-2)\pi}{n}}{1-2x\cos\frac{(n-2)\pi}{n}+x^2}\right\}$$

$$\frac{dx}{{}^{n}+1} = \frac{2dx}{n}\left\{\frac{1-x\cos\frac{\pi}{n}}{1-2x\cos\frac{\pi}{n}+x^2} + \frac{1-x\cos\frac{3\pi}{n}}{1-2x\cos\frac{3\pi}{n}+x^2} + \ldots \frac{1-x\cos\frac{(n-1)\pi}{n}}{1-2x\cos\frac{(n-1)\pi}{n}+x^2}\right\}$$

PROP. X.

To integrate $\frac{x^m dx}{bx^n \pm a}$.

Make $\frac{b}{a}x^n = y^n$: then it is changed to $\frac{1}{a}\left(\frac{a}{b}\right)^{\frac{m+1}{n}} \int \frac{y^m\,dy}{y^n \pm 1}$.

Suppose the denominator decomposed as before into its quadratic factors, and let $y^2 - 2y\cos\alpha + 1$ be one of them. Without regarding whether *a* and *b* (the roots of this equation) be real or not, put the partial fraction, omitting the co-efficient belonging to it, under the form

$$\frac{A}{y-a} + \frac{B}{y-b};$$

and consider simply one of these. Then if $\frac{U}{V} = \frac{A}{y-a} + \frac{R}{S}$, (where $\frac{R}{S}$ is the rational fraction comprising all the other partial fractions) or $U = AS + R(y-a)$; and when $y = a$, $\frac{U_1}{S_1} = A$. But $\frac{dV}{dy} = \frac{dS(y-a)}{dy} + S$; in which case, if $y = a$, $\frac{dV_1}{da} = S_1$; and $U_1 = AS_1$; or $A = \frac{U_1}{\frac{dV_1}{da}}$; and similarly $B = \frac{U_2}{\frac{dV_2}{db}}$. Hence we have the partial fraction.

But $U_1 = a^m$, and $\frac{dV_1}{da_1} = na^{n-1}$: whence $A = \frac{a^m}{na^{n-1}} = \frac{a^{m+1}}{na^n}$, and similarly $B = \frac{b^{m+1}}{nb^n}$. Or, since in these cases $a^n = \pm 1$, we have $A = \pm\frac{a^{m+1}}{n}$ and $B = \pm\frac{b^{m+1}}{n}$; and the expression is $\frac{A}{y-a} + \frac{B}{y-b} = \pm\frac{1}{n}\left\{\frac{a^{m+1}}{y-a} + \frac{b^{m+1}}{y-b}\right\}$

$$= \mp\frac{1}{n}\cdot\frac{y(a^{m+1} + b^{m+1}) - ab(a^m + b^m)}{y^2 - 2y\cos\alpha + 1}.$$

But *a*, *b* being the roots of $y^2 - 2y\cos\alpha + 1 = 0$, we have

$a = \cos\alpha + \sin\alpha . \sqrt{-1}$; and $b = \cos\alpha - \sin\alpha . \sqrt{-1}$: or
$a + b = 2\cos\alpha$; and $ab = \cos^2\alpha - \sin^2\alpha\,(-1) = 1$; and
$a^{m+1} + b^{m+1} = 2\cos(m+1)\,\alpha$ and $a^m + b^m = 2m\cos m\alpha$.

The result therefore is

$$\frac{A}{y-a} + \frac{B}{y-b} = \mp\frac{2}{n} . \frac{y\cos(m+1)\,\alpha - \cos m\alpha}{y^2 - 2y\cos\alpha + 1}.$$

All the factors of the denominator (except the real ones) take this form, and hence the expression is integrable. The general term of the integral is, hence,

$$\mp\frac{1}{n}\left\{\cos(m+1)\,\alpha\log_e(y^2 - 2y\cos\alpha + 1) - 2\sin(m+1)\,\alpha\,\tan^{-1}\frac{y-\cos\alpha}{\sin\alpha}\right\},$$

in which inserting the successive values of α, we get the whole result.

PROP. XI.

To integrate $\dfrac{x^m\,dx}{a + bx^n + cx^{2n}}$.

Put $z^{2n} = \frac{c}{a}x^{2n}$: then $x^n = \left(\frac{a}{c}\right)^{\frac{1}{2}} z^n$; $x^m dx = \left(\frac{a}{c}\right)^{\frac{m+1}{2n}} z^m dz$; and

$$\int\frac{x^m\,dx}{a + bx^n + cx^{2n}} = \frac{1}{a} . \left(\frac{a}{c}\right)^{\frac{m+1}{2n}}\int\frac{z^m dz}{1 - 2z^n\cos\beta + z^{2n}},$$

Where $\cos\beta = -\dfrac{b}{2\sqrt{ac}}$; which requires $4ac > b^2$.

If $4ac < b^2$: then the roots of $cx^{2n} + bx^n + a = 0$, treated as a quadratic, are real. Let them be r_1 and r_2; and then $cx^{2n} + bx^n + a = c\,(x^n - r_1)(x^n - r_2)$, and we have

$$\int\frac{x^m dx}{a + bx^n + cx^{2n}} = \frac{1}{c\,(r_1 - r_2)}\left\{\int\frac{x^m dx}{x^n - r_1} - \int\frac{x^m dx}{x^n - r_2}\right\},$$

the integrals being determinable by the preceding proposition.

For further details, reference must be had to larger treatises on the calculus.

EXAMPLES.

1. $\displaystyle\int\frac{dx}{1+x^3} = \frac{1}{3}\log_e\frac{1+x}{\sqrt{1-x+x^2}} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$.

The integral may also be put in a form similar to that of *ex.* 5.

2. $\displaystyle\int\frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}}\log_e\frac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}$.

3. $$\int\frac{dx}{1+x^5} = \log_e\left\{\frac{1+x}{\left(1-2x\cos\frac{\pi}{5}+x^2\right)^{\cos\frac{\pi}{5}}\left(1-2x\cos\frac{3\pi}{5}+x^2\right)^{\cos\frac{3\pi}{5}}}\right\}^{\frac{1}{5}}$$
$$+\frac{2}{5}\left\{\sin\frac{\pi}{5} . \tan^{-1}\frac{x\sin\frac{\pi}{5}}{1-x\cos\frac{\pi}{5}} + \sin\frac{3\pi}{5} . \tan^{-1}\frac{x\sin\frac{3\pi}{5}}{1-x\cos\frac{3\pi}{5}}\right\}$$

4. $\displaystyle\int\frac{dx}{1+x^6} = \frac{1}{4\sqrt{3}}\log_e\frac{1 + x\sqrt{3} + x^2}{1 - x\sqrt{3} + x^2} + \frac{1}{6}\tan^{-1}\frac{3x\,(1-x^2)}{1 - 4x^2 + x^4}$.

5. $$\int\frac{dx}{1-x^3} = -\log_e(1-x)^{\frac{1}{3}}\left(1 - 2x\cos\frac{2\pi}{3} + x^2\right)^{\frac{1}{3}\cos\frac{2\pi}{3}}$$
$$+\frac{2}{3}\sin\frac{2\pi}{3} . \tan^{-1}\frac{x\sin\frac{2\pi}{3}}{1 - x\cos\frac{2\pi}{3}}.$$

The preceding integral may also be put in a form analogous to that of *ex.* 1.

6. $\int \frac{dx}{1-x^6} = \log_e \frac{1+x}{1-x}\left(\frac{1+x+x^2}{1-x+x^2}\right)^{\frac{1}{2}} + \frac{1}{2\sqrt{3}} \tan^{-1} \frac{x\sqrt{3}}{1-x^2}.$

7. $\int \frac{dx}{1-x^4} = \log_e \left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} + \frac{1}{2} \tan^{-1} x.$

CHAPTER IV.

THE RATIONALISATION OF FUNCTIONS INVOLVING BINOMIAL SURD EXPRESSIONS.

The general problem reduces itself to rendering rational the differential expression

$$x^{\frac{\mu}{\mu_1}}\left(a + bx^{\frac{\nu}{\nu_1}}\right)^{\frac{p}{q}} dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

where μ, μ_1, ν, ν_1, p, q, are any integers, and having either the positive or negative sign.

The cases in which this can be done by a general method are very few; but by particular artifices, (mainly dependent on the principles of the Diophantine analysis,) a considerable number of particular cases not included in the general methods of solution have been rationalised.

PROP. I.

To reduce equation (1) *to a form in which the indices* $\frac{\mu}{\mu_1}$ *and* $\frac{\nu}{\nu_1}$ *are integer, and the latter positive.*

1. Let $\rho = \mu_1 \nu_1$ be the common denominator of these fractions, and assume $x = z^\rho$; then we have

$$x^{\frac{\mu}{\mu_1}} = \left(z^\rho\right)^{\frac{\mu}{\mu_1}} = z^{\frac{\rho\mu}{\mu_1}} = z^{\nu_1\mu}; \text{ and similarly, } x^{\frac{\nu}{\nu_1}} = z^{\mu_1\nu}.$$

Whence put $\nu_1\mu = m$, and $\mu_1\nu = n$, and we get

$$x^{\frac{\mu}{\mu_1}}\left(a + bx^{\frac{\nu}{\nu_1}}\right)^{\frac{p}{q}} dx = z^m\,(a + bz^n)^{\frac{p}{q}}.\,\rho z^{\rho-1} dz.$$

2. Let n be negative; then the expression may be again transformed by putting $z^{-1} = y$; and the expressions become

$$y^{-m}\,(a + by^n)^{\frac{p}{q}} \times -\frac{\rho dy}{y^{\rho+1}}, \text{ or } -\rho y^{-(m+\rho+1)}\,(a + by^n)^{\frac{p}{q}}\,dy;$$

whence the equation is reduced as was required.

We shall hence only consider the equation under the form in which m, n, are integers, $\frac{p}{q}$, fractional, and n always positive; for to this form the present proposition shows the general form is always reducible.

PROP. II.

The function $x^m (a + bx^n)^{\frac{p}{q}} dx$ *is capable of being rationalised when either of the two following conditions is fulfilled:*

(1) .... $\frac{m+1}{n}$ = *any positive or negative integer, or* 0.

(2) .... $\frac{m+1}{n} + \frac{p}{q} =$ ....

First. Put $z = a + bx^n$; then $x^m = \left(\frac{z-a}{b}\right)^{\frac{m}{n}}$; and hence

$$x^{m-1}\, dx = \frac{b^{-\frac{m}{n}}}{n} (z-a)^{\frac{m}{n}-1} dz \text{; or}$$

$$x^m (a + bx^n)^{\frac{p}{q}} dx = \frac{1}{n} b^{-\frac{m+1}{n}} (z-a)^{\frac{m+1}{n}-1} z^{\frac{p}{q}} dz \text{..........(2).}$$

Now this is integrable if $\frac{m+1}{n} = r$ be an integer or 0.

When r is *positive*, the series $(z-a)^{r-1}$ is expansible by the binomial theorem, and hence the expression is transformed to a series of monomes, each of which is integrable, by the methods of *chapter ii. prop. i.*

When r is *negative*, the expression is reduced to a rational fraction, and the integration will be effected by the appropriate methods given in *chapter iii.*

This condition is called the *first criterion of rationability.*

Secondly. Divide the factor within the vinculum by $x^{\frac{np}{q}}$, and multiply that without it by the same; then

$$x^m (a + bx^n)^{\frac{p}{q}} dx \text{ becomes } x^{m+\frac{np}{q}} (ax^{-n} + b)^{\frac{p}{q}} dx.$$

Put $n' = -n$, and $m' = m + \frac{np}{q}$: then the second form becomes

$$x^{m'} (b + ax^{n'})^{\frac{p}{q}} dx;$$

and substituting z, and reducing as in the preceding case,

$$x^{m'} (b + ax^{n'})^{\frac{p}{q}} dx = \frac{1}{n'} . a^{-\frac{m'+1}{n'}} (z-b)^{\frac{m'+1}{n'}-1} z^{\frac{p}{q}} dz.$$

$$= -\frac{1}{n} . a^{\frac{m+1}{n}+\frac{p}{q}} (z-b)^{-\frac{m+1}{n}-\frac{p}{q}-1} z^{\frac{p}{q}} dz.$$

This form will also be rational when $\frac{m+1}{n} + \frac{p}{q}$ is an integer or 0; and will be a series of monomes or a rational fraction, according as r is positive or negative. This is called the *second criterion of rationability.*

Scholium.

If neither of these criteria be fulfilled, there is no known method of rationalising the general form. Two resources, however, are still open for the integration: integration by parts, which may or may not succeed in furnishing integrable expressions; and the developement of the irrational expression into an infinite series of monomes integrating each term, and approximating to its value for a given value of z (or $x^{\frac{1}{n}}$), or between two given values of z. Upon these subjects some explanation will be given in future chapters. We shall now give a few examples of rationalisation when the general criteria are fulfilled; and then furnish a few instances of particular artifices by which rationalisation may be effected upon functions which do not come under the general solutions above.

EXAMPLES.

1. Integrate the expression $x^3(a+bx^2)^{\frac{1}{2}}dx$.

Here $\frac{m+1}{n}=\frac{3+1}{2}=2$ an integer; which is the *first criterion*. Hence the transformed expression is

$$\int\frac{1}{2b^2}(z-a)\,z^{\frac{1}{2}}\,dz=\int\frac{z^{\frac{3}{2}}dz}{2b^2}-\int\frac{az^{\frac{1}{2}}dz}{2b^2}=\frac{z^{\frac{5}{2}}}{5b^2}-\frac{az^{\frac{3}{2}}}{3b^2},$$

or putting $a+bx^2$ for z,

$$\int x^3(a+bx^2)^{\frac{1}{2}}\,dx=\frac{3bx^2-2a}{15b^2}.\{a+bx^2\}^{\frac{3}{2}}+\text{C}.$$

2. Find the integral of $\frac{dx}{x^2(a+x^3)^{\frac{5}{3}}}$, or $x^{-2}(a+x^3)^{-\frac{5}{3}}\,dx$.

Here $\frac{m+1}{n}+\frac{p}{q}=\frac{-2+1}{3}-\frac{5}{3}=-2$; or the *second criterion* is fulfilled. The transformed expression, on putting $z=ax^{-3}+1$ is found to be

$$-\int\frac{(z-1)\,z^{-\frac{5}{3}}\,dz}{3a^2}=-\frac{z^{\frac{1}{3}}+\frac{1}{2}z^{-\frac{2}{3}}}{a^2}+\text{C}.=\text{C}-\frac{3x^3+2a}{2a^2x\;(a+x)^{\frac{2}{3}}}.$$

3. $\int x^3(a^2+x^2)^{\frac{1}{3}}=\frac{3}{56}(a^2+x^2)^{\frac{4}{3}}(4x^2-3a^2)+\text{C}$ (1*st criterion*).

4. $\int\frac{adx}{(1+x^2)^{\frac{3}{2}}}=\frac{ax}{\sqrt{1+x^2}}+\text{C}$ (2*nd criterion*).

5. $\int x^5(a+bx^2)^{\frac{2}{3}}dx=\frac{3(20b^2x^4-51abx^2-9a^2)(a+bx^2)^{\frac{5}{3}}}{440}$ (1*st criterion*).

6. $\int x^{-4}\,(1+x^2)^{-\frac{1}{2}}dx=\frac{(2x^2-1)\,(1+x^2)^{\frac{1}{2}}}{3x^3}+\text{C}$ (2*nd criterion*).

7. $\int(1-x^n)^{-\frac{1}{n}}\,dx=-\int\frac{z^{n-2}}{1+z^n}$, where $z=\left(\frac{1-x^n}{x^n}\right)^{\frac{1}{n}}$; and may be integrated by partial fractions.

8. Let $x^{\frac{3}{4}}\left(1-2x^{-\frac{1}{2}}\right)^{\frac{3}{2}}dx$ be proposed for integration.

Reducing the indices $\frac{3}{4}$ and $\frac{1}{2}$ to a common denominator 4, and putting $x=y^4$, and then $y^{-1}=z$, we have

$$x^{\frac{3}{4}}\left\{1-2x^{-\frac{1}{2}}\right\}^{\frac{3}{4}}dx = y^3\,(1-2y^{-2})^{\frac{3}{4}}\,.\,4y^3\,dy = -4z^{-8}\,(1-2z^2)^{\frac{3}{4}}dz.$$

Hence, $\frac{m+1}{n}+\frac{p}{q}=\frac{-8+1}{2}+\frac{3}{2}=-2$, the *second criterion* is fulfilled; and the integral is obtained as in the former examples.

9. Given $x^{-3}\,(2-x^3)^{\frac{5}{6}}\,dx$ for integration. Here neither criterion is fulfilled, and the expression cannot be rationalized by either of the general methods. It does not, therefore, belong to this class.

10. $$\int\frac{x^{\frac{5n}{2}-1}\,dx}{\sqrt{a+bx^n}}=\left\{\frac{x^{\frac{3n}{2}}}{2b}-\frac{3ax^{\frac{n}{2}}}{4b^2}\right\}\frac{\sqrt{a+bx^n}}{n}+\frac{3a^2}{4nb^{\frac{5}{2}}}\log_e\left(b^{\frac{1}{2}}x^{\frac{n}{2}}+\sqrt{a+bx^n}\right)$$

the *second criterion* being applicable here.

11. Required $\int\frac{x^{n-1}\,dx}{(1-x^n)\,(2x^n-1)^{\frac{1}{2n}}}$. Assume $(2x^n-1)^{\frac{1}{2n}}=z$: then

$$2\,(1-x^n)=1-z^{2n}.$$

Take the differentials of the logs of this; then

$$\frac{x^{n-1}\,dx}{1-x^n}=\frac{2z^{2n-1}\,dz}{1-z^{2n}};$$

and the transformed expression is

$$\int\frac{x^{n-1}\,dx}{(1-x)\,(2x^n-1)^{\frac{1}{2n}}}=\int\frac{2z^{2n-2}dz}{1-z^{2n}}\text{; a rational form.}$$

12. In $\int\frac{dx}{(1-x^n)\,(2x^n-1)^{\frac{1}{2n}}}$, put $(2x^n-1)^{\frac{1}{2n}}=xz$; then,

$$(1-x^{-n})^2=1-z^{2n},\text{ and }\frac{2nx^{(n+1)}\,dx}{1-x^{-n}}=\frac{-2nz^{2n-1}}{1-z^{2n}}\text{; hence,}$$

$$\int\frac{dx}{(1-x^n)\,(2x^n-1)^{\frac{1}{2n}}}=\int\frac{z^{2n-2}\,dz}{1-z^{2n}}.$$

The results (11) (12) are called *Lexell's integrals*, from the circumstance of that geometer having discovered them.

13. To integrate $\sqrt{1+\cos\theta}\,d\theta$. Put $z=1+\cos\theta$; and hence $-\sin\theta\,d\theta=dz$. But $\sqrt{1+\cos\theta}=\frac{\sin\theta}{\sqrt{1-\cos\theta}}$, and $1-\cos\theta=2-z$; whence

$$\sqrt{1+\cos\theta}\,d\theta=-\frac{\sin\theta\,d\theta}{\sqrt{1-\cos\theta}}=\frac{dz}{\sqrt{2-z}}\text{; hence we have}$$

$$\int\sqrt{1+\cos\theta}\,d\theta=-\int\frac{dz}{\sqrt{2-z}}=2\sqrt{2-z}=2\sqrt{1-\cos\theta}.$$

14. To integrate $\frac{(1+x^2)\,dx}{(1-x^2)\sqrt{1+x^4}}$, put $x^2=\frac{y-1}{y+1}$: then

$$\int\frac{(1+x^2)\,dx}{(1-x^2)\sqrt{1+x^4}}=\int\frac{2y\,dy}{2\sqrt{2}\sqrt{y^4-1}}=\frac{\log_e\left\{\sqrt{y^4-1}+y^2\right\}}{2\sqrt{2}}.$$

$$=\frac{1}{2\sqrt{2}}\log_e\frac{(1+x^2)^2+2x\sqrt{1+x^4}}{(1+x^2)^2}.$$

PROP. III.

To integrate $R\left\{x, (a+bx)^{\frac{m}{n}}, (a+bx)^{\frac{m'}{n'}}, etc.\right\} dx$; *where R is the symbol of any rational function of the quantities within the vinculum.*

Reduce the indices to a common denominator δ: then assume $a + bx$[*] $= z^\delta$, we shall have on making this substitution in the given function, all the binomial terms changed to rational indices of z; and likewise $x = \frac{z^\delta - a}{b}$ is also rational in z. Moreover, $dx = \frac{\delta z^{\delta - 1} dz}{b}$; and the expression becomes

$$\int R\left\{\frac{z^\delta - a}{b}, z^{\frac{\delta m}{n}}, z^{\frac{\delta m'}{n1}}, etc.\right\} . \frac{\delta z^{\delta - 1} dz}{b}.$$

which is essentially rational.

EXAMPLES.

1. Integrate $\frac{dx}{\sqrt{a + bx}}$. Here $\delta = 2$; and hence putting $a + bx = z^2$, we have $dx = \frac{2zdz}{b}$, and $\frac{dx}{\sqrt{a+bx}} = \frac{2dz}{b}$.

$$\text{Whence } \int \frac{dx}{\sqrt{a + bx}} = \frac{2}{b}\int dz = \frac{2z}{b} + C = \frac{2\sqrt{a+bx}}{b} + C.$$

2. Integrate $\frac{xdx}{(1+x)^{\frac{3}{2}}}$. Here $z^2 = 1 + x$, or $x = z^2 - 1$ and $xdx = 2(z^2 - 1)zdz$.

$$\text{Whence} \int \frac{xdx}{(1+x)^{\frac{3}{2}}} = \int \frac{2(z^2-1)\,dz}{z^2} = 2z + \frac{2}{z} + C = \frac{2(x+2)}{\sqrt{x+1}} + C.$$

3. To integrate $\frac{xdx}{(1+x)^{\frac{1}{3}} - (1+x)^{\frac{1}{2}}}$: put $1 + x = z^{3 \cdot 2} = z^6$.

$$\text{Then} \int X dx = \int \frac{-6z^3(1-z^6)}{1-z} dz = -\left\{\frac{3}{2}z^4 + \frac{6}{5}z^5 + z^6 + \frac{6}{7}z^7 + \frac{3}{4}z^8 + \frac{2}{3}z^9\right\}.$$

4. $\int \frac{dx}{(c + ex^2)\sqrt{a + bx}}$ is proposed. Here let $z^2 = a + bx$: then we get

$$\frac{1}{\sqrt{a+bx}} = \frac{2dz}{bdx}; \text{ and}$$

$$\int X dx = \frac{2}{b}\int \frac{dz}{c + e\left(\frac{z^2-a}{b}\right)^2} = 2b \int \frac{dz}{cb^2 + ea^2 - 2eaz^2 + ez^4},$$

which is integrable by the preceding chapter.

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[*] The same process and conclusion applies to the case where $\frac{a + bx}{a_1 + b_1 x}$ is put instead of $a + bx$, as in the proposition. The value of x is, in this case $x = -\frac{a - a_1 z^\delta}{b - b_1 z^\delta}$: and the result is a rational fraction, soluble as in the preceding chapter.

$$\text{Also} \int R\left\{x^{mn}, \left(\frac{a + bx^n}{a_1 + b_1 x^n}\right)^{\frac{p}{q}}, \left(\frac{a + bx^n}{a + b_1 x^n}\right)^{\frac{p_1}{q_1}}, \ldots\right\} x^{n-1}\,dx.$$

is reduced to the same form by putting $x^n = y$, and transforming from y to z as before.

PROP. IV.

To integrate the form $R\,\{x, \sqrt{bx \pm cx^2}\}\,dx$, *where* R *denotes a rational function.*

ASSUME $x^2z^2 = bx \pm cx^2$: then $x = \dfrac{b}{z^2 \mp c}$, $dx = \dfrac{-2bzdz}{(z^2 \mp c)^2}$, and

$\sqrt{bx \pm cx^2} = \dfrac{bz}{z^2 \mp c}$. Wherefore, by substitution,

$$\int R\,\{x, \sqrt{bx \pm cx^2}\}\,dx = \int \left\{\frac{b}{z^2 \mp c}, \frac{bz}{z^2 \mp c}\right\} \cdot \frac{-2bzdz}{(z^2 \mp c)^2}$$

which is rational.

EXAMPLE.

In $\displaystyle\int \frac{dx}{(a + x)\sqrt{bx + cx^2}}$ put $bx + cx^2 = x^2z^2$: then the integral in z becomes $\displaystyle -2\int \frac{dz}{az^2 - ac + b}$.

PROP. V.

To integrate the form $R\,\{x, \sqrt{a + bx \pm cx^2}\}\,dx$,

First, let $+ c$ be taken, and make $a + bx + cx^2 = c\,(x + z)^2$; then

$$x = \frac{a - cz^2}{2cz - b},\ dx = -\frac{2c\,(cz^2 - bz + a)\,dz}{(2cz - b)^2}, \text{ and}$$

$$\sqrt{a + bx + cx^2} = \left(\frac{a - cz^2}{2cz - b} + z\right)\sqrt{c} = \frac{cz^2 - bz + a}{2cz - b}\sqrt{c}.$$

Hence all components of the given function are found in rational functions of x.

Second, let $- c$ be taken, and put r_1, r_2 the roots of the equation $a + bx - cx^2 = 0$: then $a + bx - cx^2 = -c\,(x - r_1)\,(x - r_2)$. Make $\sqrt{-c(x - r_1)(x - r_2)} = (x - r_1)\ cz$; hence $x - r_2 = -\,cz^2\,(x - r_1)$, or $x = \dfrac{cr_1z^2 + r_2}{cz^2 + 1}$; $dx = \dfrac{(r_1 - r_2)\,2czdz}{(cz^2 + 1)^2}$; $\sqrt{a + bx - cx^2} = -\dfrac{(r_1 - r_2)\,cz}{cz^2 + 1}$.

Corollary 1.

$\int R\,\{x, \sqrt{a \pm cx^2}\}\,dx$ is included in the present form, b being $= 0$.

Corollary 2.

$\int x^{m-1}.\ R\,\{x^n, \sqrt{a + bx^n + cx^{2n}}\}\,dx$ is also reducible to the same form, provided $\dfrac{m}{n}$ be integer, by making $z = x^n$.

EXAMPLE.

$$\int \frac{dx}{\sqrt{a + bx - cx}} = \int \frac{(r_1 - r_2)\,2czdz}{(cz^2 + 1)^2} \div \frac{(r_2 - r_1)\,cz}{cz^2 + 1} = \int \frac{-2dz}{1 + cz^2}$$

$$\Big[= -\frac{2}{\sqrt{c}}\tan^{-1}(z\sqrt{c}) = -\frac{2}{\sqrt{c}}\tan^{-1}\frac{1}{\sqrt{c}}\left(\frac{r_2 - x}{x - r_1}\right)^{\frac{1}{2}}.$$

PROP. VI.

To integrate $\int R\,\{x, \sqrt{a + bx}, \sqrt{a_1 + b_1x}\}\,dx$; where R denotes a rational *function.*

Put $\frac{a + bx}{a_1 + b_1x} = z^2$: then $x = -\frac{a - a_1z^2}{b - b_1z^2}$; $dx = \frac{(a_1b - ab_1)\,2zdz}{(b_1z^2 - b)^2}$

$$\sqrt{a+bx} = \frac{z\sqrt{ab_1 - a_1b}}{\sqrt{b_1z^2 - b}}, \text{ and } \sqrt{a_1 + b_1x} = \frac{\sqrt{ab_1 - a_1b}}{\sqrt{b_1z^2 - b}}.$$

These values being substituted in the given expression, it is transformed into $\int R\,\left\{z, \sqrt{b_1z^2 - b}\right\}\,dz$ which is again transformed by *prop. v.* into a rational form.

EXAMPLES.

1. $\int \frac{dx}{x\sqrt{(a + bx)\,(a_1 + bx)}} = \int \frac{2dz}{a_1z^2 - a}$; a form already known.

2. $\int \frac{dx}{\sqrt{a - x} - \sqrt{b + x}} = \int \frac{-2\sqrt{b - a}\,.\,zdz}{(z - 1)\,(z^2 + 1)^{\frac{3}{2}}}$ a form in z, falling under the conditions of the preceding problem; and hence by a second transformation will be rationalised.

This may, however, be rationalised at a single operation by putting $\sqrt{a - x} - \sqrt{b + x} = y$; upon which the transformed differential becomes

$$\frac{-(a + b)\,dy}{y\sqrt{2\,(a + b) - y^2}} + \frac{ydy}{\sqrt{2\,(a + b) - y^2}},$$

the integrals of these being forms already known.

PROP. VII.

To integrate $x^m R\,\{x^n, \sqrt{a + b^2x^{2n}}, bx^n \pm \sqrt{a + b^2x^{2n}}\}\,dx$, *where* $\frac{m + 1}{n}$ *is an integer, and R the symbol of a rational function.*

Assume $bx^n \pm \sqrt{a + b^2x^{2n}} = z$, or $x^n = \frac{z^2 - a}{2bz}$: hence $\sqrt{a + b^2x^{2n}} = \frac{z^2 + a}{2z}$, and $x^m dx = \frac{dz}{2nb}\left(\frac{z^2 - a}{2bz}\right)^{\frac{m+1}{n} - 1}\left(1 + \frac{a}{z^2}\right)$ which last is evidently rational when $\frac{m + 1}{n}$ is an integer; and hence all the terms are rational.

Corollary.

The case $\int R\,\{x, \sqrt{a + b^2x^2}, bx \pm \sqrt{a + b^2x^2}\}\,dx$ is one of frequent occurrence, and hence it may be remarked that the substitution is generally $z = bx \pm \sqrt{a + b^2x^2}$.

PROP. VIII.

To integrate $x^m\,(a + bx^n)^{\frac{p}{q}}\,R\,(x^n)\,.\,dx$.

This is the equation of *prop. ii.* multiplied by $R\,(x^n)$; and as all the terms but this one have been rationalised in that proposition, when the rationalisation is possible (or either of the two criteria fulfilled), it is clear that the expression multiplied by the rational $\mathrm{R}(x^n)$ or $\mathrm{R}\left(\frac{z-a}{b}\right)$ is also rational.

Corollary.

In precisely the same way may $\int x^m . \mathrm{R}\left\{x^m, x^n, (a+bx^n)^{\frac{p}{q}}\right\} dx$ be transformed when either of the two criteria is fulfilled.

Scholium.

The first systematic attempt to transform such expressions as those in the last six propositions was made by Sir E. Ffrench Bromhead, in the Phil. Trans. 1816, in a paper on the ***Fluents of Irrational Functions***—a series of investigations well worthy of the student's attention.

CHAPTER V.

THE CORRECTION OF INTEGRALS, AND INTEGRATION BETWEEN LIMITS.

It has been already pointed out (*page* 435) that every integration introduces an arbitrary constant: viz., that which in the assumption of the differential being obtained by a direct process, disappears from the expression. The integration, in itself, furnishes no means of restoring the original value of that constant: and hence it is termed an *arbitrary constant;* it being in all cases a constant, properly so called, but indeterminable by the mere operations of integration. The uses, however, which we have invariably to make of the integral are such as arise from some particular problem under consideration, and which, in fact, produced the differential expression (for in all the applications of the calculus, our differentials appear as the expressions of the condition of some specified problem, not as the result of mere differentiation): and hence from that problem itself may be gained the information which we require for finding the value of the arbitrary constant, or, in technical language, correcting the integral. Most commonly, this consists in our being able to discover the value of the integral corresponding to some given value of the variable; or, in symbols, if the general integral be $\int \phi(x) . dx = f(x)$, we can find the value of $f(x)$ when $x = a$. In fact, no problem is properly limited in the expression of its condition where it does not furnish the method of finding the value of the integral for some one value a of the independent variable x; provided the general integral of a differential expression be required as its final step [*].

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[*] We may recur to a very simple integral already found (*page* 438) as an illustration of this subject. It is to integrate $(a+bx)^2dx$. Now this, if treated as a binomial differential, gives $\int(a+bx)^2dx = \frac{(a+bx)^3}{3b}$; whilst, if previously expanded, $\int a^2dx + \int 2abxdx + \int b^2x^2dx =$

It will be at once apparent how to express the value of the arbitrary constant. Let b be the known value of the integral when $x = c$, and $f(x) + C$ the general form of the integral including its arbitrary constant. Then since $b = f(c) + C$, we have $C = b - f(c)$; and hence the general and corrected integral becomes $f(x) - f(c) + b$. When, therefore, we know the value of the integral function itself corresponding to any one value of the variable, the arbitrary constant can be found. It will, moreover, generally occur in practice, that in this research either $b = 0$, or $c = 0$: though this is not essential, nor yet universal. It is possible, however, universally to render $b = 0$; and under this form we shall at present view it.

From the form in which the correction makes its appearance, viz., the general form of the function $f(x)$ which constitutes the general integral, it will be obviously, not only most convenient, but most natural, in ordinary cases, to write $-f(c)$ for C. In the case of the corrections which we have given for the logarithmic integrals, this has been done for the sake of brevity; but in general expressions, we have either omitted the correction altogether (in which cases we have been governed by the breadth of the printed page), or have simply written them "C".

Again, it is very often (perhaps more frequently than otherwise) the case that the final result of a problem requires the value of the integral to be found for the interval of x varying from one value a to another b. In this case, the integral sought will evidently be $f(b) - f(a)$; and hence can always be computed, if we can find the general form of the integral.

Def. The integral $u = f(x) + C$ is called *an indefinite integral*,

.... $u = f(x) - f(c)$...... *a corrected integral*,

.... $u = f(b) - f(a)$...... *definite integral.*

The symbol $\int_a^b Xdx$ is used to express that the integral of Xdx is to be taken between the limits a, b; the value of x at the origin, which is subtractive, being written below the other. Thus if $\int Xdx = f(x) + C$: then we shall have

$$\int_a^b Xdx = f(b) - f(a).$$

EXAMPLES.

1. If when $x = 1$, $\int x^{m-1}\,dx$ has the value $\frac{1}{m}$, what is the form of the corrected integral?

The indefinite integral is $\frac{x^m}{m} + C$; and by hypothesis, when $x = 1$, $\frac{x^m}{m} + C = \frac{1^m}{m} + C = \frac{1}{m} + C$. Hence $C = 0$, and the corrected integral is $\frac{x^m}{m}$.

2. If when $x = 1$, $\int x^{m-1}\,dx$ has the value 0, what is the form of the corrected integral?

Here the indefinite integral is, as before, $\frac{x^m}{m} + C$; and $0 = \frac{1}{m} + C$, or

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$a^2x + abx^2 + \frac{1}{3}b^2x^3$. In comparing the two forms of the integral, it will be found that they differ only in the value assigned to the absolute terms; being $\frac{a^3}{3b}$ in the one case, and 0 in the other. They are capable of identification only by means of the arbitrary constant.

$C = -\frac{1}{m}$. Hence the corrected integral is $\frac{x^m}{m} + C = \frac{x^m}{m} - \frac{1}{m} = \frac{x^m - 1}{m}$.

3. When $x = a$, the same integral is known to be of the value b: give the value of the correction.

Here $\frac{a^m}{m} + C = b$; or $C = b - \frac{a^m}{m}$; and hence the corrected integral is $\frac{x^m}{m} + \left(b - \frac{a^m}{m}\right) = \frac{x^m - a^m}{m} + b$.

4. Take the example in the note (*page* 468) $\int (a + bx)^2 dx$; supposing $u = 0$ when $x = a$.

Take the *first form* of integration, $u = \frac{(a + bx)^3}{3b} + C_1$. Now when $x = a$, $u = 0$, or $0 = \frac{(a + ab)^3}{3b} + C$, or $C = -\frac{(a + ab)^3}{3b}$. Hence the corrected integral is $\frac{(a + bx)^3 - (a + ab)^3}{3b}$.

Next, take the *second form*, $u = a^2x + abx^2 + \frac{1}{3} b^2 x^3 + C_2$; which, when $x = a$ becomes $0 = a^3 + a^3b + \frac{1}{3} a^3 b^2 + C_2$, or $C_2 = -(a^3 + a^3 b + \frac{1}{3} a^3 b^2)$. Hence the corrected integral is $a^2x + abx^2 + \frac{1}{3} b^2 x^3 - (a^3 + a^3 b + \frac{1}{3} a^3 b^2)$.

Expand the former of these, and divide out the denominator, and we get

$$a^2 (x - a) + ab (x^2 - a^2) + \tfrac{1}{3} b^2 (x^3 - a^3);$$

which is evidently identical with the second form.

It will universally be found (as indeed it is obvious should be the case), that however many different forms different methods of integration may give to the uncorrected integral, the corrected ones will be identical under the same hypothesis of correction.

4. When $x = 1$, $u = 0$: find the corrected integral of $\frac{dx}{1 + x^2}$. The indefinite integral has been already found to be $u = \tan^{-1} x + C$; hence, under the conditions $0 = \tan^{-1} 1 + C$, or $C = -45°$. Hence,

$$\int \frac{dx}{1 + x^2} = \tan^{-1} x - 45° = \tan^{-1} x - \tfrac{1}{4}\varpi.$$

5. If when $x = 0$, $u = 0$; give the corrected integral of $\frac{3x^2\,dx}{x^3 + a^3}$. The indefinite integral is $\log_e (x^3 + a^3) + C$; and hence $C = -\log_e a^3$. Whence the corrected one is

$$u = \log_e (x^3 + a^3) - \log_e a^3 = \log_e \frac{x^3 + a^3}{a^3}.$$

6. Integrate $3x^2dx$ between the limits a and b. Here the indefinite integral is x^3; and hence the definite integral is $\int_a^b 3x^2dx = b^3 - a^3$.

7. Find the definite integral $\int_0^1 \frac{dx}{1 + x^2}$. The indefinite integral is $\tan^{-1} x$, as above. Hence the definite integral under the specified conditions is

$$\tan^{-1} 1 - \tan^{-1} 0 = \tfrac{1}{4}\varpi - 0 = \tfrac{1}{4}\varpi.$$

8. $\int_{-a}^{a} a e^{x\sqrt{-1}}\, dx = a \frac{\left\{e^{a\sqrt{-1}} - e^{-a\sqrt{-1}}\right\}}{\sqrt{-1}} = 2 \sin a.$

9. $\int_0^\pi \sin m\theta \cos n\theta\, d\theta = \frac{m \mp n}{m^2 - n^2}$. See *p.* 442, *Ex.* 13.

Scholium.

It will be apparent, that in all cases where we can obtain the indefinite integral, we can, with one superadded condition, find the corrected one. Also, that between any two given limits we can get the definite integral. It will appear, however, further on, that there are large classes of differentials which are of the utmost importance in the application of the calculus to geometry and physics, the integrals of which we can only obtain by infinite series, which are either divergent, or so very slowly convergent, as to be utterly useless for all the purposes of calculation. If such series, indeed, could be summed, or represented by a function containing a finite number of terms, the difficulty would be evaded; but cases of this kind are comparatively few. Analysts have, therefore, directed their attention, within the last few years, to the investigation of the definite integrals of such expressions independently of the indefinite integrals of which we have hitherto treated. Their success, though promising, has been but limited in this department of the science; but a slight notice of the methods they have employed will be given in a future volume. In the chapters which immediately follow, we shall sometimes leave the integral indefinite; at others, corrected; and at others, again, give also the definite integral between specified limits.

CHAPTER VI.

INTEGRATION BY PARTS; OR INTEGRATION BY SUCCESSIVE REDUCTION.

Let u be a function of x; then we shall have,

$$d(uv) = udv + vdu, \text{ or } \int udv = uv - \int vdu.$$

Wherefore, if any proposed differential be divisible into two factors, one of which (as u) is any function of x, and the other (dv) the differential of a known function of x; then $\int udv$ will be found if $\int vdu$ can be found, in virtue of the relation above. It is obvious that the applicability of this method requires that the function $\int vdu$ shall be less complex than the given one $\int udv$ [*].

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[*] A little reflection will make it obvious that the assumption of the product of two functions of x in the manner here given, is but a particular case of a general principle of assumption for integration by parts. It is also the most simple of all combinations of functions of x, except those of addition and subtraction, which cases are already provided for by the addition and subtraction of any integrable differential expressions to the given differential, so as to also render the entire expression integrable. This is only on the principle of restoring the terms which mutually cancelled each other in taking the differential originally. The assumption of a product of three or more functions of x, instead of two, as above, or that of other than products, as quotients, powers, roots, or more complex forms, will occasionally be useful; but, as a system, no other than the product of two has been much considered by analysts.

PROP. I.

To integrate the expression $x^{m'}(ax^{r'} + bx^{n'})^{p'}dx$ *by parts; where* m', r', n', p' *are any numbers, positive or negative, integer or fractional.*

1. This expression is at once transformable into a simpler one, by multiplying and dividing by x^{pr}; thus giving the equally general form under which we shall consider it, viz.

$$x^{m'+pr'}(a + bx^{n'-r'})^{p'}dx.$$

For brevity in writing, we shall, however, put it

$$x^m(a + bx^n)^p dx, \text{ or } x^m X^p dx.$$

2. This differential may always be transformed into two others: one of these being integrable, and the other of the same general form, having $m \pm n$ instead of m, and $p \pm 1$ instead of p.

This will be best seen by putting down in a tabular form the several forms of which $\int x^m X^p dx$ is capable. Prefixed to the forms is a table of the cases to which the several forms are applicable, as depending on the signs of m and p. Where there are only dots, the form is unaffected by this circumstance, as far as regards the letter in whose column they occur, arising from the corresponding part under the sign of integration being the same as in the given function.

	m	p	Values of $\int x^m X^p dx$.
I	.....	$+$	$\frac{x^{m+1}X^p}{pn+m+1} + \frac{anp}{pn+m+1}\int x^m X^{p-1}dx.$
II	$-$	$+$	$\frac{x^{m+1}X^p}{m+1} - \frac{bnp}{m+1}\int x^{m+n}X^{p-1}dx.$
III	.....	$-$	$-\frac{x^{m+1}X^{p+1}}{n(p+1)a} + \frac{(p+1)n+m+1}{n(p+1)a}\int x^m X^{p+1}dx.$
IV	$+$	$-$	$\frac{x^{m-n+1}X^{p+1}}{n(p+1)b} - \frac{m-n+1}{n(p+1)b}\int x^{m-n}X^{p+1}dx.$
V	$+$	.....	$\frac{x^{m-n+1}X^{p+1}}{(pn+m+1)b} - \frac{(m-n+1)a}{(pn+m+1)b}\int x^{m-n}X^p dx$
VI	$-$	.....	$\frac{x^{m+1}X^{p+1}}{(m+1)a} - \frac{(pn+n+m+1)b}{(m+1)a}\int x^{m+n}X^p dx.$

In $x^m X^p dx$ put $X^p = u$ and $x^m dx = dv$; then $v = \frac{x^{m+1}}{m+1}$, and

$$\int udv = uv - \int vdu = \frac{x^{m+1}X^p}{m+1} - \frac{pnb}{m+1}\int x^{m+1}X^{p-1}x^{n-1}dx,$$

$$\text{or} \int x^m X^p dx = \frac{x^{m+1}X^p}{m+1} - \frac{pnb}{m+1}\int x^{m+1}X^{p-1}x^{n-1}dx \ldots\ldots. \quad (a),$$

which reduces to the *second form*.

$$\text{Again,} \int x^m X^p dx = \int x^m X^{p-1}(a + bx^n)dx$$

$$= a\int x^m X^{p-1}dx + b\int x^{m+n}X^{p-1}dx \ldots\ldots\ldots\ldots\ldots \quad (b).$$

Equate this to the preceding expression (a) for $\int x^m X^p dx$; then

$$\int x^m X^{p-1}\,dx = \frac{x^{m+1}X^p}{a(m+1)} - \frac{(pn+m+1)b}{(m+1)a}\int x^{m+n}X^{p-1}\,dx \;\ldots\ldots\; (c),$$

which, when p is writen for $p-1$ is the *sixth form.*

Multiply (b) by $\frac{pn}{m+1}$, and add to (a): then

$$\int x^m X^p\,dx = \frac{x^{m+1}X^p}{pn+m+1} + \frac{anp}{pn+m+1}\int x^m X^{p-1}\,dx \;\ldots\ldots\ldots\ldots (d),$$

which is the *first form.*

Multiply (d) by its denominator, and transpose: then

$$\int x^m X^{p-1}\,dx = -\frac{x^{m+1}X^p}{apn} + \frac{pn+m+1}{apn}\int x^m X^p\,dx \;\ldots\ldots\ldots\ldots\; (e),$$

which, when p is written for $p-1$ becomes the *third form.*

Multiply (a) by its denominator, and transpose: then

$$\int x^{m+n}X^{p-1}\,dx = \frac{x^{m+1}X^p}{pnb} - \frac{m+1}{pnb}\int x^{m-n}X^{p+1}\,dx \ldots\ldots\;\ldots\ldots (f),$$

in which putting m for $m+n$, and p for $p-1$, we get the *fourth form.*

Lastly, multiply (c) by its denominator, and transpose: then

$$\int x^{m+n}X^{p-1}\,dx = \frac{x^{m+1}X^p}{(pn+m+1)b} - \frac{(m-n+1)a}{(pn+m+1)b}\int x^{m-n}X^{p-1}\,dx .. (g),$$

which, when m is written for $m+n$, becomes the *fifth form.*

Scholium.

It should be remarked that the formulæ (II, VI) fail to apply to the case where $m+1=0$, as in that case our result becomes the difference of two infinite quantities, and therefore cannot be determined. The integration may, however, be otherwise effected, since it fulfils the *first criterion* of rationability. We shall take a particular case (which will be hereafter required) as an instance of the method: but all corresponding cases may be treated in the same way. The case we shall take is $\frac{dy}{y\sqrt{a^2-y^2}}$.

Put $a^2-y^2=z^2$: then $\frac{dy}{y\sqrt{a^2-y^2}} = -\frac{dz}{a^2-z^2}$, a rational fraction. The method of indeterminate co-efficients converts this into

$$\int\frac{dy}{y\sqrt{a^2-y^2}} = -\int\frac{dz}{a^2-z^2} = -\frac{1}{2a}\int\frac{dz}{a+z} - \frac{1}{2a}\int\frac{dz}{a-z},$$

$$= -\frac{1}{2a}\log_e(a+z) + \frac{1}{2a}\log_e(a-z)$$

$$= \frac{1}{2a}\log_e\frac{a-z}{a+z} = \frac{1}{2a}\log_e\frac{a-\sqrt{a^2-y^2}}{a+\sqrt{a^2-y^2}}.$$

EXAMPLES.

1. To integrate $\frac{x^m dx}{\sqrt{1-x^2}}$, or $x^m(1-x^2)^{-\frac{1}{2}}\,dx$.

In this case $X=(1-x^2)^{-\frac{1}{2}}$ is already in an elementary form, and hence we shall only have to successively reduce the index m to lower integers. For this purpose the form V is adapted, and the index will be reduced two units at each transformation. We thus have at once

$$\int \frac{x^m dx}{\sqrt{1-x^2}} = -\frac{x^{m-1}(1-x^2)^{\frac{1}{2}}}{m} + \frac{m-1}{m}\int \frac{x^{m-2}dx}{\sqrt{1-x^2}},$$

and writing $m-2$, $m-4$, *etc.* for m, we get the successive steps of the reduction

$$\int \frac{x^{m-2}dx}{\sqrt{1-x^2}} = -\frac{x^{m-3}(1-x^2)^{\frac{1}{2}}}{m-2} + \frac{m-3}{m-2}\int \frac{x^{m-4}dx}{\sqrt{1-x^2}},$$

$$\int \frac{x^{m-4}dx}{\sqrt{1-x^2}} = -\frac{x^{m-5}(1-x^2)^{\frac{1}{2}}}{m-4} + \frac{m-5}{m-4}\int \frac{x^{m-6}dx}{\sqrt{1-x^2}},$$

....

The integral must now be completed by the substitution of the successive values of its several parts. This will present two cases, when m is even, and when m is odd; since upon these the forms of the particular final integral will depend.

(1). Let $m = 2n$, n being an integer. Also let $P_{2n} = \int \frac{x^{2n}dx}{\sqrt{1-x^2}}$ and $Q_{2n-1} = x^{2n-1}(1-x^2)^{\frac{1}{2}}$. Then

$$P_{2n} = -\frac{Q_{2n-1}}{2n} + \frac{2n-1}{2n}. P_{2n-2},$$

$$P_{2n-2} = -\frac{Q_{2n-3}}{2n-2} + \frac{2n-3}{2n-2}. P_{2n-4},$$

$$P_{2n-4} = -\frac{Q_{2n-5}}{2n-4} + \frac{2n-5}{2n-4} P_{2n-6},$$

....

$$P_2 = -\frac{Q_1}{2} + \frac{1}{2}. P_0.$$

Hence, obviously, we have $\int \frac{x^{2n}dx}{\sqrt{1-x^2}}$ expressed by

$$P_n = -\left\{\frac{Q_{2n-1}}{2n} + \frac{(2n-1)Q_{2n-3}}{2n(2n-2)} + \frac{(2n-1)(2n-3)Q_{2n-5}}{2n(2n-2)(2n-4)} + \ldots\right\}$$
$$+ \frac{(2n-1)(2n-3)(2n-5)\ldots\ldots 3.1}{2n(2n-2)(2n-4)\ldots\ldots 4.2}. \sin^{-1}x + C.$$

The correction, on the hypothesis of $P_n = 0$, when $x = 0$, gives Q_{2n-1}, Q_{2n-3}, *etc.* all 0 since x^2, which is a common factor of them, is then 0. Also $\sin^{-1} 0 = 0$; and hence $P_{2n} = 0$, and $P_{3n} = C$; or $C = 0$.

Again, between the limits 0, 1, we have for $x = 1$, all the terms Q_{2n-1}, Q_{2n-3}, *etc.*, equal to 0, since $(1-x^2)^{\frac{1}{2}}$, which is a factor of them all, is 0. Also $\sin^{-1}1 = \frac{1}{2}\pi$: whence the definite integral between 0, 1, as the values of x, is

$$\int_0^1 \frac{x^{2n}dx}{(1-x^2)^{\frac{1}{2}}} = \frac{(2n-1)(2n-3)(2n-5)\ldots\ldots 3.1}{2n(2n-2)(2n-4)\ldots\ldots 4.2}.\frac{\varpi}{2}.$$

(2). Let $m = 2n+1$, n being any integer. Then as before,

$$P_{2n+1} = -\frac{Q_{2n}}{2n+1} + \frac{2n}{2n+1} P_{2n-1},$$

$$P_{2n-1} = -\frac{Q_{2n-2}}{2n-1} + \frac{2n-2}{2n-1}. P_{2n-3},$$

....

$$P_3 = -\frac{Q_2}{3} + \frac{2}{3}. P_1,$$

$$P_1 = -\int \frac{xdx}{\sqrt{1-x^2}} = -(1-x)^{\frac{1}{2}}.$$

Wherefore, as before,

$$P_{2n+1} = -\left\{\frac{Q_{2n}}{2n+1} + \frac{2n\,Q_{2n-2}}{(2n+1)(2n-1)} + \frac{2n\,(2n-2)\,Q_{2n-4}}{(2n+1)(2n-1)(2n-3)} + \dots\right\}$$
$$- \frac{2n\,(2n-2)\,(2n-4)\,\dots\dots\,4\,.\,2\,.\,(1-x^2)^{\frac{1}{2}}}{(2n+1)(2n-1)(2n-3)\,\dots\dots\,3\,.\,1} + C.$$

If the integral commence with $x = 0$: then all the terms within the brackets become 0, and we have

$$C = \frac{2n\,(2n-2)\,(2n-4)\,\dots\dots\,4\,.\,2}{(2n+1)\,(2n-1)\,(2n-3)\,\dots\dots\,3\,.\,1}.$$

Again, if $x = 1$, the entire series becomes 0, having $\sqrt{1-x^2}$ in all its terms as a common factor. Hence the definite integral, in this case, becomes [*].

$$\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{2n\,(2n-2)\,(2n-4)\dots\dots\,4\,.\,2}{(2n+1)\,(2n-1)\,(2n-3)\dots\dots\,3\,.\,1}.$$

2. It is required to integrate $(a^2 - x^2)^{\frac{n}{2}}dx$; or $x^0\,(a^2-x^2)^{\frac{n}{2}}dx$. In this $m = 0$, and the factor x^m is hence in its elementary state. Whence, we must so choose one formula as to diminish p. To this purpose (I) is adapted. Wherefore,

$$\int x^0\,(a^2-x^2)^{\frac{n}{2}} = \frac{x\,(a^2-x^2)^{\frac{n}{2}}}{n+1} + \frac{na^2}{n+1}\int x^0\,(a-x^2)^{\frac{n-2}{2}},$$

$$\int x^0\,(a^2-x^2)^{\frac{n-2}{2}} = \frac{x\,(a^2-x^2)^{\frac{n-2}{2}}}{n-1} + \frac{(n-2)\,a^2}{n-1}\int x^0\,(a^2-x^2)^{\frac{n-4}{2}}.$$

…………………………………………………………

and hence by a process similar to that employed in the general integral, this particular case may be deduced.

As n is odd, the last integral will be $\int (a^2-x^2)^{\frac{1}{2}}\,dx$

$$= \int \frac{a^2dx}{\sqrt{a^2-x^2}} - \int \frac{x^2dx}{\sqrt{a^2-x^2}} = \tfrac{1}{2}\left\{x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right\}.$$

Now this class of integrals, like the preceding, are usually required between the limits 0 and a of the independent variable x. In such case we have

$$\int_0^a (a^2-x^2)^{\frac{1}{2}}\,dx = \frac{a^2}{2}\,.\,\frac{\pi}{2} \qquad = \quad \frac{1}{2}\,.\,\frac{\varpi a^2}{2}\dagger,$$

$$\int_0^a (a^2-x^2)^{\frac{3}{2}}\,dx = \frac{3a^2}{4}\int_0^a (a^2-x^2)^{\frac{1}{2}} = \frac{1\,.\,3}{2\,.\,4}\,.\,\frac{\varpi a^4}{2},$$

$$\int_0^a (a^2-x^2)^{\frac{5}{2}}\,dx = \frac{5a^2}{6}\int_0^a (a^2-x^2)^{\frac{3}{2}} = \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\,.\,\frac{\varpi a^6}{2},$$

……………………………………………

$$\int_0^a (a^2-x^2)^{\frac{n}{2}}\,dx = \frac{1\,.\,3\,.\,5\,\dots\dots\,(n-2)^n}{2\,.\,4\,.\,6\,\dots\dots\,(n-1)\,(n+1)}\,.\,\frac{\varpi a^{n+1}}{2},$$

---

[*] From this result may be deduced the very remarkable expression first given by Dr. Wallis, for the length of the semicircle.

Let n be infinite: then $P_{2n} = P_{2n+1}$; or inserting the values of the definite integrals above found we have

$$\frac{\varpi}{2}\,.\,\frac{1\,.\,3\,.\,5\dots\dots}{2\,.\,4\,.\,6\dots\dots} = \frac{2\,.\,4\,.\,6\dots\dots}{3\,.\,5\,.\,7\dots\dots};\ \text{or}\ \frac{\varpi}{2} = \frac{2\,.\,2\,.\,4\,.\,4\,.\,6\,.\,6\dots\dots}{1\,.\,3\,.\,3\,.\,5\,.\,5\,.\,7\dots\dots}$$

[†] At *p.* 19 (*note*), of the present volume a distinction has been proposed, as a matter of notation, between the forms π and ϖ, to which it appears desirable again to recur.

The form π has been restricted to signify the arc of a semicircle described with the scale-

3. To integrate $\frac{dx}{x^m \sqrt{1-x^2}}$. Here (VI) applies: for the differential may be written $x^{-m}(1-x^2)^{-\frac{1}{2}}\,dx$; where $(1-x^2)^{-\frac{1}{2}}$ is already in an elementary state, and we are required to increase the index of x^{-m}. The formula for reduction is, hence,

$$\int \frac{dx}{x^m \sqrt{1-x^2}} = -\frac{(1-x^2)^{\frac{1}{2}}}{(m-1)\,x^{m-1}} + \frac{m-2}{m-1}\int \frac{dx}{x^{m-2}\,(1-x^2)^{\frac{1}{2}}}$$

As is always the case in these integrals, the terminal results will be of different forms as m is odd or even. We shall therefore take the two cases separately. Instead, however, of proceeding in the manner of the first example to obtain the general integral, we shall proceed in descending steps for the value of $-m$.

(1). Let m be *odd*. Then giving it the successive values 1, 3, 5, *etc.*

$$\int \frac{dx}{x\sqrt{1-x^2}} = \mathrm{C} - \log_e \frac{1+\sqrt{1-x^2}}{x},$$

$$\int \frac{dx}{x^3\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{2x^2} + \frac{1}{2}\int \frac{dx}{x\sqrt{1-x^2}},$$

$$\int \frac{dx}{x^5\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{4x^4} + \frac{3}{4}\int \frac{dx}{x^3\sqrt{1-x^2}},$$

$$\int \frac{dx}{x^7\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{6x^6} + \frac{5}{6}\int \frac{dx}{x^5\sqrt{1-x^2}},$$

……………………………………………

$$\int \frac{dx}{x^m\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{(m-1)x^{m-1}} + \frac{m-2}{m-1}\int \frac{dx}{x^{m-2}\sqrt{1-x^2}},$$

or, again, by substitution of each of these integrals in the succeeding one, they become

---

radius unity, and viewed as the *measure of two right angles*. Now (*vol. i. p.* 421), the angle, the arc subtended, and the sector included have all the same ratio to the respective four right-angles, the entire circumference, and the entire area of the circle; and hence, in a mathematical point of view, any one may be considered as the measure of either of the others. Whenever π is used in this work it has been to denote the semicircle to radius unity, as the *measure of two right angles*; and ϖ has been used to designate the *number of diameters contained in the entire circumference*. There can be no doubt that, viewed as *numerical quantities*, these are identical: but that used for specifically different purposes, they ought to be marked by distinct characters. In the place referred to in the present volume, it is obvious that they require to be marked differently: though in most other cases, perhaps, it may be contended with some plausibility, that they *may be* represented by the same symbol. If, however, π be considered (as most mathematicians do consider it) to represent two right angles, then there cannot, it appears to me, be any propriety in connecting it with the actual number 3·14159… Moreover, into the differential and integral calculus, the idea of *angles*, viewed as such, does not enter. Whenever the sine or cosine enters into the function under consideration, it is expressly as the sine or cosine of an *arc*. This is to be always kept in mind: and it is especially necessary to view every trigonometrical function that enters into this part of the science as the function of an arc, and not of an angle, *sui generis*. Thus θ, α, or any symbol of the same kind, is to be considered as the *length* of the arc, which subtends a certain corresponding angle. As such, ϖ is of precisely the same value as π: for the semicircle to radius unity is equal to the entire circle to diameter unity. In the integrals, between limits especially, the arc π frequently appears; and to remove ambiguity from the signification of the results, I have invariably substituted the symbol ϖ instead of π as used by other authors. Whether in this or the usual notation, it never means other than the *number* 3·14159……

$$\int \frac{dx}{x\sqrt{1-x^2}} = \mathrm{C} - \log_e \frac{1+\sqrt{1-x^2}}{x},$$

$$\int \frac{dx}{x^3\sqrt{1-x^2}} = \mathrm{C} - \frac{\sqrt{1-x^2}}{2x^2} - \frac{1}{2}\log_e \frac{1+\sqrt{1-x^2}}{x},$$

$$\int \frac{dx}{x^5\sqrt{1-x^2}} = \mathrm{C} - \left\{\frac{1}{4x^4} + \frac{1\,.\,3}{2\,.\,4\,x^2}\right\}\sqrt{1-x^2} - \frac{1\,.\,3}{2\,.\,4}\log_e \frac{1+\sqrt{1-x^2}}{x}.$$

$$\frac{dx}{x^7\sqrt{1-x^2}} = \mathrm{C} - \left\{\frac{1}{6x^6} + \frac{1\,.\,5}{4\,.\,6\,.\,x^4} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6\,.\,x^2}\right\}\sqrt{1-x^2} - \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\log_e \frac{1+\sqrt{1-x^2}}{x}.$$

....

$$\frac{dx}{x^m\sqrt{1-x^2}} = -\left\{\begin{array}{l}\dfrac{1}{(m-1)\,x^{m-1}} + \dfrac{m-2}{(m-3)\,(m-1)\,x^{m-3}} + \dfrac{(m-4)\,(m-2)}{(m-5)\,(m-3)\,(m-1)\,x^{m-5}} \\ + \,\ldots\ldots\, \dfrac{1\,.\,3\,.\,5\ldots\ldots(m-2)}{2\,.\,4\,.\,6\ldots\ldots(m-1)x^2}\end{array}\right\}\sqrt{1-x^2}$$

$$-\frac{1\,.\,3\,.\,5\ldots\ldots(m-2)}{2\,.\,4\,.\,6\ldots\ldots(m-1)}\log_e \frac{1+\sqrt{1-x^2}}{x} + \mathrm{C}.$$

(2). Let m be *even*. Then giving it the values 0, 2, 4, ... we have

$$\int \frac{dx}{\sqrt{1-x^2}} = \mathrm{C} + \sin^{-1}x,$$

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = \mathrm{C} - \frac{\sqrt{1-x^2}}{x},$$

$$\int \frac{dx}{x^4\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{3x^3} + \frac{2}{3}\int \frac{dx}{x^2\sqrt{1-x^2}},$$

$$\int \frac{dx}{x^6\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{5x^5} + \frac{4}{5}\int \frac{dx}{x^4\sqrt{1-x^2}},$$

....

$$\int \frac{dx}{x^m\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{(m-1)\,x^{m-1}} + \frac{m-2}{m-1}\int \frac{dx}{x^{m-2}\sqrt{1-x^2}}.$$

Hence substituting as before, we get

$$\int \frac{dx}{\sqrt{1-x^2}} = \mathrm{C} + \sin^{-1}x,$$

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = \mathrm{C} - \frac{\sqrt{1-x^2}}{x},$$

$$\int \frac{dx}{x^4\sqrt{1-x^2}} = \mathrm{C} - \left\{\frac{1}{3x^3} + \frac{2}{1\,.\,3\,x}\right\}\sqrt{1-x^2},$$

$$\int \frac{dx}{x^6\sqrt{1-x^2}} = \mathrm{C} - \left\{\frac{1}{5x^5} + \frac{4}{3\,.\,5x^3} + \frac{2\,.\,4}{1\,.\,3\,.\,5x}\right\}\sqrt{1-x^2},$$

....

$$\frac{dx}{x^m\sqrt{1-x^2}} = \mathrm{C} - \left\{\begin{array}{l}\dfrac{1}{(m-1)x^{m-1}} + \dfrac{m-2}{(m-3)\,(m-1)\,x^{m-3}} + \dfrac{(m-4)\,(m-2)}{(m-5)(m-3)(m-1)x^{m-5}} \\ + \,\ldots\ldots\, + \dfrac{2\,.\,4\,.\,6\ldots\ldots(m-2)}{1\,.\,3\,.\,5\ldots\ldots(m-1)x}.\end{array}\right\}\sqrt{1-x^2}$$

If the integral commence when $x = 1$, and be estimated during the diminution of x, we can find the constant C. Then when $x = 1$, all the terms within the brackets are finite, but their sum multiplied by 0. That term of the integral therefore vanishes. Also $\log_e \dfrac{1+\sqrt{1-x^2}}{x}$ becomes $\log_e 1$ or 0. Hence also $C = 0$.

Again, if $x = 0$, then all the terms within the brackets are infinite, and the series multiplied by unity: likewise $\log_e \dfrac{1+\sqrt{1-x^2}}{x}$ becomes $\log_e \dfrac{2}{0} =$ infinity; and C already found to be 0. Whence the value of the integral in this case is infinite. Whence $\int_0^1 \dfrac{dx}{x^m\sqrt{1-x^2}} = -\dfrac{1}{0}$.

4. Integrate $\dfrac{x^m dx}{\sqrt{x^2-1}}$; that is, find $\int x^m (x^2-1)^{-\frac{1}{2}} dx$,

This falls under form (V), and we have to reduce the index of x without the vinculum. The process will be very similar to that employed in *Ex.* i. and is left for the student's exercise. We have, however, to remark, that the differential given in *Ex.* 3. is reducible to this form. For, put $x = z^{-1}$: then $x^{-m} = z^m$, $dx = -z^{-2}dz$, and $\sqrt{1-x^2} = \dfrac{\sqrt{z^2-1}}{z}$: whence $\dfrac{dx}{x^m\sqrt{1-x^2}} = -\dfrac{z^{m-1}\,dz}{\sqrt{z^2-1}}$, as affirmed.

5. Required the integral of $\dfrac{x^m dx}{\sqrt{2ax-x^2}}$ between the limits 0, $2a$. This may be written $\dfrac{x^{m-\frac{1}{2}}dx}{\sqrt{2a-x}}$; and we shall have to diminish the index without the vinculum. To this purpose the fifth form is adapted; and we have at once

$$\int \frac{x^{m-\frac{1}{2}}dx}{\sqrt{2a-x}} = -\frac{x^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}}}{m} + \frac{(2m-1)\,a}{m}\int \frac{x^{m-\frac{3}{2}}\,dx}{\sqrt{2a-x}};$$

or, multiplying and dividing by $x^{\frac{1}{2}}$ it is changed to

$$\int \frac{x^{m-1}\,dx}{\sqrt{2ax-x^2}} = -\frac{x^{m-1}\,(2ax-x^2)^{\frac{1}{2}}}{m} + \frac{(2m-1)\,a}{m}\int \frac{x^{m-1}\,dx}{\sqrt{2ax-x^2}};$$

under either of which forms the integration may be continued.

If we put $P_m = \int \dfrac{x^m dx}{\sqrt{2ax-x^2}}$ and $Q_{m-1} = x^{m-1}(2ax-x^2)^{\frac{1}{2}}$: then we shall have, as in the first example,

$$P_m = -\frac{Q_{m-1}}{m} + \frac{(2m-1)\,a}{m}.P_{m-1}$$

$$P_{-1} = -\frac{Q_{m-2}}{m-1} + \frac{(2m-3)\,a}{m-1}.P_{m-2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P_2 = -\frac{Q_1}{2} + \frac{a}{2}.P_1$$

$$P_1 = -\frac{Q_0}{1} + \frac{a}{1}.P_0.$$

Hence, eliminating P_{m-1}, P_{m-2}, *etc*, we get

$$P_n = -\left\{\begin{aligned} &\frac{Q_{m-1}}{m} + \frac{(2m-1)\,a}{m-1}.\frac{Q_{m-2}}{m-2} + \frac{(2m-1)\,(2m-3)\,a^2}{m\,(m-1)}.\frac{Q_{m-3}}{m-2} \\ &\qquad + \ldots \frac{(2m-1)\,(2m-3)\ldots.5.3a^{m-1}}{m\,(m-1)\ldots\ldots4.3}.\frac{Q_1}{2} \end{aligned}\right\}$$

$$\frac{(2m-1)(2m-3)(2m-5)\ldots.3.1.a^m}{m(m-1)(m-2)\ldots\ldots..2.1}\int\frac{dx}{\sqrt{2ax-x^2}}.$$

The last integral $\int\frac{dx}{\sqrt{2ax-x^2}} = \text{vers}^{-1}\frac{x}{a} + \text{C}$.

Between the limits 0, $2a$, we see that all the terms in the brackets involving powers of x, and the factor $(2ax - x^2)^{\frac{1}{2}}$, will vanish at each limit: and at these places also, $\text{vers}^{-1}\frac{0}{a} = 0$, $\text{vers}^{-1}\frac{2a}{a} = \text{vers}^{-1} 2 = \pi$. Hence, finally *,

$$\int_0^{2a}\frac{x^m dx}{\sqrt{2ax-x^2}} = \frac{1.3.5\ldots.(2m-3)(2m-1)}{1.2.3\ldots.(m-1)^m}.\pi.a^m.$$

6. $\int\frac{dx}{x^m\sqrt{x^2+1}} = -\frac{1}{m-1}.\frac{(x^2+1)^{\frac{1}{2}}}{x^{m-1}} - \frac{m-2}{m-1}\int\frac{dx}{x^{m-2}\sqrt{x^2+1}}$.

The final integral is $\int\frac{dx}{x\sqrt{x^2+1}} = \log_e\frac{x}{1+\sqrt{x^2+1}}$, when m is *odd;*

and .... $\int\frac{dx}{x^2\sqrt{x^2+1}} = -\frac{(x^2+1)^{\frac{1}{2}}}{x}$, when m is *even.*

7. $\int\frac{dx}{x^m\sqrt{x^2-1}} = +\frac{1}{m-1}.\frac{(x^2-1)^{\frac{1}{2}}}{x^{m-1}} + \frac{m-2}{m-1}\int\frac{dx}{x^{m-2}\sqrt{x^2-1}}$.

The final integral is $\int\frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x$ when m is *odd;* and

.... $\int\frac{dx}{x^2\sqrt{x^2-1}} - = \frac{(x^2-1)^{\frac{1}{2}}}{x}$ when m is *even.*

8. Given $\frac{dx}{x^m\sqrt{2ax-x^2}}$ for integration.

This may be written $\frac{dx}{x^{m+\frac{1}{2}}\sqrt{2a-x}}$; and the process of form (V) may be applied to it. A previous transformation will, however, put it in a state to render it adapted to form (IV); which will require less practical labour in the successive reductions. Put $x = \frac{1}{z}$, and $b = \frac{1}{2a}$: then, as suggested in *ex.* 4, we shall have

$$\int\frac{dx}{x^m\sqrt{2ax-x^2}} = -\int\frac{z^{m-1}dz}{\sqrt{2az-1}} = -\sqrt{b}\int\frac{z^{m-1}dz}{\sqrt{z-b}}.$$

We shall now find the formula of reduction to be by (IV)

$$\int\frac{z^{m-1}dz}{\sqrt{z-b}} = \frac{2z^{m-1}(z-b)^{\frac{1}{2}}}{2m-1} + \frac{(2m-2)b}{2m-1}\int\frac{z^{m-2}dz}{\sqrt{z-b}};$$

---

* In comparing examples (1) and (5) with example (3), the relative advantages of the two courses of proceeding will be obvious, and may be stated thus.

For the purpose of obtaining general formulæ, the method (1, 5) is more brief and more elegant than the other (3); but as it is always desirable to have the expressions computed for the earlier integer values of m, the process of (3) determines these values, *pari passu*, its greater prolixity is compensated to the young analyst by the information thus superadded to that directly sought by the proposed problem.

and the final integral will be $\int \frac{dz}{\sqrt{z-b}} = 2\sqrt{z-b}$.

9. If $\int \frac{x^m\,dx}{\sqrt{a+bx+cx^2}}$, and $\int \frac{dx}{x^m\sqrt{a+bx+cx^2}}$ were proposed, we may proceed as follows:

In the former put $x + \frac{b}{2c} = z$, $\frac{b}{2c} = \alpha$, and $\frac{a}{c} - \frac{b^2}{4c^2}$ or $\frac{4ac - b^2}{4c^2} = \pm \beta^2$: then the expression is changed to

$$\int \frac{x^m\,dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}} \int \frac{(z-\alpha)^m\,dz}{\sqrt{z^2 \pm \beta^2}}.$$

Expanding the numerator by the binomial theorem, we shall have a series of terms of the form $\frac{k.\alpha^n z^{m-n}}{\sqrt{z^2 \pm \beta^2}}$, each of which is of a known form of integral.

The second form may be reduced to the first by writing $\frac{1}{y}$ for x; thus converting it into

$$\int \frac{dx}{x^m\sqrt{a+bx+cx^2}} = -\int \frac{y^{m-1}\,dy}{\sqrt{ay^2+by+c}}.$$

10. To integrate $z^2(c^2-z^2)^{\frac{3}{2}}\,dz$. Perform the first two steps of the partition by form (I), and the third by (V)

$$\int z^2(c^2-z^2)^{\frac{3}{2}}\,dz = \frac{(c^2-z^2)^{\frac{3}{2}}z^3}{6} + \frac{c^2}{2}\int z^2(c^2-z^2)^{\frac{1}{2}}\,dz,$$

$$\int z^2(c^2-z^2)^{\frac{1}{2}}\,dz = \frac{(c^2-z^2)^{\frac{1}{2}}z^3}{4} + \frac{c^2}{4}\int z^2(c^2-z^2)^{-\frac{1}{2}}\,dz,$$

$$\int z^2(c^2-z^2)^{-\frac{1}{2}}\,dz = -\frac{(c^2-z^2)^{\frac{1}{2}}z}{2} + \frac{c^2}{2}\int (c^2-z^2)^{-\frac{1}{2}}\,dz,$$

$$\int (c^2-z^2)^{\frac{1}{2}}\,dz = \sin^{-1}\frac{z}{c}, \text{ (elementary integral).}$$

Hence, collecting the terms, we have

$$\int z^2(c^2-z^2)^{\frac{3}{2}}\,dz = -\frac{z(c^2-z^2)^{\frac{1}{2}}}{48}\{8z^4 - 14c^2z^2 + 3c^4\} + \frac{c^6}{16}\sin^{-1}\frac{z}{c} + C.$$

11. $\int x^2\sqrt{a+bx}.dx = \frac{2(8a^2 + 3abx + 15b^2x^2)}{105b^3}.(a+bx)^{\frac{3}{2}}.$

12. $\int \frac{\sqrt{a+bx}.dx}{x^2} = -\frac{(a+bx)^{\frac{1}{2}}}{x} + \frac{b}{2\sqrt{a}} - \log_e \frac{(a+bx)^{\frac{1}{2}} - a^{\frac{1}{2}}}{(a+bx)^{\frac{1}{2}} + a^{\frac{1}{2}}}.$

13. $\int \frac{x^3\,dx}{\sqrt{a+bx}} = -\frac{2(16a^3 - 8a^2bx + 6ab^2x^2 - 5b^3x^3)}{35b^4}.(a+bx)^{\frac{1}{2}}$

14. $\int \frac{dx}{x\sqrt{bx+a}} = \frac{1}{\sqrt{a}}.\log_e \frac{\sqrt{bx+a} - \sqrt{a}}{\sqrt{bx+a} + \sqrt{a}}.$

15. $\int \frac{dx}{x\sqrt{bx-a}} = \frac{2}{\sqrt{a}}.\tan^{-1}\left(\frac{bx-a}{a}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{a}}\sin^{-1}\left(\frac{bx-a}{bx}\right)^{\frac{1}{2}}.$

16. $\int \frac{x^2\,dx}{(2+3x)^{\frac{1}{3}}} = \frac{4 - 4x + 5x^2}{40}(2+3x)^{\frac{2}{3}}.$

17. $$\int \frac{x^3\,dx}{(1+x^2)^{\frac{3}{2}}} = \frac{x^2+2}{\sqrt{x^2+1}}.$$

18. $$\int \frac{x^4\,dx}{(1+x^2)^{\frac{3}{2}}} = \frac{(x^2+3)\,x}{2\sqrt{1+x^2}} - \frac{3}{2}\log_e\left(x+\sqrt{1+x^2}\right).$$

19. $$\int \frac{dx}{(2ax+x^2)^{\frac{3}{2}}} = -\frac{x+a}{a^2\sqrt{2ax+x^2}}.$$

20. $$\int \frac{dx}{(1+x+x^2)^{\frac{3}{2}}} = \frac{2\,(2x+1)}{3\sqrt{1+x+x^2}}.$$

21. $$\int \frac{dx}{(1+x+x^2)^{\frac{5}{2}}} = \frac{2}{9}\left\{\frac{1}{1+x+x^2}+\frac{8}{3}\right\}\frac{(2x+1)}{\sqrt{1+x+x^2}}.$$

22. $$\int \frac{x\sqrt{x}.dx}{1+x^2} = 2\sqrt{x} + \frac{1}{\sqrt{2}}.\log_e\frac{x+\sqrt{2x}+1}{\sqrt{1+x^2}} - \tan^{-1}\frac{\sqrt{2x}}{1-x}.$$

23. $$\int \frac{x^2\,dx}{\sqrt{a+bx+cx^2}} = \frac{2cx-3b}{4c^2}\sqrt{a+bx+cx^2}$$
$$+\frac{4ac-3b^2}{8c^2\sqrt{c}}\log_e\frac{b+2cx+2\sqrt{c\,(a+bx+cx^2)}}{2c}$$

PROP. II.

To integrate $x^m\,(a+bx+cx^2)^p\,dx$ without transformation.

Put $a+bx+cx^2 = \mathrm{X}$: then integrating the binomial $x^m\,\mathrm{X}^p\,dx$ by parts as in the preceding problem, we get

$$\int x^m\,\mathrm{X}^p\,dx = \frac{x^{m+1}\,\mathrm{X}^p}{m+1} - \frac{p}{m+1}\int x^{m+1}\,\mathrm{X}^{p-1}\,(b+2cx)\,dx.$$
$$= \frac{x^{m+1}\,\mathrm{X}^p}{m+1} - \frac{pb}{m+1}\int x^{m+1}\,\mathrm{X}^{p-1}\,dx - \frac{2pc}{m+1}\int x^{m+2}\,\mathrm{X}^{p-1}\,dx.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(a).$$

Again, since $(a+bx+cx^2)^p = a\mathrm{X}^{p-1} + bx\,\mathrm{X}^{p-1} + cx^2\,\mathrm{X}^{p-1}$, we get

$$\int x^m\,\mathrm{X}^p\,dx = a\int x^m\,\mathrm{X}^{p-1}\,dx + b\int x^{m+1}\,\mathrm{X}^{p-1}\,dx + c\int x^{m+2}\,\mathrm{X}^{p-1}\,dx$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(b)$$

Eliminate $\int x^{m+2}\,\mathrm{X}^{p-1}\,dx$ from (a) and (b), and there results

$$\int x^m\,\mathrm{X}^p\,dx = \frac{x^{m+1}\,\mathrm{X}^p}{2p+m+1} + \frac{2pa}{2p+m+1}\int x^m\,\mathrm{X}^{p-1}\,dx$$
$$+\frac{pb}{2p+m+1}\int x^{m+1}\,\mathrm{X}^{p-1}dx \;\ldots\ldots\ldots\ldots\; (\mathrm{I.})$$

To obtain another form, eliminate $\int x^m\,\mathrm{X}^p\,dx$ from (a), (b); then

$$(m+1)\,a\int x^m\,\mathrm{X}^{p-1}\,dx + (p+m+1)\,b\int x^{m+1}\,\mathrm{X}^{p-1}\,dx$$
$$+(2p+m+1)\,c\int x^{m+2}\,\mathrm{X}^{p-1}\,dx = x^{m+1}\,\mathrm{X}^p \;\ldots\ldots\; (c).$$

In which, writing $m-2$ for m, and $n+1$ for n, we shall have

$$\int x^m \mathrm{X}^p\,dx = \frac{x^{m-1}\,\mathrm{X}^{p+1}}{(2p+m+1)\,c} - \frac{(m+n)\,b}{(2p+m+1)\,c}\int x^{m-1}\,\mathrm{X}^p\,dx$$

$$-\frac{(m-1)\,a}{(2p+m+1)\,c}\int x^{m-2}\,\mathrm{X}^p\,dx \ \ldots\ldots\ldots\ldots \ (\mathrm{II.})$$

In (c) change m to $-m$, and n to $n+1$, then a formula is obtained, adapted to x^{-m}, in the expression given for integration :

$$\int \frac{\mathrm{X}^p\,dx}{x^m} = -\frac{\mathrm{X}^{p+1}}{(m-1)\,ax^{m-1}} - \frac{(m-p-2)\,b}{(m-1)\,a}\int\frac{\mathrm{X}^p\,dx}{x^{m-1}}$$

$$-\frac{(m-2p-3)\,c}{(m-1)\,a}\int\frac{\mathrm{X}^p\,dx}{x^{m-1}} \ \ldots\ldots\ldots\ldots \ (\mathrm{III.})$$

EXAMPLES.

1. $\displaystyle\int\frac{dx}{a+bx+cx^2} = \frac{2}{\sqrt{4ac-b^2}}.\tan^{-1}\frac{b+2cx}{\sqrt{4ac-b^2}}$, when $4ac-b^2=+$.

$$= \frac{1}{\sqrt{b^2-4ac}}\log_e\frac{b+2cx-\sqrt{b^2-4ac}}{b+2cx+\sqrt{b^2-4ac}},$$

when $4ac-b^2=-$.

2. $\displaystyle\int\frac{x\,dx}{a+bx+cx^2} = \frac{1}{2c}\log_e(a+bx+cx^2) - \frac{b}{2c}\int\frac{dx}{a+bx+cx^2}.$

3. $\displaystyle\int\frac{dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}}\log_e\{2cx+b+2\sqrt{c\,(a+bx+cx^2)}\}.$

4. $\displaystyle\int\frac{dx}{\sqrt{a+bx-cx^2}} = \frac{1}{\sqrt{c}}\sin^{-1}\frac{2cx-b}{\sqrt{b^2+4ac}}.$

5. $\displaystyle\int\sqrt{a+bx\pm cx^2}.\,dx = \frac{(2cx+b)\sqrt{\mathrm{X}}}{4c} + \frac{4ac-b^2}{8c}\int\frac{dx}{\sqrt{\mathrm{X}}},$

where $\mathrm{X} = a+bx\pm cx^2$, and the same in 6, 7, 8.

6. $\displaystyle\int\frac{xdx}{\sqrt{a+bx\pm cx^2}} = \frac{\sqrt{\mathrm{X}}}{c} - \frac{b}{2c}\int\frac{dx}{\sqrt{\mathrm{X}}}.$

7. $\displaystyle\int\sqrt{\mathrm{X}}\,.\,xdx = \frac{\mathrm{X}^{\frac{3}{2}}}{3c} - \frac{b}{2c}\int\sqrt{\mathrm{X}}\,.\,dx.$

8. $\displaystyle\int\frac{dx}{x\sqrt{\mathrm{X}}} = \frac{1}{\sqrt{a}}\log_e\frac{2a+bx-2\sqrt{a\mathrm{X}}}{x}.$

9. $\displaystyle\int\frac{dx}{x\sqrt{-a+bx+cx^2}} = \frac{1}{\sqrt{a}}\sin^{-1}\frac{bx-2a}{x\sqrt{b^2+4ac}}.$

10. $\displaystyle\int\frac{dx}{(a+bx+cx^2)^{\frac{3}{2}}} = -\frac{2\,(2cx+b)}{(b^2-4ac)\sqrt{a+bx+cx^2}}.$

11. $\displaystyle\int\frac{x^2dx}{\sqrt{a+bx+cx^2}} = \frac{2cx-3b}{4c^2}.\sqrt{a+bx+cx^2}$

$$+\frac{4ac-3b^2}{8c^2\sqrt{c}}\log_e\frac{2cx+b+2\sqrt{c\,(a+bx+cx^2)}}{2c}.$$

PROP. III.

To integrate $X(\log_e x)^n dx$ where X is a function of x.

PUT $Xdx = dX_1$, $\frac{X_1 dx}{x} = dX_2$, $\frac{X_2 dx}{x} = dX_3$, *etc.*

Then in the general formula $\int udv = uv - \int vdu$, put $dv = Xdx$, and $u = (\log_e x)^n$. Then we have

$$\int Xdx . (\log_e x)^n = (\log_e x)^n \int Xdx - \int \left(\int Xdx\right) . n(\log_e x)^{n-1} . \frac{dx}{x}$$

$$= X_1 (\log_e x)^n - n\int (\log_e x)^{n-1} \frac{X_1 dx}{x},$$

which is the general formula of reduction.

By applying the same system of reduction to the successive integrals, we obtain the general result

$$\int Xdx (\log_e x)^n = X_1 (\log_e x)^n - nX_2 (\log_e x)^{n-1} + n(n-1) X_3 (\log_e x)^{n-2} - \dots.$$

which, if n be a positive integer, terminates at $n(n-1) \dots 2 . 1 . X_n (\log_e x)^0$.

The applicability of the method depends entirely on the power we possess of integrating the several functions which give $X_1, X_2, \dots . X_n$. This, if X be either a rational function of x, or a binomial of the form discussed in *chapter* IV., or in the preceding problem, or the exponential and circular functions to be treated presently, can always be done; but in most other cases, the problem in the present state of analysis becomes impossible.

EXAMPLES.

1. Integrate $(\log_e x)^n x^m dx$.

In this case $X = x^m$; and hence, performing the integration for $X_1, X_2, \dots .$, we have,

$$\int Xdx = \int x^m dx = \frac{x^{m+1}}{m+1} = X_1,$$

$$\int \frac{X_1 dx}{x} = \int \frac{x^m dx}{m+1} = \frac{x^{m+1}}{(m+1)^2} = X_2,$$

$$\int \frac{X_2 dx}{x} = \int \frac{x^m dx}{(m+1)^2} = \frac{x^{m+1}}{(m+1)^3} = X_3,$$

. . . .

Inserting these values of $X_1, X_2, \dots$ we have

$$\int (\log_e x)^n x^m dx = \frac{x^{m+1}}{m+1}\left\{(\log_e x)^n - \frac{n}{m+1}(\log_e x)^{n-1} + \frac{n(n-1)}{(m+1)^2}(\log_e x)^{n-2} - \dots\right\}$$

$$+ (-1)^n . \frac{n(n-1)(n-2) \dots 2 . 1}{(m+1)^n} x^{m+1}.$$

For the limits 0, 1, we see that all the terms vanish when $x = 0$, and all but the last when $x = 1$; hence

$$\int_0^1 (\log_e x)^n . x^m dx = (-1)^n \frac{1.2.3 \dots n}{(m+1)^n}.$$

2. Integrate $(\log_e x)^2 x^3 dx$, a particular case of the last.

Here, $m = 3$, and $n = 2$, and making the substitutions, we have

$$\int (\log_e x)^2\, x^3\, dx = \frac{x^4}{4}\left\{(\log_e x)^2 - \frac{1}{2}\log_e x + \frac{1}{4}\right\} + \mathrm{C}.$$

Between the limits 0, 1, the same integral is $\frac{1.2}{4^4} = \frac{1}{128}$.

3. $\int (\log_e x)^2\, x^2\, dx = \frac{x^3}{3}\left\{(\log_e x)^2 - \frac{2}{3}\log_e x + \frac{1.2}{9}\right\}$.

PROP. IV.

To integrate $\frac{\mathrm{X}dx}{(\log_e x)^n}$, *X being a function of x and n an integer.*

If we take $d(\mathrm{X}x) = \mathrm{Y}_1 dx$, $d(\mathrm{Y}_1 x) = \mathrm{Y}_2 dx$, *etc.*, and proceed as in the last proposition, we shall obtain

$$\int \frac{\mathrm{X}dx}{(\log_e x)^n} = -\frac{\mathrm{X}x}{(n-1)(\log_e x)^{n-1}} - \frac{\mathrm{Y}_1 x}{(n-1)(n-2)(\log_e x)^{n-2}} - \frac{\mathrm{Y}_2 x}{(n-1)(n-2)(n-3)(\log_e x)^{n-3}} - -$$

the last integral of the series being $\int \frac{\mathrm{Y}_{n-1}dx}{\log x}$; which, however, cannot be found in finite terms. It may still be expanded in an infinite series capable of integration in single terms, with great ease, when Y_n is a rational function of x.

If Y_{n-1} be composed of functions of x, which may generally be represented by x, put $\log_e x = z$, and we have $x = e^z$: whence

$$\int \frac{x^p dx}{\log_e x} = \int \frac{e^{(p+1)z}\,dz}{z} = \int \left\{1 + \frac{p+1}{1}z + \frac{(p+1)^2 z^2}{1.2} + \dots\right\}\frac{dz}{z}.$$

$$= \log_e z + \frac{(p+1)z}{1} + \frac{(p+1)^2 z^2}{1.2^2} + \frac{(p+1)^3 z^3}{1.2.3^2} + \dots\dots\dots$$

$$= \log_e^2 x + \frac{(p+1)\log_e x}{1^2} + \frac{(p+1)^2(\log_e x)^2}{1.2^2} + \frac{(p+1)^3(\log_e x)^3}{1.2.3^2} + \dots$$

EXAMPLES.

1. $\int \frac{dx}{\log_e x} = \log_e^2 x + \frac{\log_e x}{1^2} + \frac{(\log_e x)^2}{1.2^2} + \frac{(\log_e x)^3}{1.2.3^2} + \dots\dots$

2. $\int \frac{x^m\,dx}{(\log_e x)^2} = -\frac{x^{m+1}}{\log_e x} + (m+1)\int \frac{x^m\,dx}{\log_e x}$.

3. $\int \frac{dx}{x(\log_e x)^n} = -\frac{1}{(n-1)(\log_e x)^{n-1}}$.

4. $\int \frac{x^m\,dx}{(\log_e x)^n} = \frac{(m+1)^{n-1}}{1.2.3\dots(n-1)}\int \frac{x^m dx}{\log_e x}$

$$-\frac{x^{m+1}}{n-1}\left\{\frac{1}{(\log_e x)^{n-1}} + \frac{m+1}{n-2}\cdot\frac{1}{(\log_e x)^{n-2}} + \frac{(m+1)^2}{(n-2)(n-3)}\;\frac{1}{(\log_e x)^{n-3}}\right\} + \dots$$

5. $\int \frac{x\log_e x dx}{\sqrt{a^2 + x^2}} = \sqrt{a^2 + x^2}\,.\log_e x - \int \sqrt{a^2 + x^2}\,.\frac{dx}{x}$

$$= \log_e\left(\frac{x}{e}\right)\sqrt{a^2 + x^2} - a\log_e\left(\frac{x}{a + \sqrt{a^2 + x^2}}\right).$$

6. $\int \log_e\left(\frac{x^2 - b^2}{b^2}\right) x\sqrt{a^2 + x^2}\,dx = \frac{1}{3}(a^2 + x^2)^{\frac{3}{2}}\log_e\left(\frac{x^2 - b^2}{b^2}\right)$

$$-\frac{2}{3}(a^2 + b^2)(a^2 + x^2)^{\frac{1}{2}} - \frac{2}{3}(a^2 + b^2)^{\frac{3}{2}}\log_e \frac{(x^2 - b^2)^{\frac{1}{2}}}{(a^2 + b^2)^{\frac{1}{2}} + (a^2 + x^2)^{\frac{1}{2}}}$$

PROP. V.

To integrate $a^x x^m dx$, where m is any integer.

PUT $u = x^m$, and $dv = a^x dx$; and hence $v = \frac{a^x}{\log_e a} = \mathrm{M}a^x$.

(1). Then the formula for partition becomes (m being *positive*),

$$\int a^x x^m dx = \frac{x^m a^x}{\log_e a} - \frac{m}{\log_e a}\int x^{m-1} a^x dx.$$

If we reduce by means of this, we get

$$\int a^x x^m dx = \mathrm{M}a^x \{x^m - \mathrm{M}mx^{m-1} + m(m-1)\mathrm{M}^2 x^{m-2} - \ldots (-1)^m . 1.2 \ldots m . \mathrm{M}^{m-1}\}.$$

(2). Let m be negative, put $u = a^x$, and $dv = x^{-m}dx$; thence $v = \frac{-1}{(m-1)x^{m-1}}$, the formula of partition then becomes in this case

$$\int \frac{a^x dx}{x^m} = -\frac{a^x}{(m-1)x^{m-1}}\left\{1 + \frac{1}{m-2}.\frac{x}{\mathrm{M}} + \frac{1}{(m-2)(m-3}.\frac{x^2}{\mathrm{M}^2} + \ldots\right\}$$

$$+ \frac{1}{1.2.3\ldots(m-1)}.\frac{1}{\mathrm{M}^{m-1}}\int \frac{a^x dx}{x}.$$

The last integral can only be obtained in an infinite series; but this by expression of a^x becomes when $\mathrm{A} = \mathrm{M}^{-1}$,

$$\int \frac{a^x dx}{x} = \left\{1 + \frac{\mathrm{A}x}{1} + \frac{\mathrm{A}^2x^2}{1.2} + \ldots .\right\}\frac{dx}{x},$$

$$= \log_e x + \frac{\mathrm{A}x}{1^2} + \frac{\mathrm{A}^2x^2}{1.2^2} + \frac{\mathrm{A}^3x^3}{1.2.3^2} + \ldots .$$

Scholium.

It will sometimes be possible, by a little algebraic skill, to transform functions which are not precisely in these forms, into a series of terms which separately possess them. This will appear in some of the following examples.

1. To integrate $da^{\mathrm{X}}dx$, where X is a function of x.
Since $da^{\mathrm{X}} = \log_e a . a^{\mathrm{X}} . d\mathrm{X}$ we have

$$\int a^{\mathrm{X}} d\mathrm{X} = \frac{a^{\mathrm{X}}}{\log_e a} + \mathrm{C}, \text{ for the general form.}$$

2. To integrate $f(a^{\mathrm{X}})\,dx$, where f is an algebraic function of a^{X}. Make $a^{\mathrm{X}} = z$: then $dx = \frac{dz}{z \log_e a}$; and hence

$$\int f(a^{\mathrm{X}})\,dx = \frac{1}{\log^e a}\int \frac{f(z)}{z}, \text{ which is an algebraic function.}$$

3. $$\int \frac{dx}{\sqrt{1 + a^{\mathrm{X}}}} = \frac{1}{\log_e a}\int \frac{d\sqrt{z}}{\sqrt{z}.\sqrt{1+z}} = \frac{2}{\log_e a}.\log_e \frac{\sqrt{z}}{1 + \sqrt{1+z}}$$

$$= \frac{2\{\log_e a^{\frac{\mathrm{X}}{2}} - \log_e (1 + \sqrt{1 + a^{\mathrm{X}}})\}}{\log_e a}.$$

4. $$\int \frac{e^{2x} - 1}{e^{2x} + 1} dx = \log_e (e^x - e^{-x}).$$

5. $\int \frac{e^x x dx}{(1 + x)^2}$ is proposed for integration: and this example is one of a kind that not unfrequently occurs.

The condition is that, in this case, $\frac{x}{(1+x)^2}$ is divisible into the sum of two quantities, one of which is the differential co-efficient of the other. For let y be one of them: then

$$d(e^x y) = e^x\left(y + \frac{dy}{dx}\right), \text{ and } e^x y = \int e^x\left(y + \frac{dy}{dx}\right).$$

In the present case, if $y = \frac{1}{1+x}$, then $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$, and the criterion $\frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2}$ is fulfilled. Whence we have

$$\int \frac{e^x x dx}{(1+x)^2} = e^x y = \frac{e^x}{1+x}.$$

6. $\int \frac{e^x(2-x)\,dx}{(1-x)\sqrt{1-x^2}} = e^x\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$, which is given by the same method.

Except, however, in very simple cases, it will be impracticable to discover the form of the function y.

7. $\int x^3 a^x dx = \mathrm{M}a^x\,\{x^3 - 3\mathrm{M}x^2 + 6\mathrm{M}^2 x - 6\mathrm{M}^3\}$.

8. $\int \frac{a^x \,.\, x dx}{(b+x)^2} = \frac{1}{a^b}\left\{\frac{b\,.\,a^{b+x}}{b+x} + (1 - b\log_e a)\int \frac{a^{b+x}\,dx}{b+x}\right\}$.

9. $\int \frac{x^m dx}{a^x} = -\frac{\mathrm{M}}{a^x}\,\{x^m + m\mathrm{M}x^{m-1} + m(m-1)\mathrm{M}^2 x^{m-2} + \ldots + 1.2\ldots m\mathrm{M}^m\}$.

PROP. VI.

To integrate the general form $\sin^m\theta\cos^n\theta\,d\theta$, m and n being any integers, positive or negative.

The following transformations are obvious:

$$\sin^m\theta\cos^n\theta\,d\theta = \sin^{m-1}\theta\cos^n\theta\sin\theta\,d\theta = -\frac{\sin^{m-1}\theta\,d(\cos^{n+1}\theta)}{n+1} \;\ldots\ldots\ldots (1),$$

$$\sin^m\theta\cos^n\theta\,d\theta = \sin^m\theta\cos^{n-1}\theta\cos\theta\,d\theta = +\frac{\cos^{n-1}\theta\,d(\sin^{m+1}\theta)}{m+1} \;\ldots\ldots\ldots (2).$$

In (1) put $\sin^{m-1}\theta = u$; $\frac{d(\cos^{n+1}\theta)}{n+1} = dv$; and hence $v = \frac{\cos^{n+1}\theta}{n+1}$. Then the formula of partition, $\int u dv = uv - \int v du$, becomes

$$\int \sin^m\theta\cos^n\theta\,d\theta = -\frac{\sin^{m-1}\theta\cos^{n+1}\theta}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2}\theta\cos^{n+2}\theta\,d\theta \;\ldots\ldots (3).$$

In (2) put $u = \cos^{n-1}\theta$; $\frac{d(\sin^{m+1}\theta)}{m+1} = dv$; and hence $v = \frac{\sin^{m+1}\theta}{m+1}$.

The formula of partition in this case becomes

$$\int \sin^m\theta\cos^n\theta\,d\theta = \frac{\sin^{m+1}\theta\cos^{n-1}\theta}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2}\theta\cos^{n-2}\theta\,d\theta \;\ldots\ldots (4).$$

The formula (3) is adapted to n negative and m positive, and (4) to m negative and n positive; since in both cases the tendency is to lessen the index of the denominator; or, in other words, to reduce the final integral to cosines or sines alone.

Again, other transformations may be effected in the following ways.

In the expression (3), put, in the term under the sign of integration, $(1 - \sin^2\theta) \cos^n\theta$ for $\cos^{m+2}\theta$, then we have that term converted into

$$\frac{m-1}{n+1}\int \sin^{m-2}\theta \cos^n\theta \, d\theta - \frac{m-1}{n+1}\int \sin^m \theta \cos^n\theta \, d\theta.$$

Transpose the second term of this to the first side of (3); and, after slight reduction, that equation (3) is converted into

$$\int \sin^m\theta \cos^n\theta \, d\theta = -\frac{\sin^{m-1}\theta \cos^{n+1}\theta}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}\theta \cos^n\theta \, d\theta \ldots\ldots\ldots (5).$$

Also, in the expression (4), put $(1 - \cos^2\theta) \sin^m\theta$ for $\sin^{m+2}\theta$, in the term under the sign of integration; then that term is

$$\frac{n-1}{m+1}\int \sin^m\theta \cos^{n-2}\theta \, d\theta - \frac{n-1}{m+1}\int \sin^m\theta \cos^n\theta \, d\theta.$$

Transpose the second part of this to the first side of (4), and reduce as before; then

$$\int \sin^m\theta \cos^n\theta \, d\theta = \frac{\sin^{m+1}\theta \cos^{n-1}\theta}{m+n} + \frac{n-1}{m+n}\int \sin^m\theta \cos^{n-2}\theta \, d\theta \ \ldots\ldots\ldots \ (6).$$

These two formulæ are adapted to the cases of m and n both positive; and respectively to the cases where it may be desirable to reduce the final integral to a cosine and to a sine.

But we may find formulæ of reduction for diminishing, *pari passu*, the powers both of the sine and cosine, when both are positive, and for increasing them when both are negative.

In (6) write $m - 2$ for m; then it becomes

$$\int \sin^{m-2}\theta \cos^n\theta \, d\theta = \frac{\sin^{m-1}\theta \cos^{n-1}\theta}{m+n-2} + \frac{n-1}{m+n-2}\int \sin^{m-2}\theta \cos^{n-2}\theta \, d\theta.$$

Insert this value of $\int \sin^{m-2}\theta \cos^n\theta \, d\theta$ in (5), and we have

$$\int \sin^m\theta \cos^n\theta \, d\theta = -\frac{\sin^{m-1}\theta \cos^{n+1}\theta}{m+n} + \frac{(m-1) \sin^{m-1}\theta \cos^{n-1}\theta}{(m+n)(m+n-2)}$$
$$+ \frac{(m-1)(n-1)}{(m+n)(m+n-2)}\int \sin^{m-2}\theta \cos^{n-2}\theta \, d\theta \ldots\ldots (7).$$

Also in (5) write $n - 2$ for n; and we similarly obtain

$$\int \sin^m\theta \cos^{n-2}\theta \ d\theta = -\frac{\sin^{m-1}\theta \cos^{n-1}\theta}{m+n-2} + \frac{m-1}{m+n-2}\int \sin^{m-2}\theta \cos^{n-2}\theta \, d\theta.$$

Insert this value of $\int \sin^m\theta \cos^{n-2}\theta \, d\theta$ in (6); then,

$$\int \sin^m\theta \cos^n\theta \, d\theta = \frac{\sin^{m+1}\theta \cos^{n-1}\theta}{m+n} - \frac{(n-1) \sin^{m-1}\theta \cos^{n-1}\theta}{(m+n)(m+n-2)}$$
$$+ \frac{(m-1)(n-1)}{(m+n)(m+n-2)}\int \sin^{m-2}\theta \cos^{n-2}\theta \, d\theta \ldots\ldots (8).$$

These two forms (7, 8) are adapted to the cotemporaneous reduction of m and n. They are not, as to the *co-efficients*, symmetrical functions of m and n in either expression, whilst the indices under the sign of integration would lead us to expect that such should be the case. Such form is, however, easily obtained by taking half the sum of (7) and (8). We thus obtain

$$\int \sin^m\theta \cos^n\theta\, d\theta = \frac{\sin^{m-1}\theta \cos^{n-1}\theta}{m+n} \cdot \frac{(m-1)\sin^2\theta - (n-1)\cos^2\theta}{m+n-2}$$
$$+ \frac{(n-1)(m-1)}{(m+n)(m+n-2)} \int \sin^{m-2}\theta \cos^{n-2}\theta\, d\theta \;\ldots\ldots\; (9).$$

This formula is symmetrical; though for the purposes of actual investigation it is less useful than either of the two from which it was deduced.

Lastly, to obtain formulæ adapted to m and n being negative. Throughout the entire previous investigations, no hypothesis has been made as to whether m or n (one or both) were positive; and hence the concluding formulæ equally apply to all cases. We have merely stated to what cases they were *usefully* adaptable. Instead of diminishing we must now increase the indices m and n; to effect which write $-m+2$, and $-n+2$, instead of m and n in (5, 6): then we get, by reducing as in (7, 8),

$$\int \frac{d\theta}{\sin^m\theta \cos^n\theta} = -\frac{1}{(m-1)\sin^{n-1}\theta \cos^{n-1}\theta} + \frac{m+n-2}{(m-1)(n-1)\sin^{m-3}\theta \cos^{n-1}\theta}$$
$$+ \frac{(m+n-2)(m+n-4)}{(m-1)(n-1)} \int \frac{d\theta}{\sin^{m-2}\theta \cos^{n-2}\theta} \;\ldots\ldots\ldots\; (10);$$

and in a similar manner we get

$$\int \frac{d\theta}{\sin^m\theta \cos^n\theta} = +\frac{1}{(n-1)\sin^{m-1}\theta \cos^{n-1}\theta} - \frac{m+n-2}{(m-1)(n-1)\sin^{m-1}\theta \cos^{n-3}\theta}$$
$$+ \frac{(m+n-2)(m+n-4)}{(m-1)(n-1)} \int \frac{d\theta}{\sin^{m-2}\theta \cos^{n-2}\theta} \;\ldots\ldots\ldots\ldots\; (11).$$

Either of these continually reduces the index towards unity, and hence they are adapted to the specified case.

If we reduce this to a symmetrical function of m and n, as was done in (9), we shall have

$$\int \frac{d\theta}{\sin^m\theta \cos^n\theta} = \frac{m\sin^2\theta - n\cos^2\theta + 1}{(m-1)(n-1)\sin^{n-1}\theta \cos^{n-1}\theta}$$
$$+ \frac{(m+n-2)(m+n-4)}{(m-1)(n-1)} \int \frac{d\theta}{\sin^{m-2}\theta \cos^{n-2}\theta} \ldots\ldots (12).$$

When $n = -m$, the expression for integration becomes $\int \tan^m\theta\, d\theta$; and the formula for partition gives at once

$$\int \tan^m\theta\, d\theta = \frac{\tan^{m-1}\theta}{m-1} - \int \tan^{m-2}\theta\, d\theta \ldots\ldots\ldots (13).$$

Moreover, when $m = 0$ and n positive or negative, we get for these cases respectively, in a somewhat simpler manner,

$$\int \cos^n\theta\, d\theta = \cos^{n-1}\theta \sin\theta + (n-1)\int \sin^2\theta \cos^{n-2}\theta\, d\theta$$
$$= \cos^{n-1}\theta \sin\theta + (n-1)\int (1-\cos^2\theta)\cos^{n-2}\theta\, d\theta$$
$$= \cos^{n-1}\theta \sin\theta + (n-1)\int \cos^{n-2}\theta\, d\theta - (n-1)\int \cos^n\theta\, d\theta.$$

Whence, transposing and simplifying,

$$\int \cos^n\theta\, d\theta = \frac{\cos^{n-1}\theta \sin\theta}{n} + \frac{n-1}{n}\int \cos^{n-2}\theta\, d\theta \ldots\ldots (14).$$

Again, for the other case multiply by $\sin^2\theta + \cos^2\theta = 1$; then

$$\int \frac{d\theta}{\cos^n\theta} = \int \frac{\sin^2\theta\, d\theta}{\cos^n\theta} + \int \frac{\cos^2\theta\, d\theta}{\cos^n\theta} = \int \frac{\sin^2\theta\, d\theta}{\cos^n\theta} + \int \frac{d\theta}{\cos^{n-2}\theta}.$$

But applying (5) to the integral $\int \frac{\sin^2\theta\, d\theta}{\cos\,\theta}$, we have, ($m$ being in this case $=2$)

$$\int \frac{\sin^2\theta\, d\theta}{\cos^n\theta} = -\frac{\sin\,\theta \cos^{-n+1}\theta}{-n+1} + \frac{2-1}{-n+1}\int \frac{d\theta}{\cos^{n-2}\theta}$$

$$= \frac{\sin\,\theta}{(n-1)\cos^{n-1}\theta} - \frac{1}{n-1}\int \frac{d\theta}{\cos^{n-2}\theta}.$$

Insert this value of the integral; then we get

$$\int \frac{d\theta}{\cos^n\theta} = \frac{\sin\,\theta}{(n-1)\cos^{n-1}\theta} + \frac{n-2}{n-1}\int \frac{d\theta}{\cos^{n-2}\theta} \ldots\ldots\ldots\ldots (15).$$

Also, by a corresponding course of reductions, we get the forms adapted to $n = 0$, and m respectively positive and negative. They are as follows:

$$\int \sin^m\theta\, d\theta = -\frac{\cos\,\theta \sin^{m-1}\theta}{m} + \frac{m-1}{m}\int \sin^{m-2}\theta\, d\theta \ldots\ldots (16).$$

$$\int \frac{d\theta}{\sin^m\theta} = -\frac{\cos\,\theta}{(m-1)\sin^{m-1}\theta} + \frac{m-2}{m-1}\int \frac{d\theta}{\sin^{m-2}\theta} \ldots\ldots (17).$$

There is another method of integrating $\sin^m\theta \cos^n\theta\, d\theta$, which is elegant in itself; and as it suggests a different method of integrating $\frac{x^m\, dx}{(1+x^2)^{\frac{r}{2}}}$, we shall add a slight notice of it here.

Put $x = \tan\,\theta$, and $n = r - m - 2$; then we shall have

$$\frac{x^m\, dx}{(1+x^2)^{\frac{r}{2}}} = \frac{\tan^m\theta\, d\tan\,\theta}{\sec^r\theta} = \frac{\sin^m\theta \cos^r\theta\, d\theta}{\cos^m\theta \cos^2\theta}$$

$$= \sin^m\theta \cos^{r-m-2}\theta\, d\theta = \sin^m\theta \cos^n\theta\, d\theta;$$

$$\text{and hence, } \int \frac{x^m\, dx}{(1+x^2)^{\frac{r}{2}}} = \int \sin^m\theta \cos^n\theta\, d\theta.$$

But if, on the contrary, we avail ourselves of this relation, we may integrate $\int \sin^m\theta \cos^n\theta\, d\theta$. For integrating $\int \frac{\tan^m\theta\, d\tan\theta}{(1+\tan^2\theta)^{\frac{r}{2}}}$ by parts, we get either of the two following forms:

$$\int \frac{\tan^m\theta\, d\tan\,\theta}{(1+\tan^2\theta)^{\frac{r}{2}}} = \frac{\tan^{m+1}\theta \cos^r\theta}{m-r+1} - \frac{r}{m-r+1}\int \frac{\tan^m\theta\, d\tan\,\theta}{(1+\tan^2\theta)^{\frac{r+2}{2}}}$$

$$= \frac{\tan^{m+1}\theta \cos^{r-2}\theta}{r-2} - \frac{m-r+3}{r-2}\int \frac{\tan^m\theta\, d\tan\,\theta}{(1+\tan^2\theta)^{\frac{r-2}{2}}}.$$

The former of which is adapted to r being negative, and the latter to r being positive.

Scholium.

Other methods of integration adapted to these cases are easily devised; but it is found that they are more laborious than those just given. Instead of pointing them[*] out, we shall leave it to the student to make the trial; and merely put down the general expressions for the integrals, which arise from the methods of partition employed above.

$$\text{I. } \int \sin^m\theta \cos^n\theta \, d\theta = -\frac{\cos^{n+1}\theta}{m+n}\left\{\sin^{m-1}\theta + \frac{m-1 \sin^{m-3}\theta}{m-2+n} + \frac{(m-1)(m-3)\sin^{m-5}\theta}{(m-2+n)(m-4+n)} + \ldots\right.$$

$$+ \frac{(m-1)(m-3)(m-5)\ldots\ldots\ldots\ldots}{(m+n)(m-2+n)(m-1+n)\ldots} \times \begin{cases}\int \sin\theta \cos^n\theta . d\theta, & (m \textit{ odd})\\ \text{or} \int \cos^n\theta \, d\theta, & (m \textit{ even}).\end{cases}$$

$$\text{II. } \int \sin^m\theta \cos^n\theta \, d\theta = \frac{\sin^{m+1}\theta}{m+n}\left\{\cos^{n-1}\theta + \frac{n-1 \cos^{n-3}\theta}{n-2+m} + \frac{(n-1)(n-2)\cos^{n-4}\theta}{(n-2+m)(n-4+m)} + \ldots\right\}$$

$$+ \frac{(n-1)(n-3)(n-5)\ldots\ldots\ldots\ldots}{(n+m)(n-2+m)(n-4+m)\ldots} \times \begin{cases}\int \cos\theta \sin^m\theta \, d\theta & (m \textit{ odd})\\ \text{or} \int \sin^m\theta \, d\theta, & (m \textit{ even}).\end{cases}$$

$$\text{III. } \int \sin^m\theta \, d\theta = -\frac{\cos\theta}{m}\left\{\sin^{m-1}\theta + \frac{m-1}{m-2}\sin^{m-3}\theta + \frac{(m-1)(m-3)}{(m-2)(m-4)}\sin^{m-5}\theta + \ldots\right\}$$

$$+ \frac{(m-1)(m-3)(m-5)\ldots\ldots}{m(m-2)(m-4)\ldots\ldots\ldots\ldots} \times \begin{cases}\int \sin\theta \, d\theta, & (m \textit{ odd})\\ \text{or} \int d\theta, & (m \textit{ even}).\end{cases}$$

$$\text{IV. } \int \cos^n\theta \, d\theta = \frac{\sin\theta}{n}\left\{\cos^{n-1}\theta + \frac{n-1}{n-2}\cos^{n-3}\theta + \frac{(n-1)(n-3)}{(n-2)(n-4)}\cos^{n-5}\theta + \ldots\right\}$$

$$+ \frac{(n-1)(n-3)(n-5)\ldots\ldots}{n(n-2)(n-4)\ldots\ldots} \times \begin{cases}\int \cos\theta \, d\theta, & (n \textit{ odd})\\ \text{or} \int d\theta, & (n \textit{ even}).\end{cases}$$

$$\text{V. } \int \frac{d\theta}{\sin^m\theta} = -\frac{\cos\theta}{m-1}\left\{\frac{1}{\sin^{m-1}\theta} + \frac{m-2}{(m-3)\sin^{m-3}\theta} + \frac{(m-2)(m-4)}{(m-3)(m-5)\sin^{m-5}\theta} + \ldots\right\}$$

$$+ \frac{(m-2)(m-4)(m-6)\ldots\ldots}{(m-1)(m-3)(m-5)\ldots\ldots} \times \begin{cases}\log\tan\tfrac{1}{2}\theta, & (m \textit{ odd})\\ \text{or } \theta, & (m \textit{ even}).\end{cases}$$

$$\text{VI. } \int \frac{d\theta}{\cos^n\theta} = \frac{\sin\theta}{n}\left\{\frac{1}{\cos^{n-1}\theta} + \frac{n-2}{(n-3)\cos^{n-3}\theta} + \frac{(n-2)(n-4)}{(n-3)(n-5)\cos^{n-5}\theta} \ldots\right\}$$

$$+ \frac{(n-2)(n-4)(n-6)\ldots}{(n-1)(n-3)(n-5)\ldots} \times \begin{cases}\log\tan\tfrac{1}{2}\left(\frac{\pi}{2}+\theta\right) & (n \textit{ odd}),\\ \text{or } \theta, & (n \textit{ even}).\end{cases}$$

$$\text{VII. } \int \tan^n\theta d\theta = \frac{\tan^{n-1}\theta}{n-1} - \frac{\tan^{n-3}\theta}{n-3} + \frac{\tan^{n-5}\theta}{n-5} - \ldots\ldots \pm \begin{cases}\tan\theta d\theta, & (n \textit{ even}),\\ \text{or } d\theta & (n \textit{ odd}).\end{cases}$$

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[*] Perhaps we ought to make an exception as to the case of m and n being positive; and especially where one of them is 0, and the other positive: for we can then expand $\sin^m\theta$ and

Scholium 2.

The final integrals that are given in the preceding tables are thus found :—

$\int \sin^p \theta d\theta$ is reduced to (III); and $\sin \theta d\theta = - d \cos \theta$,

$\int \cos^q \theta d\theta$ is .... (IV); and $\cos \theta d\theta = d \sin \theta$.

$$\int \frac{d\theta}{\sin \theta} = \int \frac{\sin \theta d\theta}{1-\cos^2\theta} = - \frac{d \cos \theta}{1-\cos^2\theta} = \log_e \left(\frac{1-\cos \theta}{1+\cos \theta}\right)^{\frac{1}{2}} = \log_e \tan \tfrac{1}{2} \theta,$$

$$\int \frac{d\theta}{\cos \theta} = \int \frac{\cos \theta d\theta}{1-\sin^2 \theta} = \frac{d \sin \theta}{1-\sin^2 \theta} = \log_e \left(\frac{1+\cos \theta}{1-\cos \theta}\right)^{\frac{1}{2}} = \log_e \tan \tfrac{1}{2} (\tfrac{1}{2} \pi + \theta),$$

$\int \frac{d\theta}{\sin^2\theta} = - \cot\theta$: and $\int \frac{d\theta}{\cos^2\theta} = + \tan \theta$ (see *page* 364),

$$\int \frac{\cos\theta d\theta}{\sin \theta} = \int \frac{d \sin \theta}{\sin \theta} = \log_e \sin \theta.$$

$$\int \frac{\sin \theta d\theta}{\cos \theta} = - \int \frac{d \cos \theta}{\cos \theta} = \log_e \sec \theta.$$

Scholium 3.

The preceding investigations require that m and n shall be integers. It is, however, possible under particular circumstances, that they may be fractions.

For, put $\cos \theta = x$: then $\sin \theta = \sqrt{1 - x^2}$, and $d\theta = - \frac{dx}{\sqrt{1-x^2}}$; whence

$$\int \sin^m \theta \cos^n \theta d\theta = - \int x^n (1 - x^2)^{\frac{m-1}{2}} dx;$$

which is capable of being rationalised when $\frac{m+1}{2}$, or $\frac{m+n}{2}$ is an integer.

Again, put $\sin \theta = y$: then $\cos \theta = \sqrt{1 - y^2}$, and $d\theta = \frac{dy}{\sqrt{1 - y^2}}$;

$$\int \sin^m \theta \cos^n \theta d\theta = \int y^m (1 - y^2)^{\frac{n-1}{2}} dy;$$

which is integrable when $\frac{n-1}{2}$, or $\frac{m+n}{2}$ is integer.

Wherefore if $\frac{m-1}{2}$, $\frac{n-1}{2}$, or $\frac{m+n}{2}$ be integer, the integral can be obtained in a rational form by *chap*. IV.

EXAMPLES.

1. $\int \sin^4\theta \cos^3 \theta d\theta = \frac{\sin^5 \theta}{5} - \frac{\sin^7 \theta}{7}$.

$\cos^n \theta$ in terms of multiple arcs, as at *page* 438, *vol. i.*, where the general form is given. Thus,

$\sin^2\theta = - \frac{1}{2} \cos 2\theta + \frac{1}{2}$	$\cos^2\theta = \frac{1}{2} \cos 2\theta + \frac{1}{2}$
$\sin^3\theta = - \frac{1}{4} \sin 3\theta + \frac{3}{4} \sin \theta$	$\cos^3\theta = \frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta$
$\sin^4\theta = \frac{1}{8} \cos 4\theta - \frac{1}{2} \cos 2\theta + \frac{3}{8}$	$\cos^4\theta = \frac{1}{8} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8}$
etc. etc.	*etc. etc.*

2. $\int \sin^5\theta \cos^4\theta d\theta = -\frac{\cos^5\theta}{9}\left\{\sin^4\theta + \frac{4\sin^2\theta}{7} + \frac{4\,.\,2}{7\,.\,5}\right\}.$

3. $\int \sin^3\theta d\theta = -\frac{\cos\theta}{3}\{\sin^2\theta + 2\}.$

4. $\int \sin^3\theta \cos^2\theta d\theta = -\frac{\cos^3\theta}{5}\left\{\sin^2\theta + \frac{2}{3}\right\}.$

5. $\int \sin^3\theta \cos^n\theta d\theta = -\frac{\cos^{n+1}\theta}{n+3}\left\{\sin^2\theta + \frac{2}{n+1}\right\}.$

6. $\int \cos^3\theta \sin^m\theta d\theta = \frac{\sin^{m+1}\theta}{m+3}\left\{\cos^2\theta + \frac{2}{m+1}\right\}.$

7. $\int \sin^6\theta d\theta = -\frac{\cos\theta}{6}\left\{\sin^5\theta + \frac{5\sin^3\theta}{4} + \frac{5\,.\,3}{4\,.\,2}\sin\theta\right\} + \frac{5\,.\,3\,.\,1}{6\,.\,4\,.\,2}.$

8. $\int \frac{\cos^7\theta d\theta}{\sin\theta} = \frac{\cos^6\theta}{6} + \frac{\cos^4\theta}{4} + \frac{\cos^2\theta}{2} + \log_e \sin\theta.$

9. $\int \frac{\cos^6\theta d\theta}{\sin\theta} = \frac{\cos^5\theta}{5} + \frac{\cos^3\theta}{3} + \frac{\cos\theta}{1} + \log_e \tan\tfrac{1}{2}\theta.$

10. $\int \frac{\sin^4\theta d\theta}{\cos^5\theta} = \frac{1}{\cos^4\theta}\left\{\sin^3\theta - \frac{5\sin\theta}{4}\right\} + \frac{3\,.\,1}{4\,.\,2}\left\{\frac{\sin\theta}{\cos^2\theta} + \log_e \tan\tfrac{1}{2}(\tfrac{1}{2}\pi+\theta)\right\}.$

11. $\int \frac{d\theta}{\cos^4\theta} = \frac{\sin\theta}{3\cos\theta} + \frac{2\tan\theta}{3}.$

12. $\int \frac{\sin^3\theta d\theta}{\cos^2\theta} = \cos\theta + \sec\theta.$

13. $\int \sin^2\theta d\theta = -\frac{\sin 2\theta}{4} + \frac{\theta}{2}.$

14. $\int \sin^3\theta d\theta = \frac{\cos 3\theta}{12} - \frac{3\cos\theta}{4}.$

15. $\int \sin^4\theta d\theta = \frac{\sin 4\theta}{32} - \frac{\sin 2\theta}{4} + \frac{3\theta}{8}.$

16. $\int \cos^2\theta d\theta = \frac{\sin 2\theta}{4} + \frac{\theta}{2}.$

17. $\int \cos^3\theta d\theta = \frac{\sin 3\theta}{12} + \frac{3\sin\theta}{4}.$

18. $\int \cos^4\theta d\theta = \frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3\theta}{8}.$

19. $\int \frac{d\theta}{\sin^3\theta \cos^2\theta} = \frac{1}{\cos\theta} - \frac{\cos\theta}{2\sin^2\theta} + \frac{\log_e \tan\frac{1}{2}\theta}{2}.$

20. $\int \frac{d\theta}{\sin^4\theta \cos\theta} = -\frac{1}{\sin\theta} - \frac{1}{3\sin^3\theta} + \log_e \tan\tfrac{1}{2}(\tfrac{1}{2}\pi + \theta).$

21. $\int \frac{d\theta}{\sin^2\theta \cos^4\theta} = \frac{\sin\theta}{3\cos^3\theta} + \frac{2\tan\theta}{3} - \frac{2\cot 2\theta}{1}.$

22. $\int \sin^m\theta \cos^m\theta d\theta = \frac{1}{2^{m-1}}\int \sin^m(2\theta)\, d(2\theta).$

23. $\int \frac{d\theta}{\sin^m\theta \cos^m\theta} = 2^{m-1}\int \frac{d(2\theta)}{\sin^m 2\theta}.$

24. $\int_0^{\frac{\pi}{2}} \sin^{2m}\theta d\theta = \frac{1\,.\,3\,.\,5 \ldots\ldots (2m-1)}{2\,.\,4\,.\,6 \ldots\ldots 2m}\,\frac{\pi}{2}.$

$$\sin(a+m\theta)\cos(\beta+n\theta)d\theta=\tfrac{1}{2}\int\sin\{a+\beta+(m+n)\theta\}d\theta+\tfrac{1}{2}\int\sin\{a-\beta+(m-n)\}d\theta.$$

$$\sin(a+m\theta)\cos(\beta+n\theta)\,d\theta=\frac{m\sin(a+m\theta)\cos(\beta+n\theta)-n\cos(a+n\theta)\sin(\beta+n\theta).}{m^2-n^2}$$

$$\cos^n\theta d\theta=\frac{1}{2^{n-1}}\left\{\frac{\sin n\theta}{n}+\frac{n}{1}\cdot\frac{\sin(n-2)\theta}{n-2}+\frac{n(n-1)}{1.2}\cdot\frac{\sin(n-4)\theta}{n-4}+\ldots.\right\}$$
$$+\begin{cases}\dfrac{1.3.5\ldots.(n-1)}{1.2.3\ldots.\ \frac{1}{2}n}\cdot\dfrac{\theta}{2^{\frac{n}{2}}},\ (n\ \textit{even})\\[2ex] \dfrac{n(n-1)\ \ldots.\ \frac{1}{2}(n+3)}{1.2.3\ \ldots.\ \frac{1}{2}(n-1)}\cdot\dfrac{\sin\theta}{2^{n-1}},\ (n\ \textit{odd}).\ \text{ib.}\end{cases}$$

$$\sin^m\theta d\theta=\frac{1}{(-1)^{\frac{m}{2}}\,2^{m-1}}\left\{\frac{\sin m\theta}{m}-\frac{m}{1}\cdot\frac{\sin(m-2)\theta}{m-2}+\frac{m(m-1)}{1.2}\cdot\frac{\sin(m-4)\theta}{m-4}\ \ldots.\right\}$$
$$(m\ \textit{even})\qquad+\frac{m(m-1)(m-2)\ \ldots.\ (\frac{1}{2}m+1)}{1.2.3\ldots.\ \frac{1}{2}m}\cdot\frac{\theta}{2^m}.$$

$$\sin^m\theta d\theta=-\frac{1}{(-1)^{\frac{m-1}{2}}\,2^{m-1}}\left\{\frac{\cos m\theta}{m}-\frac{m}{1}\cdot\frac{\cos(m-2)\theta}{m-2}+\frac{m(m-1)}{1.2}\ \frac{\cos(m-4)\theta}{m-4}-\ \ldots.\right\}$$
$$(m\ \textit{odd})\qquad-\frac{m(m-1)(m-2)\ \ldots.\ \frac{1}{2}(m+3)}{1.2.3\ \ldots.\ \frac{1}{2}(m-1)}\cdot\frac{\cos 2\theta}{2^m}.$$

PROP. VII.

To find the integrals of mixed functions, composed from algebraic, trigonometrical, logarithmic or exponential elements.

$$\begin{array}{l|l|l|l} a^{m\theta}\sin^n\theta d\theta & \text{X}\sin^{-1}xdx & (\sin^{-1}x)^m\,dx & z^m\sin^n zdz\\ a^{m\theta}\cos^n\theta dz & \text{X}\cos^{-1}xdx & (\cos^{-1}x)^m\,dx & z^m\cos^n zdz, \end{array}$$

where X is an algebraic function of x, and a, m, n, any constants, and the two last a positive integer.

(1). We have, obviously, $a^{m\theta}\sin^n\theta d\theta=-a^{m\theta}\sin^{n-1}\theta d\cos\theta$.

Put $u=-a^{m\theta}\sin^{n-1}\theta$, $v=\cos\theta$; and hence $dv=d\cos\theta$, and

$$du=-m\log_e a\,.\,a^{m\theta}\sin^{n-1}\theta d\theta+(n-1)a^{m\theta}\sin^{n-2}\theta\cos\theta d\theta;$$

wherefore the formula $\int udv=uv-\int vdu$ becomes

$$-\int a^{m\theta}\sin^{n-1}\theta d\cos\theta=-a^{m\theta}\sin^{n-1}\theta\cos\theta+m\log_e a\int a^{m\theta}\sin^{n-1}\theta\cos\theta d\theta$$
$$+(n-1)\int a^{m\theta}\sin^{n-2}\theta\cos^2\theta d\theta\ \ldots\ldots\ldots\ldots.\ (a).$$

But $\int a^{m\theta}\sin^{n-1}\theta\cos\theta d\theta=\frac{1}{n}\int a^{m\theta}d\sin^n\theta$, or, by parts again,

$$=\frac{a^{m\theta}\sin^n\theta}{n}-\frac{m\log_e a}{n}\int a^{m\theta}\sin^{n\theta}d\theta\ldots\ldots\ldots\ldots(b).$$

Also $\int a^{m\theta}\sin^{n-2}\theta\cos^2\theta d\theta=\int a^{m\theta}\sin^{n-2}\theta\,(1-\sin^2\theta)\,d\theta$

$$=\int a^{m\theta}\sin^{n-2}\theta d\theta-\int a^{m\theta}\sin^n\theta d\theta\ldots\ldots\ldots\ldots(c).$$

Substitute (b) and (c) in (a); transpose and reduce: then

$$\int a^{m\theta}\sin^n\theta d\theta=\frac{a^{m\theta}\sin^{n-1}\theta}{(m\log_e a)^2+n^2}\left\{m\log_e a\sin\theta-n\cos\theta\right\}$$
$$+\frac{(n-1)n}{(m\log_e a)^2+n^2}\cdot\int a^{m\theta}\sin^{n-2}\theta d\theta\ldots\ldots\ldots\ldots.\ (d),$$

The integral is thus reduced to dependence on one of the same form, but in which the index n is two units lower. By the same mode of reduction, we shall get to a final integral presenting two cases, according as n is odd or even.

If *even*, then that integral is $\int a^{m\theta}\, d\theta = \dfrac{a^{m\theta}}{m \log_e a}$ .... (*e*).

If *odd*, then that integral is $\int a^{m\theta} \sin \theta d\theta$, which vanishes, since $n = 1$, and $n - 1$, the co-efficient, vanishes. We shall thus obtain

$$\int a^{m\theta} \sin\ \theta d\theta = \frac{a^{m\theta}}{(m \log_e a)^2 + 1} \left\{ m \log_e a \,.\, \sin\theta - \cos\theta \right\} \ldots\ldots (f).$$

(2). By a course of investigation in all respects similar to the preceding, we obtain

$$\int a^{m\theta} \cos^n \theta d\theta = \frac{a^{m\theta} \cos^{n-1}\theta}{(m \log_e a)^2 + n^2} \left\{ m \log_e a \,.\, \cos\theta + n\ \sin\ \theta \right\}$$
$$+ \frac{n\,(n-1)}{(m \log_e a)^2 + n^2} \int a^{m\theta} \cos^{n-2}\theta d\theta \ldots\ldots\ldots\ldots\ldots (g).$$

The same remarks apply to this as to the preceding, and the final integrals are according as n is odd or even

$$\int a^{m\theta} \cos\theta\, d\theta = \frac{a^{m\theta}}{(m \log_e a)^2 + 1} \left\{ m \log_e a \,.\cos\theta + \sin\ \theta \right\} \ \ldots\ldots (h)$$

$$\text{and} \int a^{m\theta}\, d\theta = \frac{a^{m\theta}}{m \log_e a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i).$$

(3, 4, 5, 6.) Put $u = \sin^{-1} x$, and $v = \int \mathrm{X} dx$: then, $du = \dfrac{dx}{\sqrt{1 - x^2}}$; and integrating by parts,

$$\int \mathrm{X} \sin^{-1} x\, dx = \sin^{-1} x \int \mathrm{X}\ dx - \int \frac{\int \mathrm{X} dx}{\sqrt{1 - x^2}} \,.\, dx,$$

and since X is an algebraic function, the only integrals required to be found then are those of algebraic functions Xdx, and $\dfrac{\int \mathrm{X} dx}{\sqrt{1 - x^2}} \,.\, dx$. The same transformation applies when any one else of the inverse trigonometrical functions of x takes the place of $\sin^{-1} x$ in (5, 6). Put $\sin^{-1} x = z$: then $x = \sin z$ and $dx = \cos z dz$: whence the expression is changed to $\int z^m \cos z dz$.

In the same way $\int (\cos^{-1} x)^m dx = -\int z^m \sin\ z dz$, when $z = \cos^{-1} x$; any other inverse trigonometrical functions may be reduced.

Now, integrating these by parts, we shall have as our final result

$$\int z^m \cos z dz = \sin z\ \left\{ z^m - m\,(m-1) z^{m-2} + m\,(m-1)(m-2)\,(m-3)\, z^{m-4} - \ldots \right\}$$
$$+ \cos z\ \left\{ m z^{m-1} - m\ (m-1)\ (m-2) z^{m-3} + \ldots\ldots\ldots\ldots\ldots\ldots \right\}$$
$$\int z^m \sin z dz = -\cos z\ \left\{ z^m - m(m-1) z^{m-2} + m(m-1)\,(m-2)\,(m-3) z^{m-4} - \ldots \right\}$$
$$+ \sin\ z\ \left\{ m z^{m-1} - m\ (m-1)\ (m-2)\ z^{m-3} + \ \ldots\ldots\ldots\ldots\ldots \right\}.$$

The last terms within the brackets are respectively

$$(-1)^{\frac{m-1}{2}} m\,(m-1) \ldots . 3 \,.\, 2 \,.\, 1\ z, \text{ and } (-1)^{\frac{m-1}{2}} m\ (m-1) \ldots . 3 \,.\, 2 \,.\, 1\ (n \text{ odd}),$$

$$(-1)^{\frac{m}{2}}\ m\,(m-1) \ldots . 3 \,.\, 2 \,.\, 1, \text{ and } -\ (^-1)^{\frac{m}{2}} m\,(m-1) \ldots\ 3 \,.\, 2 \,.\, 1\ z, (n \text{ even}).$$

If m be negative, we shall have

$$\int z^{-m} \cos z dz = -\ \sin z \left\{ \frac{z^{-m+1}}{m-1} - \frac{z^{-m+3}}{(m-1)\ (m-2)\ (m-3)} + \ldots\ldots\ldots\ldots\ldots \right\}$$
$$-\ \cos\ z \left\{ \frac{z^{-m+2}}{(m-1)\ (m-2)} - \frac{z^{-m+4}}{(m-1)(m-2)(m-3)(m-4)} + \ldots \right\}$$

and similarly for $\int z^{-m} \sin z dz$. The last terms will involve $\int \frac{\sin z dz}{z}$, and $\int \frac{\cos z dz}{z}$; the numerators of which must be expanded in series (*vol. i. p.* 435) and the resulting terms integrated separately.

(7, 8.) The forms $x^m \cos^n x dx$, $x^m \sin^m x dx$, are only capable of integration by expanding $\cos^n x$, and $\sin^n x$ in multiple arcs, and integrating each term of the result as in the preceding case.

It must be recollected in this integration, that if we have to find $\int x^m \sin rx dx$, it must be put under the form $\frac{1}{r^{m+1}} \int (rx)^m \sin rx \,.\, drx$; and so with other cases.

EXAMPLES.

1. $\frac{x^3 dx}{\sqrt{1-x^2}} \sin^{-1} x$ given for integration.

$$\text{Here} \int \text{X} dx = \int \frac{x^3 dx}{\sqrt{1-x^2}} = -\left\{\frac{x^2}{3} + \frac{1 \,.\, 2}{2 \,.\, 3}\right\} \sqrt{1-x^2},$$

$$\text{and} \int \frac{\int \text{X} dx}{\sqrt{1-x^2}} dx = \int \left\{\frac{x^2}{3} + \frac{1 \,.\, 2}{2 \,.\, 3}\right\} dx = \frac{x^3}{9} + \frac{2x}{3}.$$

$$\text{Hence} \int \frac{x^3 dx}{\sqrt{1-x^2}} \sin^{-1} x = -\left\{\frac{x^2}{3} + \frac{1 \,.\, 2}{2 \,.\, 3}\right\} \sin^{-1} x + \frac{x^3}{9} + \frac{2x}{3}.$$

2. To integrate $x^2 \sin^{-1} x dx$. Here $\int \text{X}\, dx = \int x^2 dx = \frac{x^3}{3}$;

$$\text{and} \int \frac{\int \text{X} dx}{\sqrt{1-x^2}} dx = \frac{1}{3} \int \frac{x^3 dx}{\sqrt{1-x^2}} = -\frac{1}{3} \left\{\frac{x^2}{3} + \frac{2}{3}\right\} \sqrt{1-x^2}.$$

Wherefore

$$\int x^2 \sin^{-1} x dx = \frac{x^3 \sin^{-1} x}{3} - \frac{1}{3} \left\{\frac{x^2}{3} + \frac{2}{3}\right\} \sqrt{1-x^2}.$$

3. $\int \frac{x^4 dx}{\sqrt{1-x^2}} \sin^{-1} x = -\left\{\left(\frac{x^3}{4} + \frac{3x}{2 \,.\, 4}\right) \sqrt{1-x^2} - \frac{3 \sin^{-1} x}{16}\right\} + \frac{x^4}{16} + \frac{3x^2}{16}.$

4. $\int x^4 \tan^{-1} x dx = \frac{x^5 \tan^{-1} x}{5} - \frac{1}{5} \left\{\frac{x^4}{4} - \frac{x^2}{2} + \log_e (1+x^2)\right\}.$

5. $\int x^m \sin^{-1} x dx = \frac{x^{m+1} \sin^{-1} x}{m+1} - \frac{1}{m+1} \int \frac{x^{m+1} dx}{\sqrt{1-x^2}}.$

6. $\int x^m \tan^{-1} x dx = \frac{x^{m+1} \tan^{-1} x}{m+1} - \frac{1}{m+1} \int \frac{x^{m+1} dx}{1+x^2}.$

7. $\int e^x \sin^2 x dx = \frac{e^x \sin x}{1+2^2} \{\sin x - 2 \cos x\} + \frac{2}{1+2^2} \int e^x dx.$

$$= \frac{e^x \sin x}{5} \{\sin x - 2 \cos x\} + \frac{2}{5} e^x.$$

8. $\int e^{ax} \cos x dx = \frac{e^{ax}}{a^2+1} \{a \cos x + \sin x\}.$

9. $\int e^x \sin^3 x dx = \frac{e^x}{10} \{\sin^3 x + 3 \cos^2 x + 3 \sin x - 6 \cos x\}.$

10. $\int e^{nx} \cos mx\, dx = .\, \frac{e^{nx} (n \cos mx + m \sin mx)}{m^2 + n^2}.$

11. $\int e^{nx} \sin mx \, dx = \dfrac{e^{nx}(n \sin mx - m \cos mx)}{m^2 + n^2}.$

12. $\int e^{ax} \sin mx \cos nx \, dx = \dfrac{1}{2}\int e^{ax}\left\{\sin(m+n)x + \sin(m-n)x\right\} dx,$

and the integrals can be found as in the two preceding examples.

13. $\int x^2 \cos^3 x \, dx = \dfrac{1}{4}\int x^2 (\cos 3x + 3 \cos x) \, dx,$

$$= \frac{1}{108}\int (3x)^2 \cos 3x \, d(3x) + \frac{3}{4}\int x^2 \cos x dx.$$

$$= \frac{1}{108}\int \left\{(3x)^2 \sin 3x + 2 \,.\, 3x \,.\, \cos 3x - 2 \sin 3x\right\},$$

$$+ \frac{3}{4}\left\{x^2 \sin x + 2x \cos x - 2 \sin x\right\}.$$

14. $\int \dfrac{\sin^{-1} x dx}{(1-x^2)^{\frac{3}{2}}} = \dfrac{x \sin^{-1} x}{\sqrt{1-x^2}} + \log_e \sqrt{1-x^2}.$

15. $\int \dfrac{x \sin^{-1} x dx}{(1-x^2)^{\frac{1}{2}}} = x - \sqrt{1-x^2} \,.\, \sin^{-1} x.$

16. $\int \sin^{-1}\left(\dfrac{x}{a+x}\right)^{\frac{1}{2}} dx = (x+a) \sin^{-1}\left(\dfrac{x}{a+x}\right)^{\frac{1}{2}} - \sqrt{ax}.$

17. $x \sin^{-1} \frac{1}{2}\left(\dfrac{2a-x}{a}\right)^{\frac{1}{2}} dx = \dfrac{x^2}{2} \sin^{-1}\left(\dfrac{2a-x}{a}\right)^{\frac{1}{2}} + \dfrac{a^2}{2} \sin^{-1} \dfrac{x}{2a} - \dfrac{x}{8}\sqrt{4a^2-x^2}.$

PROP. VIII.

To integrate $\dfrac{d\theta}{a + b \cos \theta}$ *and other analogous forms.*

1. *First*, let a be greater than b: then we have for the given form,

$$\int \frac{d\theta}{a + b \cos \theta} = \int \frac{d\theta}{a(\cos^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\theta) + b(\cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta)}$$

$$= \int \frac{\sec^2 \frac{1}{2}\theta d\theta}{(a+b)+(a-b)\tan^2 \frac{1}{2}\theta} = 2\int \frac{d \tan^2 \frac{1}{2}\theta}{a + b + (a-b)\tan^2 \frac{1}{2}\theta}$$

$$= \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left\{\tan \tfrac{1}{2}\theta \sqrt{\frac{a-b}{a+b}}\right\}.$$

Second, let a be less than b: then the given form becomes

$$\int \frac{d\theta}{a + b \cos \theta} = 2\int \frac{d \tan \frac{1}{2}\theta}{b + a - (b-a)\tan^2 \frac{1}{2}\theta}$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \,.\, \log_e \left\{\frac{\sqrt{b+a} + \sqrt{b-a} \,.\, \tan \frac{1}{2}\theta}{\sqrt{b+a} - \sqrt{b-a} \,.\, \tan \frac{1}{2}\theta}\right\}.$$

2. Again, for the form $\dfrac{d\theta}{a + b \sin \theta}$ we may pursue the following method.

First, let a be greater than b: then

$$\int \frac{d\theta}{a + b \sin \theta} = \int \frac{d\theta}{a(\cos^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\theta) + 2b \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta}$$

$$= \int \frac{a \sec^2 \frac{1}{2}\theta}{a^2 + a^2 \tan^2 \frac{1}{2}\theta + 2ab \tan \frac{1}{2}\theta} = \int \frac{d(b + a \tan \frac{1}{2}\theta)}{a^2 - b^2 + (b + a \tan \frac{1}{2}\theta)^2}$$

$$= \frac{2}{\sqrt{a^2 - b^2}} \,.\, \tan^{-1}\left\{\frac{b + a \tan \frac{1}{2}\theta}{\sqrt{a^2 - b^2}}\right\}$$

Second, let b be greater than a: then

$$\int \frac{d\theta}{a + b\sin\theta} = 2\int \frac{d\,(b + a\tan\frac{1}{2}\theta)}{(b + a\tan\frac{1}{2}\theta)^2 - (b^2 - a^2)}$$

$$= \frac{1}{\sqrt{b^2 - a^2}}\,.\log_e \left\{\frac{b + a\tan\frac{1}{2}\theta - \sqrt{b^2 - a^2}}{b + a\tan\frac{1}{2}\theta + \sqrt{b^2 - a^2}}\right\}.$$

3. The form $\int \frac{d\theta}{a + b\cos\theta + c\cos\theta}$ is reducible to the preceding cases.

For make $e = \sqrt{b^2 + c^2}$ and $\frac{c}{b} = \tan\alpha$; then

$$\int \frac{d\theta}{a + b\cos\theta + c\sin\theta} = \int \frac{bd\,(\theta - \alpha)}{a + e\cos(\theta - \alpha)}.$$

4. Again, let $\frac{d\theta}{(a + b\cos\theta)^n}$ be proposed for integration.

Now, $\int \frac{d\theta}{(a + b\cos\theta)^n} = \int \frac{\sin\theta\, d\theta}{\sin\theta\,(a + b\cos\theta)^n} = \frac{1}{b}\int \frac{d\,(a + b\cos\theta)}{\sin\theta\,(a + b\cos\theta)^n}$;
and putting $\Theta = a + b\cos\theta$, it takes the form

$$\int \frac{d\theta}{(a + b\cos\theta)^n} = \frac{1}{b}\int \frac{d\Theta}{\sin\theta.\Theta^n} = -\frac{1}{(n - 1)\,b}\int \frac{d\Theta^{-n+1}}{\sin\theta.}.$$

Integrating this by parts, we have

$$\int \frac{d\theta}{(a + b\cos\theta)^n} = -\frac{b\sin\theta}{(n - 1)(a^2 - b^2)\,\Theta^{n-1}} + \frac{(2n - 3)\,a}{(n - 1)(a^2 - b^2)}\int \frac{d\theta}{\Theta^{n-1}}$$

$$- \frac{n - 2}{(n - 1)(a^2 - b^2)}\int \frac{d\theta}{\Theta^{n-2}};$$

which, when n is a positive integer, will finally reduce to the first example at the head of this proposition.

As a special example, let $n = 2$; then

$$\int \frac{d\theta}{(a + b\cos\theta)^2} = \frac{1}{a^2 - b^2}\left\{\frac{-\,b\sin\theta}{a + b\cos\theta} + \frac{2a}{\sqrt{a^2 - b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\tfrac{1}{2}\theta\right)\right\}$$

5. Again, had $\int \frac{(a_1 + b_1\cos\theta)\,d\theta}{(a + b\cos\theta)^n}$, or $\int \frac{(a_1 + b_1\sin\theta)\,d\theta}{(a + b\cos\theta)^n}$ been proposed, we should still be able to reduce them to the preceding case. For,

$$\int \frac{(a_1 + b_1\cos\theta)\,d\theta}{(a + b\cos\theta)^n} = \int \frac{\left\{a_1 - \frac{b_1 a}{b} + \frac{b_1}{b}(a + b\cos\theta)\right\}d\theta}{(a + b\cos\theta)^n}$$

$$= \frac{ba_1 - ab_1}{b}\int \frac{d\theta}{(a+b\cos\theta)^n} + \frac{b_1}{b}\int \frac{d\theta}{(a+b\cos\theta)^{n-1}}$$

and in a similar way,

6. $$\int \frac{(a_1 + b_1\sin\theta)\,d\theta}{(a + b\cos\theta)^n} = \int \frac{b_1\sin\theta\,d\theta}{(a + b\cos\theta)^n} + a_1\int \frac{d\theta}{(a + b\cos\theta)^n}$$

$$= \frac{b_1}{(n - 1)\,b}\cdot\frac{1}{(a + b\cos\theta)^{n-1}} + a_1\int \frac{d\theta}{(a + b\cos\theta)^n}$$

7. $$\int \frac{d\theta}{a\cos^2\theta + b\sin^2\theta} = \int \frac{\sec^2\theta\,d\theta}{a + b\tan^2\theta} = \int \frac{d\tan\theta}{a + b\tan^2\theta} = \frac{\tan^{-1}\left(\tan x\sqrt{\frac{b}{a}}\right)}{\sqrt{ab}}$$

8. $$\int\frac{d\theta}{a+b\tan\theta}=\frac{1}{a^2+b^2}\int\left\{b\,.\,\frac{-a\sin\theta+b\cos\theta}{a\cos\theta+b\sin\theta}+a\right\}d\theta$$

$$=\frac{b\log_e(a\cos\theta+b\sin\theta)+a\theta}{a^2+b^2}.$$

9. $$\int\frac{d\theta}{a+b\sin^2\theta}=\int\frac{-d\cot\theta}{a+b+a\cot^2\theta}=\frac{1}{\sqrt{a}\,(a+b)}.\cot^{-1}\frac{\cot\theta\sqrt{a}}{\sqrt{a+b}}.$$

10. $$\int\frac{\sin\theta\cos^2\theta\,d\theta}{1+c^2\cos^2\theta}=\frac{1}{c^2}\int\frac{\sin\theta\,\{(1+c^2\cos^2\theta)-1\}d\theta}{1+c^2\cos^2\theta}$$

$$=\frac{1}{c^2}\int\left\{\sin\theta\,d\theta+\frac{d\cos\theta}{1+c^2\cos^2\theta}\right\}=-\frac{\cos\theta}{c^2}+\frac{\tan^{-1}(c\cos\theta)}{c^3}$$

11. $$\int\frac{\cos^3\theta\,d\theta}{1-c^2\cos^2\theta}=\frac{1}{c^2}\int\frac{\cos\theta\,\{1-(1-c^2\cos^2\theta)\}\,d\theta}{1-c^2\cos^2\theta}$$

$$=\frac{1}{c^2}\int\frac{d\sin\theta}{1-c^2\cos^2\theta}-\frac{1}{c^2}\int\cos\theta.d\theta$$

$$=\frac{1}{c^2\sqrt{1-c^2}}\tan^{-1}\frac{c\sin\theta}{\sqrt{1-c^2}}-\frac{\sin\theta}{c^2}.$$

12. $$\int\frac{(a+b\sin^2\theta)^{\frac{1}{2}}\,d\theta}{\sin\theta}=\int\frac{(a+b\sin^2\theta)\,d\theta}{\sin\theta\sqrt{a+b\sin^2\theta}}$$

$$=\int\frac{a\,d\theta}{\sin^2\theta\sqrt{b+a\operatorname{cosec}^2\theta}}+\int\frac{b\sin\theta\,d\theta}{\sqrt{a+b\sin^2\theta}}$$

$$=-\int\frac{a\,d\cot\theta}{\sqrt{a+b+a\cot^2\theta}}-\int\frac{b\,d\cos\theta}{\sqrt{a+b-b\cos^2\theta}}$$

$$=-\sqrt{a}\,.\log_e\{\sqrt{a}\cot\theta+\sqrt{a\operatorname{cosec}^2\theta+b}\}+\sqrt{b}\,.\cos^{-1}\frac{\sqrt{b}\,.\cos\theta}{\sqrt{a+b}}.$$

13. $\int\frac{dx}{(b^2+x^2)\sqrt{a^2+x^2}}$, an algebraic function is proposed. Put $\frac{x}{b}=\tan\theta$.

$$\int\frac{dx}{(b^2+x^2)\sqrt{a^2+x^2}}=\frac{1}{b}\int\frac{d\tan^{-1}\frac{x}{b}}{\sqrt{a^2+x^2}}=\frac{1}{b}\int\frac{d\theta}{\sqrt{a^2+b^2\tan^2\theta}}$$

$$=\frac{1}{b}\int\frac{d\sin\theta}{\sqrt{a^2-(a^2-b^2)\sin^2\theta}}=\frac{1}{b\sqrt{a^2-b^2}}.\sin^{-1}\frac{\sin\theta\sqrt{a^2-b^2}}{a}$$

$$=\frac{1}{b\sqrt{a^2-b^2}}.\sin^{-1}\frac{x\sqrt{a^2-b^2}}{a\sqrt{x^2+b^2}}.$$

14. Let $\frac{(1-x^2)^{\frac{1}{2}}dx}{(1+x)^2}$ be proposed; and put $x=\cos\theta$: then

$$\int\frac{(1-x^2)^{\frac{1}{2}}dx}{(1+x)^2}=\int\frac{1-x}{1+x}.\frac{dx}{\sqrt{1-x^2}}=-\int\tan^2\tfrac{1}{2}\theta\,d\theta$$

$$=\int(1-\sec^2\tfrac{1}{2}\theta)\,d\theta=\theta-2\tan\tfrac{1}{2}\theta=\cos^{-1}x-2\sqrt{\frac{1-x}{1+x}}.$$

The two last examples illustrate a method which can often be employed for transforming algebraical to trigonometrical forms, so as greatly to facilitate the integration. All the forms given in this proposition deserve the student's careful attention, from their very frequent occurrence in the application of analysis to physical inquiries.

CHAPTER VII.

INTEGRATION BY INFINITE SERIES.

When an expression is given for integration to which none of the preceding methods are applicable, we have no alternative but to expand it into an infinite series of monomes, integrate each term separately, and finally to sum, or approximate to the sum of, this integrated series *.

All the functions that ordinarily occur in analysis being such that we know how to develope by appropriate rules, there can arise no great difficulty in the application of this method: though, at the same time, by the proper combination of the elements of the function, the developement may be made to take a more or less simple form. Attentive observation, alone, and that often the result of several successive experiments, can guide the student in his choice of such combinations as are likely to lead to the most elegant and simple results. Rules are useless, because none of them can be rendered sufficiently general. There is, however, one very remarkable theorem discovered by John Bernoulli, for the *expansion of an integral by successive differentiation:* and it has been characterised as playing the same part in the integral developement that Taylor's theorem does in the differential. On account, however, of its most generally giving either a very slowly convergent series (sometimes a neutral one), or a divergent series, it is not often useful in practice: but it shows at least that every integral of a single variable is determinable in a series.

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* The summation of series is a branch of analysis which has been only very partially developed; and with the exception of a comparatively small number of cases, the attempts that have been made, have been upon mere isolated and small classes of series, and those of not very frequent occurrence in the applications of analysis to physical enquiries. However, though such attempts have generally failed in giving the function, which, being itself in a finite form, becomes, upon expansion, that series, the circumstance is of less real importance than might at first sight seem to be the case: for in the great majority of cases, it is not the general corrected integral, but the integral between given limits, that is required in physical research. Wherefore, since we can generally express these in a *numerical* infinite series, it will be sufficient to be able to approximate to these with a degree of rapidity which shall not require more than a moderate degree of labour. To meet the difficulty under this point of view, several of the most eminent analysts have paid much attention to the transformation of slowly converging series into others which shall be more rapid. Into none of these subjects, however, shall we be here able to enter, even in the slightest manner. They would, in fact, be in a great degree without interest, except in connexion with the problems which give rise to the special cases; and in works where such inquiries naturally arise, they are always treated with sufficient amplification.

We shall commence by investigating Bernoulli's series[*] for the integral $\int X dx$: viz.

$$\int X dx = X \cdot \frac{x}{1} - \frac{dX}{dx} \cdot \frac{x^2}{1 \cdot 2} + \frac{d^2X}{dx^2} \cdot \frac{x^3}{1 \cdot 2 \cdot 3} - \frac{d^3X}{dx^3} \cdot \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots .$$

X being any developable function of x.

Let us denote $\int X dx$ by $f(x)$: then, by Taylor's theorem,

$$f(x-h) = \int X dx - X \cdot \frac{h}{1} + \frac{dX}{dx} \cdot \frac{h^2}{1 \cdot 2} - \frac{d^2X}{dx^2} \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \ldots \ldots$$

which is a developement for all values of h. Let then $h = x$, and we have, since $f(x-x) = f(0)$ in this case.

$$f(0) = \int X dx - X \cdot \frac{x}{1} + \frac{dX}{dx} \cdot \frac{x^2}{1 \cdot 2} - \frac{d^2X}{dx^2} \cdot \frac{x^3}{1 \cdot 2 \cdot 3} - \ldots \ldots$$

$$\text{or } \int X dx = X \cdot \frac{x}{1} - \frac{dX}{dx} \cdot \frac{x^2}{1 \cdot 2} + \frac{d^2X}{dx^2} \cdot \frac{x^3}{1 \cdot 2 \cdot 3} - \ \ldots + f(0),$$

where $f(0)$ is a constant, and evidently the correction or arbitrary constant C already spoken of.

Scholium.

Professor Young (*Integral calculus*, p. 81) has given the following more useful series for the integral $\int X dx$: viz.

$$\int X dx = U_0 + U_1 \cdot \frac{x}{1} + U_2 \cdot \frac{x^2}{1 \cdot 2} + U_3 \cdot \frac{x^3}{1 \cdot 2 \cdot 3} + \ldots .$$

where U_0, U_1, U_2, ... are what $\int X dx$, X, $\frac{dX}{dx}$, ... become when $x = 0$.

This is derived from Maclaurin's series in nearly the same way that the preceding one was from Taylor's. For the application of this series we must refer

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[*] This may also be elegantly found in the following manner by partition :—

$$\int X dx = Xx - \int \frac{dX}{dx} \cdot x dx,$$

$$\int \frac{dX}{dx} \cdot x dx = \frac{x^2}{2} \cdot \frac{dX}{dx} - \frac{1}{2} \int \frac{d^2X}{dx^2} \cdot x^2 dx,$$

$$\int \frac{d^2X}{dx^2} \cdot x^2 dx = \frac{x^3}{3} \cdot \frac{d^2X}{dx^2} - \frac{1}{3} \int \frac{d^3X}{dx^3} \cdot x^3 dx,$$

$$\int \frac{d^3X}{dx^3} \cdot x^3 dx = \frac{x^4}{4} \cdot \frac{d^3X}{dx^3} - \frac{1}{4} \int \frac{d^4X}{dx^4} \cdot x^4 dx,$$

and so on: whence the series reduces to

$$\int X dx = \frac{x}{1} X - \frac{x^2}{1 \cdot 2} \cdot \frac{dX}{dx} + \frac{x^3}{1 \cdot 2 \cdot 3} \cdot \frac{d^2X}{dx^2} - \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{d^3X}{dx^3} + \ldots \pm \int \frac{d^nX}{dx^n} \frac{x^n}{1 \cdot 2 \cdot 3 \ldots\ldots n}$$

The following example will illustrate the method implied by this formula.

To integrate $X dx = ax^3 dx + bx^2 dx + cx dx + e dx$.

$$\frac{dX}{dx} = 3ax^2 + 2bx + c,$$

$$\frac{d^2X}{dx^2} = 6ax + 2b,$$

$$\frac{d^3X}{dx^3} = 6a,$$

$$\frac{d^4X}{dx^4} = 0.$$

Hence

$$\int X dx = (ax^3 + bx^2 + cx + e)\, x - \frac{x^2}{2}(3ax^2 + 2b\, x + c) + \frac{x^3}{6}(6ax + 2b) - \frac{x^4}{24} \cdot 6a$$

$$= \frac{ax^4}{4} + \frac{b\ \ {}^3}{3} + \frac{cx^2}{2} + \frac{ex}{1}.$$

to Mr. Young's work: at the same time expressing our belief that a more ample developement of this part of the subject may lead to important results.

We now proceed to give a few examples of integration by series, in several of which we shall determine the more simple expressions for the values of the integrals between assigned limits.

EXAMPLES.

1. To integrate $\frac{\sqrt{1-e^2x^2}}{\sqrt{1-x^2}}.dx$, which does not fall under any preceding case.

Expand the numerator by the binomial theorem: then the differential will be composed of a series of terms of the form $\frac{x^m dx}{\sqrt{1-x^2}}$, each of which is separately integrable. Now this expansion being performed, we have

$$\frac{\sqrt{1-e^2x^2}}{\sqrt{1-x^2}}\,dx = \frac{dx}{\sqrt{1-x^2}} - \frac{e^2x^2dx}{2\sqrt{1-x^2}} - \frac{e^4x^4dx}{2.4\sqrt{1-x^2}} - \frac{e^6x^6dx}{2.4.6\sqrt{1-x^2}} - \ldots.$$

Then integrating each term, we have

$$\int\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x$$

$$-\int\frac{e^2x^2dx}{2\sqrt{1-x^2}} = \frac{e^2}{2}\left\{\frac{x\sqrt{1-x^2}}{2} - \frac{\sin^{-1}x}{2}\right\}$$

$$-\frac{e^4x^4dx}{2.4\sqrt{1-x^4}} = \frac{e^4}{2.4}\left\{\left(\frac{x^3}{4} + \frac{1.3x}{2.4}\right)\sqrt{1-x^2} - \frac{1.3.\sin^{-1}x}{2.4}\right\}$$

$$-\int\frac{e^6x^6dx}{2.4\,6\sqrt{1-x^2}} = \frac{1.3\,e^6}{2.4.6}\left\{\left(\frac{x^5}{6} + \frac{1.5\,x^3}{4.6} + \frac{1.3.5x}{2.4.6}\right)\sqrt{1-x^2} - \frac{1.3.5\sin^{-1}x}{2.4\,6}\right\},$$

etc. *etc.* *etc.*

whose sum is the integral required.

Between the limits 0, 1, as values of x, we can readily find

$$\int_0^1\frac{\sqrt{1-e^2x^2}.dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\left\{1 - \frac{1.e^2}{2^2} + \frac{1.3e^4}{2^2.4^2} - \frac{1.3.5e^6}{2^2.4^2.6^2} + \ldots\right\}$$

since $x = 0$ renders every term 0, and $x = 1$ renders $\sin^{-1}x = \frac{1}{2}\pi$ and $\sqrt{1-x^2} = 0$.

2. To integrate $\frac{a^x dx}{1-x}$ by series.

Since $\frac{1}{1-x} = 1 + x + x^2 + \ldots.$ (by synthetic division),

and $a^x = 1 + \frac{Ax}{1} + \frac{A^2x^2}{1.2} + \ldots.$ (*vol. i. p.* 247, $A = \log_e a$), we have by actual multiplication

$$\frac{a^x}{1-x} = 1 + \left(1 + \frac{A}{1}\right)x + \left(1 + \frac{A}{1} + \frac{A^2}{1.2}\right)x^2 + \left(1 + \frac{A}{1} + \frac{A^2}{1.2} + \frac{A^3}{1.2.3}\right)x^3 + \ldots$$

Hence, multiplying by dx, and integrating term by term,

$$\int\frac{a^x\,dx}{1-x} = x + \left(1 + \frac{A}{1}\right)\frac{x^2}{2} + \left(1 + \frac{A}{1} + \frac{A^2}{1.2}\right)\frac{x^3}{3} + \left(1 + \frac{A}{1} + \frac{A^2}{1.2} + \frac{A^3}{1.2.3}\right)\frac{x^4}{4} + \ldots$$

Between the limits 0, 1, as values of x, this integral is infinitely great.

3. $$\int\frac{dx}{\sqrt{2x-x^2}} = \left\{2 + \frac{1}{2.3}\cdot\frac{x}{2^0} + \frac{1.3}{2.4.5}\cdot\frac{x^2}{2^1} + \frac{1.3.5}{2.4.6.7}\cdot\frac{x^3}{2^2} + \ldots\right\}\sqrt{\frac{x}{2}}.$$

This integral is also one of the elementary integrals, $\text{versin}^{-1}x$.

4. $\int \frac{dx}{\sqrt{2x+x^2}} = \left\{1 - \frac{1}{2\,3}\cdot\frac{x}{2} + \frac{1.3}{2.4.5}\cdot\frac{x^2}{2^2} - \frac{1.3.5}{2.4.6.7}\cdot\frac{x^3}{2^3} + \ldots\right\}\sqrt{2x}$; which is also equal to $\log_e(x + 1 + \sqrt{2x - x^2})$, as may be investigated in the appropriate manner.

5. $\int (a+bx^n)^{\frac{p}{q}}\, x^{m-1}\, dx = a^{\frac{p}{q}} \left\{\frac{x^m}{m} + \frac{pb}{qa}\cdot\frac{x^{m+n}}{m+n} + \frac{p\,(p-q)\,b^2}{1.2.q^2a^2}\cdot\frac{x^{m+2n}}{m+2n} + \ldots\right\}$

6. $\int (a+bx^n)^{\frac{p}{q}}\, x^{m-1}\, dx = \int \left(b + \frac{a}{x^n}\right)^{\frac{p}{q}} x^{\frac{mq+np}{q}-1}\, dx.$

$$= qb^{\frac{p}{q}} \left\{\frac{x^{\frac{mq+np}{q}}}{mq+np} + \frac{pa}{qb}\cdot\frac{x^{mq+n(p-q)}}{mq+n(p-q)} + \frac{p\,(p-q)\,a^2}{1.2.q^2b^2}\cdot\frac{x^{mq+n(p-2q)}}{mq+n\,(p-2q)} + \ldots\right\}$$

7. $\int \frac{dx}{\log_e x} = \log_e{}^2 x + \log_e x + \frac{1}{1.2}\cdot\frac{\log_e{}^2 x}{2} + \frac{1}{1.2.3}\cdot\frac{\log_e{}^3 x}{3} + \ldots$

8. To integrate $\frac{dx}{\sqrt{(h-x)\,(2ax-x^2)}}$ between 0 and h.

Put it in the form $\frac{1}{\sqrt{2a}}\cdot\frac{dx}{\sqrt{hx-x^2}}\cdot\left\{1 - \frac{x}{2a}\right\}^{-\frac{1}{2}}$, and develope by the binomial theorem; and we shall obtain a series of terms of the general form (abating the co-efficients) $\frac{x^m\, dx}{\sqrt{hx-x^2}}$, which are separately integrable by the preceding methods.

Now,

$$\left(1 - \frac{x}{2a}\right)^{-\frac{1}{2}} = 1 + \frac{1}{2}\cdot\frac{x}{2a} + \frac{1.3}{2.4}\cdot\frac{x^2}{4a^2} + \frac{1.3.5}{2.4.6}\cdot\frac{x^3}{8a^3} + \ldots$$

Multiply each term by $\frac{dx}{\sqrt{2a}.\sqrt{hx-x^2}}$, and integrate; then we have, as the elements of the result, putting $X = \sqrt{hx - x^2}$,

$$\begin{array}{l|l}
\int \frac{dx}{X} = \text{vers}^{-1}\,\frac{2x}{h}. & \int_0^h \frac{dx}{X} = \varpi \\
\int \frac{xdx}{X} = -X + \frac{h}{2}\int\frac{dx}{X} & \int_0^h \frac{xdx}{X} = \frac{h}{2}.\varpi \\
\int \frac{x^2dx}{X} = -\frac{xX}{2} + \frac{3h}{4}\int\frac{xdx}{X} & \int_0^h \frac{x^2dx}{X} = \frac{1.3.h^2}{2.4}.\varpi \\
\int \frac{x^3dx}{X} = -\frac{x^2X}{3} + \frac{5h}{6}\int\frac{x^2dx}{X} & \int_0^h \frac{x^3dx}{X} = \frac{1.3.5.h^3}{2.4.6}.\varpi \\
\quad \textit{etc.} \qquad\qquad \textit{etc.} & \quad \textit{etc.} \qquad\qquad \textit{etc.}
\end{array}$$

Hence the definite integral between the assigned limits is

$$\left\{1 + \left(\frac{1}{2}\right)^2\cdot\frac{h}{2a} + \left(\frac{1.3}{2.4}\right)^2\cdot\frac{h^2}{(2a)^2} + \left(\frac{1.3.5}{2.4.6}\right)^2\cdot\frac{h^3}{(2a)^3} + \ldots\right\}\frac{\varpi}{\sqrt{2a}}.$$

8. To definitely integrate $du = \frac{-\,rdx}{\sqrt{(r^2-x^2)\,(c+4mx)-c'^2}}$, the limits being a and b, determined as follows.

Assume $(r^2 - x^2)(c + 4mx) - c'^2 = (a - x)(x - b)(4mx + f)$, where a, b, f, are determinable from the three equations respectively,

$$f = \frac{4m(r^2 + ab)}{a + b};\ c = \frac{4m(r^2 - a^2 - ab - b^2)}{a + b};\ c'^2 = \frac{4m(r^2 - a^2)(r^2 - b^2)}{a + b}$$

Take $\sin\theta = \sqrt{\frac{a - x}{a - b}}$, and hence $\cos\theta = \sqrt{\frac{x - b}{a - b}}$; wherefore,

$$x = a\cos^2\theta + b\sin^2\theta = a - (a - b)\cos^2\theta = b + (a - b)\sin^2\theta.$$

Hence,

$$dx = -2(a - b)\sin\theta\cos\theta\, d\theta$$

$$a - x = (a - b)\sin^2\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a),$$

$$x - b = (b - a)\cos^2\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b),$$

$$4mx + f = 4m\{a - (a - b)\sin^2\theta\} + f;$$

and making these substitutions, the differential becomes

$$du = \frac{2r\,d\theta}{\sqrt{4m\{a - (a - b)\sin^2\theta\} + \frac{r^2 + ab}{a + b}}};\ \text{or}\ \left(f\gamma^2 = \frac{a^2 - b^2}{a^2 + 2ab + r^2}\right)$$

$$= \frac{r\sqrt{a + b}}{\sqrt{m(a^2 + 2ab + r^2)}} \cdot \frac{d\theta}{\sqrt{1 - \gamma^2\sin^2\theta}}.$$

Develope the differential factor of this by the binomial theorem; then we have

$$du = \left\{1 + \frac{1}{2}\cdot\gamma^2\sin^2\theta + \frac{1.3}{2.4}\gamma^4\sin^4\theta + \frac{1.3.5}{2.4.6}\gamma^6\sin^6\theta + \ldots\right\}d\theta.$$

For the limits a and b we have the corresponding limits of θ, viz. 0 and $\frac{1}{2}\pi$; since for $x = a$ and $x = b$, we have, from $x = a\cos^2\theta + b\sin^2\theta$, respectively $\sin\theta = 0$ and $\cos\theta = 0$, or $\theta = 0$ and $\theta = \frac{1}{2}\pi$. Taking, therefore, the general integrals of the terms of the series, and likewise their values between these limits, we have

$$\int d\theta = \theta$$

$$\int \sin^2\theta\, d\theta = -\cos\theta \cdot \frac{\sin\theta}{2} + \frac{\theta}{2}.$$

$$\int \sin^4\theta\, d\theta = -\cos\theta\left\{\frac{\sin^3\theta}{3} + \frac{2\sin\theta}{1.3}\right\} + \frac{1.3}{2.4}.\theta$$

$$\int \sin^6\theta\, d\theta = -\cos\theta\left\{\frac{\sin^5\theta}{5} + \frac{4\sin^3\theta}{3.5} + \frac{2.4\sin\theta}{1.3.5}\right\} + \frac{1.3.5}{2.4.6}\theta,$$

etc. etc. etc.

The definite integral, therefore, becomes

$$u = \frac{\varpi}{2}\cdot\frac{r\gamma}{\sqrt{m(a - b)}}\left\{1 + \left(\frac{1}{2}\right)^2\gamma^2 + \left(\frac{1.3}{2.4}\right)^2\gamma^4 + \left(\frac{1.3.5}{2.4.6}\right)^2\gamma^6 + \ldots\right\}$$

9. $\int_0^\pi e^{2a\cos\theta}\, d\theta = \pi\left\{1 + \frac{a^2}{1} + \left(\frac{a^2}{1.2}\right)^2 + \left(\frac{a^2}{1.2.3}\right)^3 + \ldots\right\}$

10. $\int_1^0 \frac{\log_e x\,dx}{\sqrt{1 - x^2}} = 1 + \frac{1}{2.3^2} + \frac{1.3}{2.4.5^2} + \frac{1.3.5}{2.4.6.7^2} + \cdots$

11. To establish the relation $\int_0^1 \frac{x^{p-1}\,dx}{(1-x^n)^{\frac{n-q}{n}}} = \int_1^0 \frac{x^{q-1}\,dx}{(1-x^n)^{\frac{n-p}{n}}}$.

Put $1 - x^n = y^n$; then the limits are interchanged.
Also, under the same substitutions we have

$(1-x^n)^{\frac{n-q}{n}} = y^{n-q}$; $x^p = (1-y^n)^{\frac{p}{n}}$; and $x^{p-1}\,dx = -\,y^{n-1}\,(1-y^n)^{\frac{p-n}{n}}\,dy$.

Wherefore, generally,

$$\int \frac{x^{p-1}\,dx}{(1-x^n)^{\frac{n-q}{n}}} = -\int \frac{(1-y^n)^{\frac{p-n}{n}}\,y^{n-1}\,dy}{y^{n-q}} = -\int \frac{y^{q-1}\,dy}{(1-y^n)^{\frac{n-p}{n}}}.$$

Taking these respectively at their prescribed limits, we get the theorem in question.

12. $\int_0^a \frac{dx}{\sqrt{x-x^3}} = \int_1^{a'} \frac{dx}{\sqrt{x-x^3}}$, where $a' = \frac{1-a}{1+a}$.

13. From the integrated series establish the following:

$$\int_0^{\frac{1}{0}} \frac{dx}{1+x^2} = \frac{\pi}{2};\ \int_0^{\frac{1}{0}} \frac{dx}{1+x^3} = \frac{2\pi}{3\sqrt{3}};\ \int_0^{\frac{1}{0}} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}};\ \int_0^{\frac{1}{0}} \frac{dx}{1+x^6} = \frac{\pi}{3}.$$

CHAPTER VIII.

ON SUCCESSIVE INTEGRATION.

It has been seen (page 435) that by every successive differentiation one arbitrary constant (or constant co-efficient) disappears from the expression, and that the nth differential co-efficient has n fewer such constants than the primitive function had: and it hence follows, conversely, that in returning by successive integrations from the nth differential co-efficient to the primitive function or final integral, we must restore the n arbitrary constants, one at each step of the process.

Let u, a function of x, denote the primitive function of x, the exact form of which is at present unknown, but such that $\frac{d^n u}{dx^n} = \mathrm{X}$, where X, a function of x, is actually given. Then, since in forming this differential co-efficient dx has been treated as constant, or $d^2x = 0$, we may write the given equation

$$\frac{d^n u}{dx^n} = \mathrm{X} \text{ in the form } \frac{d^n u}{dx^{n-1}} = \mathrm{X}\,dx, \text{ or } d\,\frac{d^{n-1}u}{dx^{n-1}} = \mathrm{X}\,dx.$$

Whence, integrating, and considering dx^{n-1} constant, we have

$$\int \frac{d^n u}{dx^{n-1}} = \int d\,\frac{d^{n-1}u}{dx^{n-1}} = \int \mathrm{X}dx = \mathrm{X}_1 + c_1,$$

where X_1 is written briefly to represent $\int \mathrm{X}dx$, and c_1 is the arbitrary constant introduced by integration. In the same way we have *, putting $X_2 = \int X_1 dx$, and c_2 the constant introduced,

$$\int^2 \frac{d^n u}{dx^{n-2}} = \int d \frac{d^{n-2}u}{dx^{n-2}} = \int X_1 dx + c_1 \int dx = X_2 + c_1 x + c_2.$$

$$\int^3 \frac{d^n u}{dx^{n-3}} = \int d \frac{d^{n-3}u}{dx^{n-3}} = \int X_2 dx + c_1 \int x dx + c_2 \int dx$$

$$= X_3 + \frac{c_1 x^2}{2} + \frac{c_2 x}{1} + c_3.$$

Proceeding thus, we finally obtain the nth integral, viz.

$$\int^n d^n u = X_n + \frac{c_1 x^{n-1}}{1.2...(n-1)} + \frac{c_2 x^{n-2}}{1.2...(n-2)} + \dots \frac{c_{n-2} x^2}{1.2} + \frac{c_{n-1} x}{1} + c_n.$$

Corollary 1.

In the applications of this process, any problem which gives rise to the nth differential co-efficient as a part of the expression of its conditions, must also involve amongst its conditions such other relations as shall enable us to determine the n arbitrary constants, which integration introduces into the final expression. Such problem is otherwise not completely limited in its conditions.

Corollary 2.

We may readily obtain the developement of $\int^n X dx^n$ as follows: —

By Maclaurin's theorem.

$$\int^n X dx^n = U_{-n} + U_{-(n-1)} . \frac{x}{1} + U_{-(n-2)} . \frac{x^2}{1.2} + \dots U_{-1} . \frac{x^{n-1}}{1.2...(n-1)}$$

$$+ U_0 . \frac{x^n}{1.2...n} + U_1 . \frac{x^{n+1}}{1.2...(n+1)} + U_2 . \frac{x^{n+2}}{1.2...(n+2)} + \dots$$

in which

$U_{-n}, U_{-(n-1)} \dots, U_{-1}\ U_0, U_1, U_2, \dots$ are the particular values of

$\int^n X dx^n, \int^{n-1} X dx^{n-1}, \dots \int X dx, X, \frac{dX}{dx}, \frac{d^2X}{dx^2}$, when $x = 0$.

The functions $U_{-1}, U_{-2}, \dots$ are evidently the constants $c_1, c_2, \dots c_n$ introduced by integration.

This formula shows that when $\int X\, dx$, or X, is developable in increasing positive powers of x, the function $\int^n X\, dx^n$ can also be developed in the same manner; and hence, that in such cases it is scarcely more laborious to obtain the nth integral $\int^n X\, dx^n$ than to obtain $\int X\, dx$ itself. The following rule expresses the process implied in the formula.

Develope X by the process appropriate to its form, and then for x^0, x^1, x^2, x^3, etc. given by that developement, write

$$\frac{x^n}{1.2...n}, \quad \frac{x^{n+1}}{1.2...(n+1)}, \quad \frac{x^{n+2}}{1.2...(n+2)}, \quad \frac{x^{n+3}}{1.2...(n+3)}, \textit{ etc.}$$

and then annex the terms dependent on the arbitrary constants, as found in the general proposition itself.

(*Young, Integ. Calc. p.* 91.)

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* $\int^2, \int^3, \dots \int^n$ denote the second, third, ... nth integrals of the differential expression to which they are prefixed,

Scholium.

Since whatever be the form of X, provided only that it be a function of a single variable x, the corrections only introduce an algebraic and integer function of the $(n-1)^{\text{th}}$ degree, we are entitled to infer that direct differentiation never cancels any constants, but the co-efficients of the algebraic parts of the primitive function, and of these none higher than the $(n-1)^{\text{th}}$ powers of x. By recurring to the elementary integrals of the exponential, logarithmic, and trigonometrical functions (the only ones yet introduced into analysis bearing on ultimate calculation, and referring to a single variable), we shall see that, *à priori*, such a conclusion might have been inferred.

Of functions of two or more independent variables, and the arbitrary functions of those variables, by which the arbitrary corrections may be expressed, this is not the place to speak; as they, and especially such applications as could give interest to the subject, are foreign to the objects of this volume.

EXAMPLES.

1. Let $\frac{d^4u}{dx^4} = \frac{1}{\sqrt{1+x^2}}$: required to find the primitive function.

Here $\mathrm{X} = \frac{1}{\sqrt{1+x^2}} = 1 - \frac{1}{2} \cdot x^2 + \frac{1.3}{2.4} x^4 - \frac{1.3.5}{2.4.6} \cdot x^6 + \ldots.$

Insert in this series, according to the rule, for $x^0, x^2, x^4, x^6 \ldots$ the quantities,

$$\frac{x^4}{1.2.3.4}, \quad \frac{x^6}{3.4.5.6}, \quad \frac{x^8}{5.6.7.8}, \quad \frac{x^{10}}{7.8.9.10}, \ldots; \text{ then,}$$

$$u = \int^4 \frac{d^4u}{dx^4} = \frac{x^4}{2.3.4} - \frac{1.x^6}{2.3.4.5.6} + \frac{1.3.x^8}{2.4.5.6.7.8} - \frac{1.3.5.x^{10}}{2.4.6.7.8.9.10} + \ldots$$

$$+ \frac{c_1x^3}{1.2.3} + \frac{c_2x^2}{1.2} + \frac{c_3x}{1} + c_4.$$

2. Let $\frac{d^3u}{d\theta^3} = \sin\theta$: required the primitive function.

In this case, to avoid the infinite series for $\sin\theta$, let us actually integrate: then,

$$\int d\frac{d^2u}{d\theta^2} = \int \sin\theta\, d\theta = -\cos\theta,$$

$$\int d\frac{du}{d\theta} = -\int \cos\theta\, d\theta = -\sin\theta,$$

$$\int du = -\int \sin\theta\, d\theta = +\cos\theta;$$

and hence, completing the integral by annexing the parts depending on the constants, we have

$$u = \int^3 \sin\theta\, d\theta^3 = \cos\theta + \frac{c_1\theta^2}{1.2} + \frac{c_2\theta}{1} + c_3.$$

But we may, by means of Young's rule, effect the integration with great simplicity. For,

$$\frac{d^3u}{d\theta^3} = \sin\theta = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2\ldots5} - \frac{\theta^7}{1.2\ldots7} + \frac{\theta^9}{1.2\ldots9} - \ldots$$

Applying that rule, and putting the terms dependent on the arbitrary constants first, we have

$$u = c_3 + \frac{c_2\theta}{1} + \frac{c_1\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \frac{\theta^6}{1.2\ldots6} + \frac{\theta^8}{1.2\ldots8} - \ldots.$$

$$= (c_3 - 1) + \frac{c_2\theta}{1} + \frac{(c_1 + 1)\,\theta^2}{1.2} + \left\{1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2..4} - \frac{\theta^6}{1.2\ldots6} + \ldots\right\}$$

$$= (c_3 - 1) + \frac{c_2\theta}{1} + \frac{(c_1 + 1)\,\theta^2}{1.2} + \cos\theta, \text{ the same general form.}$$

3. Let $\frac{d^2u}{dx^2} = \frac{a^2 - x^2}{(x^2 + a^2)^2}$: find the primitive function.

In this case we have, by the method of rational fractions

$$\frac{du}{dx} = \int \frac{(x^2 - a^2)\,dx}{(x^2 + a^2)^2} = -2\int \frac{a^2\,dx}{(x^2 + a^2)^2} + \int \frac{dx}{x^2 + a^2}$$

$$= -\frac{x}{x^2 + a^2} - \int \frac{dx}{x^2 + a^2} + \int \frac{dx}{x^2 + a^2} + c_1$$

$$= -\frac{x}{x^2 + a^2} + c_1. \text{ Wherefore,}$$

$$u = -\int \frac{x\,dx}{x^2 + a^2} + \int c_1\,dx = -\log_e \sqrt{x^2 + a^2} + c_1 x + c_2.$$

The student is recommended to apply Young's rule to this, as an exercise; though it may be remarked, that the great excellence of that rule is its facile application to cases in which the function cannot be integrated without infinite series. However, even in such cases as the present, it is only the accidental circumstance of the infinite series being expressed by one of the logarithmic, exponential, or trigonometrical functions, that we are enabled to avoid the direct and formal use of the series. In fact, they are series, though of peculiar kinds, which we are enabled to avoid in their expanded state by means of certain properties of the differentials of their developements.

4. Show that

$$\int^4 \cos\theta\,d\theta^4 = \cos\theta + \frac{c_1\theta^3}{1.2.3} + \frac{c_2\theta^2}{1.2} + \frac{c_3\theta}{1} + c_4.$$

5. If $\frac{d^4u}{dx^4} = 0$: then $u = \frac{c_1x^3}{1.2.3} + \frac{c_2x^2}{1.2} + \frac{c_3x}{1} + c_4$.

For, $\int d\,\frac{d^3u}{dx^3} = \int 0.dx$, or $\frac{d^3u}{dx^3} = c_1$; and hence the rest follows.

6. $$\int^4 \frac{dx^4}{\sqrt{1 - x^2}} = \frac{x^4}{2.3.4} + \frac{1.x^6}{2.3\ldots6} + \frac{1.3.x^8}{2.3\ldots8} + \frac{1.3.5.x^{10}}{2.3\ldots.10} + \ldots$$

$$+ \frac{c_1x^3}{1.2.3} + \frac{c_2\,x^3}{1.2} + \frac{c_3x}{1} + c_4.$$

7. $\int^3 \frac{4dx^3}{\sqrt{1 + x^3}} = \sec^{-1} x.$

8. $\int^n \frac{e^{mx}\sin(x + n\cot^{-1} m)\,dx^n}{\sin^n \cot^{-1} m} = e^{mx}\sin x.$

9. If $d^2u = \frac{u\,(u - 2x)\,dx^2}{(u - x)^3}$: then $u^2 - 2ux + a^2 = 0$.

CHAPTER IX.

THE LENGTHS AND AREAS OF CURVES.

PROP. I.

To find the length of a given curve.

By (*p.* 404) the rectangular differential of the length is

$$ds = \sqrt{dx^2 + dy^2} \dots\dots (1),$$

and (*p.* 404) the polar expression for the same differential is

$$ds = \sqrt{dr^2 + r^2 d\theta^2} \dots\dots (2).$$

Another formula for the length of a polar curve, which is little known, but often useful, may be added. It is easily deduced from (2) by simple transformation.

Let $r\theta$ be the current polar co-ordinates, p the perpendicular from the polar origin on the tangent at $r\theta$, and t the portion of the tangent intercepted between the perpendicular and the point of contact.

$$ds = pd\theta \pm dt \dots\dots (3).$$

By (*p.* 397) we have

$$p = \frac{r^2 d\theta}{ds},\ t = \frac{\pm rdr}{ds} \text{ and } r^2 = p^2 + t^2.$$

From the first and third of these,

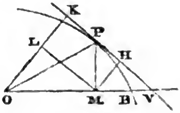
$$ds = \frac{r^2 d\theta}{p} = \frac{p^2 d\theta}{p} + \frac{t^2 d\theta}{p} = pd\theta + \frac{t^2 d\theta}{p}.$$

Also from the first and second,

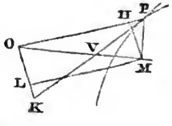
$$\frac{t^2 d\theta}{p} = \frac{r^2 dr^2 d\theta}{ds^2} \cdot \frac{ds}{r^2 d\theta} = \frac{dr^2}{ds} = d\left(\frac{rdr}{ds}\right) = \pm\, dt.$$

Whence $ds = pd\theta + \dfrac{t^2 d\theta}{p} = pd\theta \pm dt.$

The upper or lower sign to be used according as the radius-vector and perpendicular are on different sides of the polar axis or not.



Instead of the angle θ, or POV, of this theorem, we may substitute the angle VOK: for it will be evident that a differential change in θ makes only a differential change in VOK (the β of our former equations), of equal value. It may, however, be well to investigate it under this form from some fundamental equation. We shall deduce it from the rectangular equation instead of the polar, for the sake of exhibiting variety of method.



Draw MH, ML perpendicular to KP and OK. Then

OK $=$ OL $\pm$ LK $=$ OM cos MOL $\pm$ PM sin MPH;

and

KP $=$ KH $\mp$ HP $=$ OM sin MOL $\mp$ PM cos MPH;

$$\text{or } p = x\cos\beta \pm y\sin\beta, \text{ and } t = x\sin\beta \mp y\cos\beta.$$

Also $\dfrac{dy}{dx} = \mp \cot\beta$, and $\dfrac{ds}{dx} = \mp \operatorname{cosec}\beta$, by the general property.

Again, from the first of the above equations, and using the three others for reduction, we have

$$\frac{dp}{d\beta} = \cos\beta\,\frac{dx}{d\beta} - x\sin\beta - \cot\beta\,\frac{dx}{d\beta}\,.\,\sin\beta \pm y\cos\beta.$$

$$= - x\sin\beta \pm y\cos\beta = -t.$$

$$\frac{d^2p}{d\beta^2} = - x\cos\beta - \sin\beta\,.\,\frac{dx}{d\beta} \mp y\sin\beta - \cot\beta\,.\,\frac{dx}{d\beta}\,.\,\cos\beta$$

$$= - p - \operatorname{cosec}\beta\,.\,\frac{dx}{d\beta} = - p \pm \frac{ds}{dd}.$$

Whence, integrating, we have

$\frac{dp}{d\beta} = -\int p\,d\beta \pm s$; and since $\frac{dp}{d\beta} = -t$, this changes to

$$t \pm s = \int p\,d\beta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

in which β takes the place of θ in the former result.

Another useful expression is that between the radius-vector and perpendicular, on account of the simplicity of the equations of many curves when so referred.

$$ds = \frac{rdr}{\sqrt{r^2 - p^2}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots(5).$$

For,

$$ds = \frac{r^2d\theta}{p} = \frac{rdr\,.\,\frac{rd\theta}{dr}}{p} = \frac{r\,.\,\frac{p}{t}}{p} = \frac{r}{t} = \frac{rdr}{\sqrt{r^2 - p^2}} \;\ldots\ldots\ldots\ldots \quad (6).$$

PROP. II.

To find the area of a given curve.

By (*page* 404) the rectilinear differential of the area is

$$dA = ydx\sin a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

a being the angle of ordination.

By (*page* 405) the polar expression is

$$dA = \tfrac{1}{2}\,r^2d\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

As the length, when a curve is given by an equation to trace r and p, so also the area may be found in terms of those co-ordinates without recurring to $r\theta$, or xy. For inserting the value of $d\theta$ found at *page* 397, in $\frac{1}{2}r^2d\theta$, we have

$$A = \tfrac{1}{2}\int r^2d\theta = \tfrac{1}{2}\int \frac{prdr}{\sqrt{r^2 - p^2}} \;\ldots\ldots\ldots\ldots\ldots \;\ldots (3).$$

Again, a formula adapted to some particular cases, and especially for loops, may be thus investigated.

Put $\frac{y}{x} = \tan\theta$; and hence $d\theta = \cos^2\theta d\tan\theta$. Wherefore

$$A = \tfrac{1}{2}\int r^2d\theta = \tfrac{1}{2}\int x^2\sec^2\theta d\theta = \tfrac{1}{2}\int x^2d\tan\theta \ldots\ldots\ldots\ldots\ldots(4).$$

Scholium.

It is to be remarked that the area might be found, in those cases where the expressions are more easily integrated, by means of the formula $xdy = dA_1$, where $A + A_1 = xy$. For this only amounts to finding the other segment into which the rectangle xy is divided by the arc of the curve: or again, to an interchange of the axes X and Y, and estimating the curve according to the new system of co-ordinates.

I. The Circle.

The perimeter.—Several formulæ for the perimeter of the circle have been given in *vol. i. pp.* 475—6 (*note*), all of which, except the last, are deduced from that of Gregorie, by methods which are there explained. We shall here deduce Gregorie's series by means of the calculus, and add a few particulars respecting the number ϖ.

Since (*p.* 305), we have

$$d\tan^{-1}x = \frac{dx}{1+x^2} = (1 - x^2 + x^4 - x^6 + \ldots)\, dx,$$

we obtain by integration the value of s or $\tan^{-1}x$;

$$\tan^{-1}x = s = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\ldots\ldots\ldots\ldots (a),$$

as given at *vol. i. p.* 475, and at *vol. ii. p.* 370.

No constant is required, since when $x = 0$, $s = 0$.

The succeeding transformations given in the former place being independent of the calculus, and merely trigonometrical, need not, perhaps, be recapitulated in connexion with the present mode of treating the subject. As, however, the exhibition of a general method of making the successive deductions may be useful to the student, we shall resume that part which gradually leads to the final formula in that note.

Put $\tan\alpha = a_n$, $\tan\beta = b_n$, and $\tan(\alpha + \beta) = c_n$: then
$\alpha = \tan^{-1}a_n$, $\beta = \tan^{-1}b_n$, and $\alpha + \beta = \tan^{-1}c_n$; and

$$\tan^{-1}a_n + \tan^{-1}b_n = \tan^{-1}c_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots (b),$$

$$\text{or } a_n = \frac{c_n - b_n}{1 + b_n c_n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c).$$

Put $c_1 = 1$ and $b_1 = \frac{1}{5}$: then from (c), $a_1 = \frac{2}{3}$; and hence, from (b),

$$\tan^{-1}1 = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{2}{3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, put $c_2 = \frac{2}{3}$, and $b_2 = \frac{1}{5}$: then $a_2 = \frac{7}{17}$; or

$$\tan^{-1}\frac{2}{3} = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{7}{17}.$$

Insert this in (1): then

$$\tan^{-1}1 = 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{7}{17} \ldots\ldots\ldots\ldots\ldots (2).$$

Put $c_3 = \frac{7}{17}$ and $b_3 = \frac{1}{5}$: then $a_3 = \frac{9}{46}$, or

$$\tan^{-1}\frac{7}{17} = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{9}{46}.$$

Insert this in (2): then

$$\tan^{-1}1 = 3\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{9}{46} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Put $c_4 = \frac{9}{46}$ and $b_4 = \frac{1}{5}$: then $a_4 = -\frac{1}{239}$;

and insecting this in (3), we have

$$\tan^{-1} 1 = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} \dots\dots\dots\dots\dots \quad (4).$$

Put $c_3 = \frac{1}{239}$ and $b_3 = \frac{1}{70}$: then $a_3 = -\frac{1}{99}$;
and inserting in (4), we get

$$\tan^{-1} 1 = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{70} + \tan^{-1} \frac{1}{99} \dots\dots\dots\dots\dots \quad (5).$$

From this specimen it will be obvious how to conduct such enquiries as to the transformation of the expressions concerned. The great object is to obtain such fractions on the right side as when inserted in the general series (a) shall be rapidly convergent, and at the same time easy of computation. The equation (5) possesses these requisites in an eminent degree, since 5, 70, and 99 are numbers, to divide by which will cause less labour than perhaps any others that fulfil the condition of giving the left side of the equation either the entire circumference, or any rational specific part of it that may be discoverable. In the discovering of the formula itself, Mr. Rutherford had been anticipated by Euler; but Euler does not appear to have considered it so valuable as that which usually bears his name, probably from its requiring three series to be computed instead of two. Mr. Rutherford, on the contrary, has employed it (since the publication of our former volume) in the calculation of ϖ, which he has carried to the extraordinary extent of 208 places of figures. Availing himself of the principle of *synthetic division* for computing the powers of $\frac{1}{99}$, and many other ingenious contrivances, it was yet a work of prodigious labour. His *results* are printed in the *Phil. Trans.* 1842: and we annex them here.

$$\begin{aligned} \frac{1}{4}\varpi = 4 &\left\{ \frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \dots \right\} \\ &- \left\{ \frac{1}{70} - \frac{1}{3 \cdot 70^3} + \frac{1}{5 \cdot 70^5} - \frac{1}{7 \cdot 70^7} + \dots \right\} \\ &+ \left\{ \frac{1}{99} - \frac{1}{3 \cdot 99^3} + \frac{1}{5 \cdot 99^5} - \frac{1}{7 \cdot 99^7} + \dots \right\} \end{aligned}$$

$$\begin{aligned} \varpi = 3{\cdot}14159\ &26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971 \\ 69399\ &37510\ 58209\ 74944\ 59230\ 78164\ 06286\ 20899 \\ 86280\ &34825\ 34211\ 70679\ 82148\ 08651\ 32823\ 06647 \\ 09384\ &46095\ 50582\ 23172\ 53594\ 08128\ 48473\ 78139 \\ 20386\ &33830\ 21574\ 73996\ 00825\ 93125\ 91294\ 01832 \\ 80651\ &744 \dots \end{aligned}$$

(2). *The area.* Since $y = \sqrt{a^2 - x^2}$, the reference being to the centre as origin, we have $dA = \sqrt{a^2 - x^2} \cdot dx = a\sqrt{1 - \frac{x^2}{a^2}} \cdot dx$.

Expand the radical by the binomial theorem: then

$$dA = a\left\{1 - \frac{x^2}{2a^2} - \frac{x^4}{8a^4} - \frac{x^6}{16a^6} - \frac{5x^8}{128a^8} - \dots \right\} dx, \text{ or,}$$

$$A = ax\left\{1 - \frac{x^2}{6a^2} - \frac{x^4}{40a^4} - \frac{x^6}{112a^6} - \dots \right\},$$

from which, for any specified value of x, the area of the corresponding segment of the circle may be computed. If taken between the limits $x = 0$ and $x = a$, the value of the corresponding quadrantal area will be

$$A = a^2\left\{1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \dots \right\}.$$

Availing ourselves, however, of the previously computed perimeter of the circle, we may compute the area very simply. For instance, in rectangular co-ordinates:—

$$\int\sqrt{a^2-x^2}\,.\,dx=\int\frac{(a^2-x^2)\,dx}{\sqrt{a^2-x^2}}=\int\frac{a^2dx}{\sqrt{a^2-x^2}}-\int\frac{x^2dx}{\sqrt{a^2-x^2}}.$$

$$=a^2\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}-\frac{a^2}{2}\sin^{-1}\frac{x}{a}=\frac{1}{2}\left\{a^2\sin^{-1}\frac{x}{a}-x\sqrt{a^2-x^2}\right\},$$

and hence $\int_0^a\sqrt{a^2-x^2}\,.\,dx=a^2\,.\,\frac{\varpi}{4}=$ quadrantal area.

The entire area is, therefore, $a^2\varpi$, as assigned at *p.* 477, *vol. i.*

Or again, by polar co-ordinates, we have, referred to centre,

$dA=\frac{1}{2}\int r^2d\theta$, and r constant: whence $A=\frac{1}{2}r^2\theta$. Let $\theta=2\pi$: then the entire area is, as before, $A=.\,\frac{1}{2}r^2\,.\,2\varpi=r^2\varpi$.

II. The Parabola.

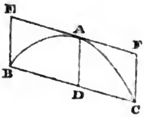
(1). *The length of the Apollonian parabola.* In this case we have $y^2=4ax$, and hence $s=\int\sqrt{dx^2+dy^2}=\frac{1}{2a}\int\sqrt{y^2+4a^2}\,.\,dy$

$$=\frac{y}{4a}\sqrt{y^2+4a^2}+a\log_e(y+\sqrt{y^2+4a^2})+c.$$

Now, when $x=0$, $s=0$ also: and hence substituting these values, we have $0=a\log 2a+c$, or $c=-a\log_e 2a$; wherefore the corrected integral is

$$s=\frac{y}{4a}\sqrt{y^2+4a^2}+a\log_e\frac{y+\sqrt{y^2+4a^2}}{2a}.$$

(2). *The area of the parabola.* We shall take this as the part cut off by any chord, BC, whatever.



Refer it to the tangent EF drawn parallel to the chord and the diameter AD through the point of contact, as co-ordinate axes, making the angle a with each other.

Then the equation of the parabola is $y^2=4a_1x$; and the formula (*p.* 509) becomes $dA=ydx.\sin a$. Integrating,

$$A=\int ydx\sin a=\frac{2}{a_1}\int y^2dy\ \sin a=\frac{2}{3}.\frac{y^3}{a_1}\sin a=\frac{2}{3}.xy\sin a.$$

This agrees with the result found at *page* 122: the entire area BAC being the double of ADC, just found.

Scholium.

The curves represented by the general equation $y^{m+n}=a^mx^n$ have been called *parabolas of the higher orders.* They admit of quadrature in all cases, and of rectification in several.

$$\text{For}\int ydx=\frac{m+n}{na^{\frac{m}{n}}}\int y^{\frac{m+n}{n}}dy=\frac{m+n}{m+2n}\cdot\frac{y^{\frac{m+2n}{n}}}{a^{\frac{m}{n}}}=\frac{m+n}{m+2n}.xy.$$

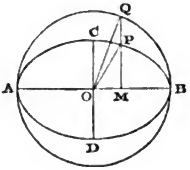
$$\text{Also, } ds = \sqrt{dx^2 + dy^2} = \left\{1 + \left(\frac{m+n}{n}\right)^2 \left(\frac{y}{a}\right)^{\frac{2m}{n}}\right\}^{\frac{1}{2}} dy\,;$$

which is integrable in finite terms when either $\frac{n}{2m}$ or $\frac{n+m}{2m}$ is an integer; that is, when $\frac{m+n}{m}$ is $\frac{3}{2}, \frac{5}{4}, \frac{7}{6}$, *etc.*, or $\frac{m+n}{m}$ is $\frac{2}{1}, \frac{4}{3}, \frac{6}{5}$, *etc.*

The *common*, the *cubical*, and the *semicubical* parabolas, are included amongst these cases.

III. The Ellipse.

(1.) *To find the length of the arc CP of an ellipse, AB, CD, being the rectangular conjugate diameters.*



On AB describe the circle AQB meeting the ordinate MP in Q, and join OP, OQ. Make QOC $= \phi$. Then,

$$x = \text{OM} = a \sin\phi, \text{ and}$$

$$y = \text{PM} = \frac{b}{a}.\text{QM} = b\cos\phi\,; \text{ and}$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{a^2\cos^2\phi + b^2\sin^2\phi}.d\phi = a\sqrt{1 - c^2\sin^2\phi}.d\phi,$$

$$\text{where } c^2 = \frac{a^2 - b^2}{a^2}.$$

Expand the radical in a series, and integrate the successive terms separately; then we get

$$\text{CP} = a\left\{\phi - \frac{c^2}{2}\left(\frac{1}{2}\phi - \frac{1}{2}\cos\phi\sin\phi\right) - \frac{1.c^4}{2.4}\left(\frac{1.3}{2.4}\phi - \frac{1.3}{2.4}\cos\phi\sin\phi - \frac{1}{4}\cos\phi\sin^3\phi\right) - \ldots\right\}$$

This taken between the limits 0, $\frac{1}{2}\pi$, as values of ϕ, gives for the length of the elliptic quadrant,

$$\text{CB} = \left\{1 - \left(\frac{1}{2}\right)^2 \cdot \frac{c^2}{1} - \left(\frac{1.3}{2.4}\right)^2 \cdot \frac{c^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{c^6}{5} - \ldots\right\} \cdot \frac{a\varpi}{2}.$$

If we put $n = \frac{1 - \sqrt{1 - c^2}}{1 + \sqrt{1 - c^2}}$, we shall have $1 + n = \frac{2a}{a + b}$; and the series becomes,

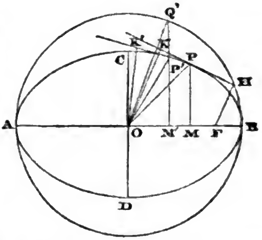
$$\text{CB} = \left\{1 + \left(\frac{1}{2}\right)^2 n^2 + \left(\frac{1.1}{2.4}\right)^2 n^4 + \left(\frac{1.1.3}{2.4.6}\right)^2 n^6 + \ldots\right\} \cdot \frac{(a+b)\varpi}{4};$$

which converges rapidly when c^2 is less than $\frac{1}{2}$.

The reader who is desirous of considering this interesting and important problem more in detail, is referred to the following, amongst the many places in which it is discussed:

Ivory, *Edinb Trans. vol. iv.*; Euler, *Novi Comm. Petrop tom. xvi.*, and *Opuscula, tom. ii.*; Legendre, *Mém. de l'Acad.;* and *Exercises du Calcul Integral;* Wallace, *Edinb. Trans. vol. v.* and *Conic Sections;* and Woodhouse, *Phil. Trans.* 1804.

(2). *If a point P in the elliptic quadrant BPC be given, it is required to find another point P′ in the same quadrant; such that arc CP′ − arc BP = a constructible straight line.* (Fagnani's theorem).



Constr. On the principal axis AB describe the semicircle, and make the angle COQ′ equal to the angle MPH formed by the ordinate and tangent at P; draw Q′M′ perpendicular to AB, meeting the ellipse at P′: then P′ is the point required.

For, from centre O and focus F draw perpendiculars to the tangent at P; and denote the angle MPH by θ and OK by p. Then, the remaining notation being as in the preceding proposition, we shall have (since OH = OB, *pp.* 130, 333),

$$p^2 = OK^2 = OH^2 - HK^2 = OB^2 - OF^2 \sin^2\theta; \text{ or } p = a\sqrt{1 - c^2\sin^2\theta};$$

$$\text{and } (p.\ 508,\ eq.\ 3),\ BP + PK = \int p\,d\theta = a\int\sqrt{1 - c^2\sin^2\theta}\,d\theta.$$

But since COQ = MPH = θ, we have OM′ $= a\sin\theta$; and hence, by the last example, CP′ $= a\int\sqrt{1 - c^2\sin^2\theta}\,d\theta$: wherefore,

$$\text{arc BP} + \text{line PK} = \text{arc CP}'; \text{ or arc CP}' - \text{arc BP} = \text{line PK}.$$

Let now, as before, OM $= x = a\sin\phi$, and MP $= y = b\cos\phi$: then,

$$\cot\theta = -\frac{dy}{dx} = \frac{b\sin\phi}{a\cos\phi} = \frac{b}{a}\tan\phi; \text{ or } \sin\phi = \frac{\cos\theta}{\sqrt{1 - c^2\sin^2\theta}}.$$

$$\text{Also, KP} = t = -\frac{dp}{d\theta} = \frac{ac^2\sin\theta\cos\theta}{\sqrt{1 - c^2\sin^2\theta}} = ac^2\sin\theta\sin\phi.$$

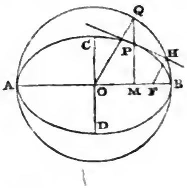
But $a\sin\theta$ = OM′, and $a\sin\phi$ = OM; wherefore,

$$t = ac^2\sin\theta\sin\phi = \frac{c^2}{a}.\text{OM}.\text{OM}'.$$

The line t, therefore, is a constructible straight line, since OM′ is given by the preceding construction, and all the other parts, c, a, and OM, are amongst the data of the problem.

It also appears, that since PK is a symmetrical function of ϕ and θ, P′K′ must be a similar function of θ and ϕ; and hence, that P′K′ = PK, in all corresponding positions of P and P′.

(3). *To divide the elliptic quadrant into two parts, the arcs of which the difference shall be equal to an assignable straight line.*



Since the points P and P′ of the preceding proposition coalesce, we have COQ = MPH, or $\theta = \phi$. Whence, $\cot^2\theta = \frac{b}{a}$; and the co-ordinates of P are

$$x = a\sin\theta = a\sqrt{\frac{a}{a+b}}, \text{ and}$$

$$y = b\cos\theta = b\sqrt{\frac{b}{a+b}}.$$

$$\text{Hence, arc BP} - \text{arc AP} = ac^2\sin^2\theta = \frac{a^2 - b^2}{a+b} = a - b.$$

The point P, therefore, being taken with the above co-ordinates, is such that it divides the quadrant into two arcs whose difference is equal to the difference of the semi-axes.

(4). *To find the area of an ellipse.*

The equation of the ellipse referred to the centre is $y = \frac{b}{a}\sqrt{a^2 - x^2}$; and hence the area is

$$A = \frac{b}{a}\int\sqrt{a^2 - x^2}\,dx = \frac{b}{a}.(\text{circle on AB}) = \frac{b}{a}.a^2\varpi = \varpi ab.$$

IV. The Hyperbola.

(1). *The length of an hyperbolic arc.*

The equation of the curve being $y = \frac{b}{a}\sqrt{x^2 - a^2}$, we have

$$dy = \frac{b}{a}\cdot\frac{xdx}{\sqrt{x^2 - a^2}}, \text{ and } ds = \sqrt{dx^2 + dy^2} = \sqrt{\frac{c^2x^2 - a^2}{x^2 - a^2}}\,dx;$$

or putting $x = az$, we get

$$s = \int\sqrt{\frac{c^2x^2 - a^2}{x^2 - a^2}}\,dx = a\int\sqrt{\frac{c^2z^2 - 1}{z^2 - 1}}\,dz = ac\int\frac{\sqrt{1 - \frac{1}{c^2z^2}}}{\sqrt{z^2 - 1}}\,dz$$

$$= a\int\frac{cz\,dz}{\sqrt{z^2 - 1}}\left\{1 - \frac{1}{2}\left(\frac{1}{cz}\right)^2 - \frac{1.1}{2.4}\cdot\left(\frac{1}{cz}\right)^4 - \frac{1.1.3}{2.4.6}\left(\frac{1}{cz}\right)^6 - \frac{1.1.3.5}{2.4.6.8}\left(\frac{1}{cz}\right)^8 - \dots\right\}$$

Wherefore, every term except the first depends upon the integration of the general form $\frac{dz}{z^m\sqrt{z^2 - 1}}$, where m is always odd. The integration of the series can therefore be effected; the general term for reduction being

$$\int\frac{dz}{z^m\sqrt{z^2 - 1}} = \frac{1}{m - 1}\cdot\frac{\sqrt{z^2 - 1}}{z^{m-1}} + \frac{m - 2}{m - 1}\int\frac{dz}{z^{m-2}\sqrt{z^2 - 1}}.$$

(2.) *To find the difference between the lengths of the asymptote and the infinite arc of the hyperbola.*

By the preceding investigation, taking z between the limits 1 and $\frac{1}{0}$, we have

$$\int_0^{\frac{1}{0}}\frac{dz}{z\sqrt{z^2 - 1}} = \sec^{-1}\frac{1}{0} = \frac{\varpi}{2};$$

$$\int_0^{\frac{1}{0}}\frac{dz}{z^3\sqrt{z^2 - 1}} = \frac{1}{2}\int_0^{\frac{1}{0}}\frac{dz}{z\sqrt{z^2 - 1}} = \frac{1}{2}\frac{\varpi}{2};$$

$$\int_0^{\frac{1}{0}}\frac{dz}{z^5\sqrt{z^2 - 1}} = \frac{3}{4}\int_0^{\frac{1}{0}}\frac{dz}{z^3\sqrt{z^2 - 1}} = \frac{1.3}{2.4}\cdot\frac{\varpi}{2};$$

$$\int_0^{\frac{1}{0}}\frac{dz}{z^7\sqrt{z^2 - 1}} = \frac{5}{6}\int_0^{\frac{1}{0}}\frac{dz}{z^5\sqrt{z^2 - 1}} = \frac{1.3.5}{2.4.6}\cdot\frac{\varpi}{2};$$

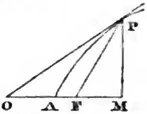
etc. *etc.*

Moreover, for the first term we have $ae\int\frac{zdz}{\sqrt{z^2 - 1}} = ae\sqrt{z^2 - 1} = aez$, between the limits $z = 0$ and z infinitely great.

But the length of the asymptote corresponding to any value of x is $\sqrt{x^2 + \frac{b^2x^2}{a^2}} = x\sqrt{\frac{a^2+b^2}{a^2}} = cx = aez$; which for a value of z infinitely great, becomes identical with the first term of the series as just determined. Whence the asymptote exceeds the hyperbolic arc by the series

$$\left\{\frac{1}{2}\cdot\frac{1}{c} + \frac{1.1}{2^2 4}\cdot\frac{1}{c^3} + \frac{1.1.3^2}{2^2.4^2.6}\cdot\frac{1}{c^5} + \frac{1.1.3^2.5^2.}{2^2.4^2.6^2.8}\cdot\frac{1}{c^7} + \ldots\right\}\frac{a\varpi}{2}.$$

Many other forms for the hyperbolic arc may be found in different works; but as they are of little use, except as exercises, we shall not insert them here.



(3). *To find the area of an hyperbolic area APM.*

Refer to the centre: then $y = \frac{b}{a}\sqrt{x^2 - a^2}$, and hence

$$\text{A} (= \text{AMP}) = \frac{b}{a}\int\sqrt{x^2-a^2}\,dx = \frac{b}{a}\left\{\int\frac{x^2\,dx}{\sqrt{x^2-a^2}} - \int\frac{a^2\,dx}{\sqrt{x^2-a^2}}\right\}$$

$$= \frac{b}{a}\left\{x\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}\,dx - a^2\log_e\left(x+\sqrt{x^2-a^2}\right)\right\}$$

$$= \frac{b}{a}\left\{\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log_e\left(x+\sqrt{x^2-a^2}\right)\right\} + \text{C}.$$

But when $x = a$, $\text{A} = 0$; whence in this case

$$0 = -\frac{b}{a}\frac{a^2}{2}\log_e a + \text{C}, \text{ or } \text{C} = \frac{ab}{2}\log_e a.$$

Insert this value of C; then we get

$$\text{A} (= \text{AMP}) = \frac{xy}{2} - \frac{ab}{2}\log_e\frac{x+\sqrt{x^2-a^2}}{a}$$

$$= \frac{xy}{2} - \frac{ab}{2}\log_e\left(\frac{x}{a}+\frac{y}{b}\right).$$

(4). *The sectoral area OAP is required.*

This is at once obtained from the preceding investigation; for

$$\text{OAP}=\text{OMP}-\text{AMP}=\frac{xy}{2} - \left\{\frac{xy}{2} - \frac{ab}{2}\log_e\left(\frac{x}{a}+\frac{y}{b}\right)\right\} = \frac{ab}{2}\log_e\left(\frac{x}{a}+\frac{y}{b}\right).$$

If we put $\log_e\left(\frac{x}{a}+\frac{y}{b}\right) = \theta$, then $\frac{x}{a}+\frac{y}{b} = e^\theta$. Also, since from the equation of the curve, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have by division $\frac{x}{a} - \frac{y}{b} = e^{-\theta}$; wherefore, by addition and subtraction,

$$x = \frac{a}{2}\left(e^\theta + e^{-\theta}\right), \text{ and } y = \frac{b}{2}\left(e^\theta - e^{-\theta}\right).$$

By a comparison of these forms with those at *p.* 437, *vol. i.*, we shall be struck with the remarkable analogy. If the hyperbola be equilateral, then $b = a$, and we have $x = \frac{a}{2}\left(e^\theta + e^{-\theta}\right)$, $y = \frac{a}{2}\left(e^\theta - e^{-\theta}\right)$, which differ in no respect from what takes place in the circle, except the absence of the symbol $\sqrt{-1}$: for if x, y, be the co-ordinates of a point in the circle ($rad = a$), and θ, the arc corresponding to them, we have

$$x = \frac{a}{2}\left\{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}\right\}, \text{ and } y = \frac{a}{2\sqrt{-1}}\left\{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}\right\}.$$

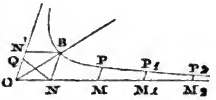
This analogy led Lambert to propose a system of *hyperbolic trigonometry;* and it was carried out to some extent by Sauri and some others, soon after. It has been lately revived by the continental writers, of which some account is given by Dr. Peacock in the Reports of the second meeting of the British Association. Since that time, a paper, remarkable for its elementary elegance and generality of result, has been given by Dr. Wallace in the *Edinb. Trans. vol. xiv.* to which the inquiring student is referred.

(5). *In the hyperbola referred to its asymptotes as co-ordinate axes, two ordinates are drawn: it is required to find an expression for the area contained between them.*

If a be the angle of ordination, and a, b, the semi-axes of the curve, we have (*p.* 325) the equation of the curve

$$xy = \tfrac{1}{4}(a^2 + b^2), \text{ or}$$

$$A = \sin a \int y dx = \tfrac{1}{4}(a^2 + b^2) \sin a \int \frac{dx}{x} = \tfrac{1}{4}(a^2 + b^2) \sin a \log_e x + C,$$



To find C, draw the ordinate BN to the vertex B of the hyperbola, and let the area be estimated from this line in the direction MP, M_1P_1, M_2P_2, *etc.* Then, if ON $(= \tfrac{1}{2}\sqrt{a^2 + b^2})$ be taken as the unit of the scale, we shall have

$$0 = \sin a \log_e 1 + C, \text{ or } C = 0;$$

and the expression is reduced in its corrected state to

$$A = \sin a \log_e x.$$

Let, then, x_1, x_2, be any values of x corresponding to the points M_1, M_2: we shall have

$$M_1M_2P_2P_1 = \sin a (\log_e x_2 - \log_e x_1) = \sin a \log_e \left(\frac{x_2}{x_1}\right).$$

If, now, to the unit ON, there be taken any numbers represented by OM, OM_1, OM_2, *etc.*, upon X; then the spaces NP, NP_1, NP_1, NP_2, *etc.* will be proportional to, or represent the logarithms of, OM, OM_1, OM_2, *etc.*, QN or $\sin a$ to ON as radius, being the modulus of the system. These spaces, therefore, may represent every system of logarithms by giving a suitable inclination to the asymptotes, that is, a suitable value to $\sin a$.

When $a = \tfrac{1}{2}\pi$, or the hyperbola equilateral, $\sin a = 1$, and hence the modulus being unity, we have a representation of the *Naperian system.* In fact, from Napier having *illustrated (not investigated)* his method by means of the equilateral hyperbola, his numbers have been very generally called the hyperbolic logarithms: but it will be obvious, that as all logarithms may be represented by hyperbolic spaces, the use of the term hyperbolic to signify one particular system, is somewhat inappropriate. The common logarithms, for instance, give $M_{10} = \cdot 43429448$, or $a = 25^\circ 55' 16''$. See, also, *p.* 162.

(6). *To find the area of an hyperbolic sector, the focus being the pole. See fig. to* (3) *preceding page.*

Let $OA = a$, $OF = ae$, $OM = x$, $MP = y$. Then, since e is greater than unity, assume $x = a \sec \phi$, and therefore $y = b \tan \phi$. Now we have

area OMP = area AMP — triangle FMP; whence, differentiating,

$$d\,(\text{area OMP}) = y\,dx + \frac{1}{2} \cdot d\,\{(ae - x)\,y\} = \frac{1}{2}\,\{y dx + (ae - x)\,dy\},$$

or, $\frac{d\,(\text{area OMP})}{d\phi} = \frac{1}{2}\left\{ y\,\frac{dx}{d\phi} + (ae - x)\frac{dy}{d\phi}\right\}$, or again,

$$d\,(\text{area OMP}) = \frac{1}{2}\,\{ab\tan^2\phi\sec\phi + ab\,(e - \sec\phi)\sec^2\phi\}\,d\phi.$$

$$= \tfrac{1}{2}\,ab\,\{e\sec^2\phi - \sec\phi\}\,d\phi, \text{ and integrating, } etc.,$$

$$\text{area OMP} = \tfrac{1}{2}\,ab\,\{e\tan\phi - \log_e\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi)\}.$$

Let now the angle AMP $= \theta$: then $r = ex - a$, and hence

$$\tan\theta = \frac{\text{PM}}{\text{FM}} = \frac{r}{y} = \frac{ex - a}{y} = \frac{a\,(\sec\phi - 1)}{b\tan\phi} = \frac{a}{b}\tan\tfrac{1}{2}\phi, \text{ or}$$

$$\tan\tfrac{1}{2}\theta = \sqrt{\frac{e+1}{e-1}}\,.\,\tan\tfrac{1}{2}\phi,$$

which inserted in the expression for OMP gives the result required.

V. THE CYCLOÏD.

(1). *The length.* Here $\frac{dy}{dx} = \sqrt{\frac{2a - x}{x}}$: hence $1 + \frac{dy^2}{dx^2} = \frac{2a}{x}$.

Hence $ds = \int dx\,\sqrt{\frac{2a}{x}} = \sqrt{2a}\int\frac{dx}{\sqrt{x}} = 2\sqrt{2ax} + \text{C}$,

and since x and s commence together, $\text{C} = 0$.

The length of the arc of the cycloïd estimated from the vertex is equal to the chord of the arc of the cycloid: the circle and cycloïd being cut off by the same parallel to the base of the figure.

Hence, if the radius of the generating circle be a, the cycloïd itself is $4a$; since the semi-cycloïd is equal to the diameter of the generating circle: which accords with what was shown at *page* 413.

(2). *The area.* Here $dy = \sqrt{2ax - x^2}\,.\,x^{-1}dx$ is the differential equation of the curve. Also, generally,

$$\int y\,dx = xy - \int x\,dy = xy - \int\sqrt{2ax - x^2}\,.\,dx.$$

$$= xy - \frac{1}{2}\,(x - a)\,\sqrt{2ax - x^2} - \frac{1}{2}\,a^2\int\frac{dx}{\sqrt{2ax - x^2}}$$

$$= xy - \frac{1}{2}\,(x - a)\,\sqrt{2ax - x^2} - \frac{1}{2}\,a^2\,\text{versin}^{-1}\frac{2x}{a} + \text{C}.$$

When $x = 0$, then $\text{A} = 0$, and hence $\text{C} = 0$, Also, when $x = 2a$, we have $y = a\varpi$; and hence $xy = 2a^2\varpi$; and

$$\sqrt{2ax - x^2} = 0\text{; and } -\frac{1}{2}\,a^2\,\text{versin}^{-1}\frac{2x}{a} = -\frac{1}{2}\,a^2\varpi. \text{ Whence}$$

$$\int_0^{2a} y\,dx = 2a^2\varpi - \frac{1}{2}\,a^2\varpi = \frac{3}{2}\,a^2\varpi.$$

Doubling this, to include the whole area, we find that the area of the cycloïd is three times that of its generating circle.

(3). *The area of the companion to the cycloïd.* In terms of θ, we have $y = a\theta$, and $x = a\,(1 - \cos\theta)$: whence

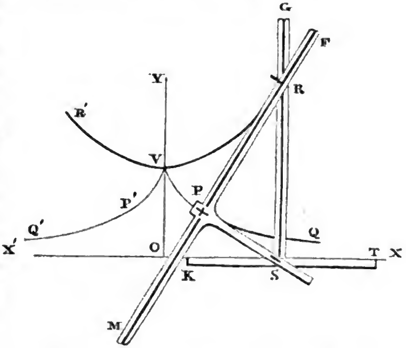
$$\int y\,dx = xy - \int x\,dy = a^2\,(\sin\theta - \theta\cos\theta) + \text{C}.$$

The whole area is $2a^2\varpi$: or twice the generating circle; or two-thirds of the cycloïd itself.

VI. THE TRACTORY AND CATENARY.

(1). *Genesis of the curves.* The equations of these curves have been exhibited under several forms by different writers, and hence it may be advisable to describe them particularly.

The *catenary* is the curve assumed by a perfectly flexible chain or cord suspended by its two extremities, the whole of the particles being subjected to the action of terrestrial attraction. The *tractory* is the path pursued by a body drawn by an inextensible string, the other end of which moves along a straight line—the tendency to motion, except that produced by the *co-temporaneous* pull of the string, being supposed to be continually destroyed by the friction of the surface on which it moves. Our reason for combining the two curves under one head is, that the catenary is the involute of the tractory; and the mechanical genesis of both curves is effected by a single construction. The construction itself is, I believe, the invention of Sir John Leslie, who gave it in his *Geometry of Curve Lines, pp.* 397—8: though, by some oversight, the figure which is engraved for its illustration differs so much in its lettering from that required by the text, as to render his description almost unintelligible—the reason, probably, of its never having been noticed by subsequent writers.



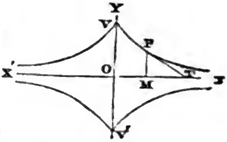
Let a thin bar of wood or brass, SG with a groove running along the middle, have a perpendicular arm, KT, projecting each way: to this frame annex a similar piece MPSF, with holes drilled in PS, having a small sharp-edged wheel at P, turning about an axis in the same plane, but at right angles to PS; one of these holes being passed over a pin fixed at S in the direction of the groove GS, and a pencil, secured in a slider, being inserted in the concourse of both grooves at R: let KST be slid along the directrix X′X, and the wheel at P gently pressed, it will describe the *tractory;* whilst the pencil carried in the intersection of the perpendiculars PF, SG will trace the *catenary.*

This instrument may be called *Leslie's Catenarian trammel.*

When P and R coalesce, as at V, then PS will be perpendicular to X′X, and OV, (which is then the position of PS), will be equal to PS. Both the curves will, therefore, pass through this point. The catenary will continually diverge from the line X′X, and the tractory continually approach towards it; and by moving the bar GS to the left of OV, there will be branches formed precisely similar, equal, and symmetrically situated with respect to OV, as there were on the right. We shall first obtain the equations of these curves, and then develope a few of their properties.

2. *The equation of the tractory.* The general construction already given implies that the wheel P always moves in a direction towards S; and agrees with the ordinary previous definition of the curve. This construction implies that the string in one case, and the rod PS in the other, is a tangent to the curve: and, by hypothesis, the tangents are all equal.

Let P be a point in the curve, PT the tangent meeting the directrix X′X in T, O the origin, and MP the ordinate at P. Then if xy be the co-ordinates of P,



and $x_1 y_1$ the current co-ordinates of the tangent PT, we have the equation of the tangent (putting $P = a$),

$$y_1 - y = \frac{dy}{dx}(x_1 - x), \text{ and } MT = x_1 - x = \frac{ydx}{dy}.$$

$$\text{Wherefore } PT^2 = y^2\left(1 + \frac{dx^2}{dy^2}\right) = a^2;$$

or, in another form,

$$dx = \frac{dy\sqrt{a^2 - y^2}}{y} = \frac{a^2 dy}{y\sqrt{a^2 - y^2}} - \frac{ydy}{\sqrt{a^2 - y^2}}.$$

Whence, integrating, the first term being integrated at *page* 473,

$$x = \sqrt{a^2 - y^2} + \frac{a}{2}\log_e \frac{a - \sqrt{a^2 - y^2}}{a + \sqrt{a^2 - y^2}}.$$

which is the equation between x and y referred to the origin O.

3. *The length of the tractory.* The equation in y and s, whilst it *rectifies the curve*, is also much more elegant and useful.

Resume the expression for the tangent, and write $ds^2 - dy^2$ for dx^2: then we have

$$y^2dy^2 + y^2dx^2 = a^2dy^2, \text{ or } y^2ds^2 = a^2dy^2, \text{ or } ds = -\frac{ady}{y}.$$

Whence $s = -a\log_e y + C$. When $s = 0$, M is at O; and hence $x = 0$, and $y = a$. Therefore $C = a\log_e a$. The corrected integral is, therefore,

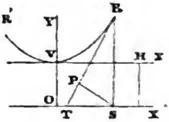
$$s = a\log_e \frac{a}{y}.$$

3. *The area of the tractory.* From the differential equation of the curve, we have at once $ydx = \sqrt{a^2 - y^2}\,.\,dy$; or

$$A = a^2\sin^{-1}\frac{y}{a}; \text{ or } A_0^a = \frac{1}{4}a^2\,.\,\varpi.$$

The whole area included by the four branches (a curve being formed below XX in the same manner as that above) is, hence, equal to the circle described with centre O, and radius OV or a.

4. *The equation of the catenary.* The construction obviously shows that PR is a tangent to the curve at R, and that the perpendicular SP from the foot of the ordinate RS is of the constant length SP or a.



Now, the equation of the tangent at xy of any curve, $x_1 y_1$ being the current co-ordinates of the tangent, is

$$y_1 - y = \frac{dy}{dx}(x_1 - x)$$

and the length of the perpendicular to the tangent from the point S $(x, 0)$ is,

$$a = \frac{ydx\,.}{\sqrt{dx^2 + dy^2}},$$

which is the differential equation of the curve, a being the constant tangent. This transforms to

$$dx = \frac{ady}{\sqrt{y^2 - a^2}}; \text{ or, integrating, } x = a\log_e \frac{y + \sqrt{y^2 - a^2}}{a},$$

which is the equation of the catenary between the co-ordinates of the curve referred to OX, OY.

At *p.* 344, the equation of the catenary is given in a different form; but it is easily deduced from the one just given.

For, taking the inverse function, $\log_e{}^{-1}$, we have

$$ae^{\frac{x}{a}} - y = \sqrt{y^2 - a^2},\text{ or squaring, cancelling, } etc.$$

$$ae^{\frac{2x}{a}} + a = 2e^{\frac{x}{a}}\, y,\text{ or } y = \frac{1}{2}\, a\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}.$$

Transpose the origin to the point V, (or write $y + a$ for y) the axis VX′ being parallel to OX, we have the expression changed to

$$y + a = \frac{1}{2}\, a\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\};$$

or again, interchanging the axes of co-ordinates, it becomes precisely as given in the place referred to.

It may be remarked generally, that the line OV is frequently taken as the axis of x, and V as origin: which, in reading works where the catenary is treated, should be kept in mind.

5. *The length of the catenary.* From the differential equation of the catenary, and the general formula for the length of the arc,

$$a^2 ds^2 = y^2\,(ds^2 - dy^2)\,;\text{ or } ds = \frac{y dy}{\sqrt{y^2 - a^2}}.\text{ Hence}$$

$$s = \sqrt{y^2 - a^2} + \mathrm{C},\text{ is the length of the curve.}$$

Now, when $y = a$, the curve is at its commencement, and hence $\mathrm{C} = 0$; or the length is, simply, when referred to the system of axes OX, OY,

$$s^2 = y^2 - a^2.$$

$$\text{Also, } s^2 = y^2 - a^2 = \frac{1}{4} a^2\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}^2 - a^2 = \frac{1}{4} a^2\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}^2,$$

$$\text{and hence } s = \frac{1}{2}\, a\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}.$$

If it be referred to VX′, VY: then $s^2 = 2ay + y^2$.

6. *The area of the catenary.* By the differential equation $ydx = ads$; we have $\mathrm{A} = as$, the constant being 0, as before. Hence, if on VX we take VH equal to the arc VR, and complete the rectangle OKHV: it will be equal to the area OSRV.

But $s = \sqrt{y^2 - a^2}$; and hence the area OSRV is, also,

$$\mathrm{A} = a\sqrt{y^2 - a^2}.$$

Also, since $\mathrm{A} = a\sqrt{y^2 - a^2}$, we readily get

$$\left(\frac{\mathrm{A}}{a}\right)^2 + a^2 = y^2 = \frac{a^2}{4}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}^2,\text{ or}$$

$$\mathrm{A} = \frac{a^2}{2}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}.$$

When referred to VX′, VY, we must subtract the rectangle $\mathrm{OL} = ax$; and hence so referred

$$\mathrm{A} = a\left\{\sqrt{y^2 - a^2} - x\right\}.$$

7. *The catenary is the evolute of the tractory.*

Let $x_1 y_1$ be the co-ordinates of the centre of curvature of the tractory: then substituting the values of the differential co-efficients of the curve in the appropriate values (*page* 408), we have

$$y_1 = \frac{a^2}{y},\text{ and } x_1 = x + \frac{a\sqrt{y_1{}^2 - a^2}}{y_1}.$$

Eliminating x, y, between these and the equation of the tractory, we get

$$x_1 = a \log_e \frac{y_1 + \sqrt{y_1^2 - a^2}}{a},$$

which is the first finite form in which we have found the equation of the catenary.

The genesis of the curves, indeed, points out this to be the case: for the tangent PR of the catenary is always perpendicular to the tangent PS of the tractory.

VII. The Logarithmic Curve.

The equation in *a homogeneous form* is $y = ae^{\frac{x}{a}}$: and the curve is, in fact, that whose *subtangent* is constant and equal to a. To establish this, let us take the differential expression for the subtangent, and equate it to a. Then

$$a = \frac{ydx}{dy} = \frac{\frac{y}{a}\,.\,dx}{d\left(\frac{y}{a}\right)}; \text{ or } d\log_e\left(\frac{y}{a}\right) = \frac{dx}{a}; \text{ whence}$$

$$\log_e\left(\frac{y}{a}\right) = \frac{x}{a}, \text{ or } y = ae^{\frac{x}{a}}.$$

This is the proper form of the equation of the logarithmic curve.

1. *The area.* Here $ydx = ae^{\frac{x}{a}}\,dx = a^2e^{\frac{x}{a}}\,d\left(\frac{x}{a}\right)$: whence

$$\text{A} = \int ydx = a^2\,e^{\frac{x}{a}} + \text{C} = ay + \text{C}.$$

Now if we consider the area to commence when $x = 0$, we have

$$0 = a^2e^0 + \text{C}, \text{ or } \text{C} = -a^2.$$

Whence $\text{A} = a^2\left\{e^{\frac{x}{a}} - 1\right\} = a\,(y - a)$.

2. *The length.* From the differential equation we have

$$ads = \sqrt{y^2 + a^2}\,.\,dy = \frac{a^2dy}{\sqrt{y^2 + a^2}} + \frac{y^2dy}{\sqrt{y^2 + a^2}}; \text{ or}$$

$$as = \frac{1}{2}\left\{y\sqrt{y^2 + a^2} + a^2\log_e\left(y + \sqrt{y^2 + a^2}\right)\right\} + \text{C}.$$

Now, if the origin of s, like that of a, be taken where $x = 0$, we have

$$0 = \frac{1}{2}a\sqrt{2} + \frac{1}{2}a\log_e\left(a + a\sqrt{2}\right), \text{ or } \text{C} = -\frac{1}{2}a\sqrt{2} - \frac{1}{2}a\log_e a\left(1 + \sqrt{2}\right),$$

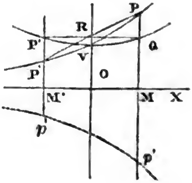
and the corrected integral for the length of the curve becomes

$$s = \frac{1}{2}\left\{\frac{y}{a}\sqrt{y^2 + a^2} - a\sqrt{2}\right\} + \frac{1}{2}a\log_e\frac{y + \sqrt{y^2 + a^2}}{a\left(1 + \sqrt{2}\right)}.$$

3. *The characters of the curve.* Refer it to the axes OX, OY: then

(1), at the origin O, we have $y = ae^0 = a$.

(2). Since e is greater than unity, the positive values of x will render y greater and greater as x is taken greater and greater: and as x approaches to infinity, y also approaches infinity as its limit.



Whence the curve extends in the angular space XOY to an infinite distance.

(3). For negative values of x, we have $y = ae^{-\frac{x}{a}}$; and hence the factor $e^{-\frac{x}{a}}$, or $\frac{1}{e^{\frac{x}{a}}}$ decreases towards zero as x approaches towards infinity negative: and at that limit, $y = 0$. The axis OX *negative*, is therefore an asymptote to the curve.

(4). If OM′, OM be taken equal to each other on opposite sides of the origin O, upon the axis X, and P′, P the corresponding points of the logarithmic curve be joined by a line cutting OY in R: then OR is the ordinate *catenary* corresponding to the abscissas OM′, OM; and if through R a parallel to x be drawn it will cut OM′, OM in points O′, P, which are in the curve of the catenary. For $\text{OR} = \frac{1}{2}(\text{MP} + \text{M}'\text{P}') = \frac{1}{2}a\,(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$; which is characteristic of the catenary.

(5). As amongst the successive values of $\frac{x}{a}$, *all fractions in their lowest terms* occur, there will for each of those whose denominators are *even*, be two values of $e^{\frac{x}{a}}$, and hence two values of y: but their signs will be different. In all these cases, there will be points on *both sides* of the axis X,—as at P, P′ on the positive, and at p, p' on the negative side in the figure. Also, between these are interposed innumerable fractional values of $\frac{x}{a}$ whose denominators are *odd*, as well as innumerable transcendental values. To these sometimes only one, and often many real values of $e^{\frac{x}{a}}$ corresponds: and hence the path on the negative side is composed of a series of *isolated points*, instead of a continuous curve; and indeed innumerable dotted branches, all isolated, may occur.

Into this subject, however, we cannot enter here; but must refer the inquiring student to a paper by M. Vincent in the *Annales des Mathematiques, tom. xvii.* and to De Morgan's *Differential Calculus, page* 382.

(6). If parts be taken on the axis of x in arithmetical progression, the ordinates corresponding to them will be in geometrical progression; or in other words, and taken generally, the abscisses are the logarithms of the ordinates to the modulus a (the subtangent):—being in this respect analogous to the hyperbolic spaces contained between the curve and the asymptotes.

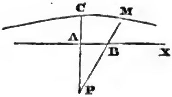
(7). A little more *generality of form* may be given to the equation of the curve (but the character and features will not be altered) if we introduce a second parameter. The form would be $y = ae^{\frac{x}{b}}$, or $y = ae^{m\,.\,\frac{x}{b}}$: the system of logs would then be changed into others whose moduli were $\frac{a}{b}$ and m respectively; and the resulting expressions rendered a little more complex by the proposed change. It may be worth the student's while to work out the rectification and quadrature under this form of the equation.

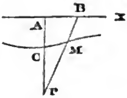
VIII. The Conchoïd.

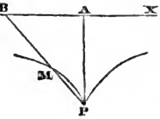
The equation of this curve has been given at *pages* 342—3.

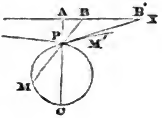
1. *The form of the conchoïd.* The four cases as to the position of the tracing

point on the moving radius-vector will be exhibited in the four following figures.









In the *first* figure the point M is taken in PB produced beyond the directrix AX in respect to the pole P; and BM of *any length*.

In the *second*, M is on the same side of AX with P; but BM *less than* PA.

In the *third*, the same relative positions exist, but PM = PA.

In the *fourth*, the same relative positions as in the two preceding; but MB *greater than* PA.

In all cases the directrix is an asymptote: in the first and second the character is sufficiently shown by the figures: in the third, P is a cusp, AP being tangent to each branch: and in the fourth there is a loop PMC formed by the two branches of the curve intersecting at P, and that point being a double point of the first species.

The slightest observation shews that the polar equation, from its greater simplicity, is better adapted for finding the length and area of the conchoïd, than the rectangular one.

2. *The area of the conchoïd.* Here $r = \pm b + a \sec\theta$, the upper sign applying to the inferior (*figs.* 1, 2), and the lower to the superior (*figs.* 3, 4) conchoïd: and hence

$$\begin{aligned}\frac{1}{2}\int r^2 d\theta &= \frac{1}{2}\int (b^2 \pm 2ab\sec\theta + a^2\sec^2\theta)\, d\theta \\ &= \frac{1}{2}\int\left(b^2 d\theta \pm 2ab\frac{d\theta}{\cos\theta} + a^2\frac{d\theta}{\cos^2\theta}\right) \\ &= \frac{1}{2}\left\{b^2\theta \pm 2ab\log_e\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + a^2\tan\theta\right\} + \mathrm{C}.\end{aligned}$$

Also, since θ and A begin together, we have $\mathrm{C} = 0$; and the expression for the area described about the pole P is found.

This, when $\theta = \frac{1}{2}\pi$, becomes infinite. It does not, however, follow from the previous investigation that the area between the curve and asymptote is infinite: though the following investigation by means of rectangular co-ordinates shows that such is the case.

The rectangular equation is $xy = (a+y)\sqrt{b^2 - y^2}$.

Now, $\frac{d\mathrm{A}}{dy} = \frac{d\mathrm{A}}{dx}\cdot\frac{dx}{dy} = \frac{ydx}{dy}$; and from the curve $\frac{dx}{dy} = -\frac{ab^2 + y^3}{y^2\sqrt{b^2 - y^2}}$.

Hence, $d\mathrm{A} = ydx = -\frac{(ab^2 + y^3)\,dy}{y\sqrt{b^2 - y^2}}$.

Whence, integrating by parts, we have

$$\begin{aligned}\mathrm{A} &= -\int\frac{ab^2dy}{y\sqrt{b^2-y^2}} - \int\frac{y^2dy}{\sqrt{b^2-y^2}} \\ &= -ab\log_e\frac{y}{b+\sqrt{b^2-y^2}} + \frac{1}{2}y\sqrt{b^2-y^2} - \tfrac{1}{2}b^2\sin^{-1}\frac{y}{b} + \mathrm{C}.\end{aligned}$$

To find C, we must remark that when $A = 0$, $y = b$; and the equation becomes $0 = -\frac{b^2}{2}\cdot\frac{\varpi}{2} + C$; and hence the correct integral of the area of the conchoïd is

$$A = \frac{b^2}{2}\left\{\frac{\varpi}{2} - \sin^{-1}\frac{y}{b}\right\} - ab \log_e \frac{y}{b + \sqrt{b^2 - y^2}} + \frac{1}{2}y\sqrt{b^2 - y^2}.$$

When $y = 0$, this is reduced to $A = \frac{b^2\varpi}{4} - \log_e 0 =$ infinity.

3. *To find the area by means of equation* 4 (*page* 509), $r^2 = x^2 + y^2 = x^2(1 + \tan^2\theta) = b^2 \pm 2ab \sec\theta + a^2\sec^2\theta$; wherefore putting $\tan^2\theta = t$, we have

$$x^2 = \frac{b^2}{1 + t^2} \pm \frac{2ab}{\sqrt{1 + t^2}} + a^2; \text{ and hence}$$

$$A = \int x^2 dt = b^2\int\frac{dt}{1 + t^2} \pm 2ab\int\frac{dt}{\sqrt{1 + t^2}} + a^2\int dt,$$

which, integrated, gives the same result as the first form, for the area.

IX. The Lemniscate of Bernoulli.

1. *The length.* Take the polar equation, $r^2 = a^2\cos 2\theta$. Then,

$$r dr = -a^2 \sin 2\theta . d\theta = -a^2 d\theta . \sqrt{1 - \frac{r^4}{a^4}} = -\sqrt{a^4 - r^4} . d\theta.$$

Hence $ds = \sqrt{dr^2 + r^2 d\theta^2} = \frac{a^2 dr}{\sqrt{a^4 - r^4}}$.

To integrate, put $r = az$: then we have

$$s = a\int\frac{dz}{\sqrt{1 - z^2}}.\frac{1}{\sqrt{1 + z^2}} = a\int\frac{1}{\sqrt{1 - z^2}}\left\{1 - \frac{1}{2}z^2 + \frac{1.3}{2.4}z^4 - \frac{1.3.5}{2.4.6}z^6 + \dots\right\}.$$

Taking the integral of this between the limits 0, 1, as values of z (which correspond to the limits 45°, 0 as values of θ), we have

$$\begin{array}{c|c} \int_0^1 \frac{dz}{\sqrt{z^2 - 1}} = \frac{\varpi}{2} & \int_0^1 \frac{z^4 dz}{\sqrt{z^2 - 1}} = \frac{1 . 3}{2 . 4}.\frac{\varpi}{2} \\ \int_0^1 \frac{z^2 dz}{\sqrt{z^2 - 1}} = \frac{1}{2}.\frac{\varpi}{2} & \int_0^1 \frac{z^6 dz}{\sqrt{z^2 - 1}} = \frac{1 . 3 . 5}{2 . 4 . 6}.\frac{\varpi}{2}, \end{array}$$

etc. etc.

Hence, collecting the terms and quadrupling the result for the whole length,

$$s = 2\varpi a\left\{1 - \frac{1^2}{2^2} + \frac{1^2 . 3^2}{2^2 . 4^2} - \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} + \dots\dots\right\}$$

2. *The area.* Here $\frac{1}{2}r^2 d\theta = \frac{1}{2}a^2\cos 2\theta d\theta = \frac{1}{4}a^2\cos 2\theta\, d(2\theta)$, and hence $\frac{1}{2}\int r^2 d\theta = \frac{1}{4}a^2\sin 2\theta + C$. Also when $\theta = 0$, we have $A = 0$, and hence $C = 0$: wherefore the area of one-fourth of the curve is $\frac{1}{4}a^2$, or the entire area is, simply, a^2.

X. The Trigonometrical Curves.

1. *The area of the curve of sines, whose equation is* $y = a\sin\frac{x}{a}$. Here

$$\int y dx = a^2\int\sin\frac{x}{a}.d\left(\frac{x}{a}\right) = -a^2\cos\frac{x}{a} + C.$$

Also, when $\frac{x}{a} = 0, \cos\frac{x}{a} = 1$; and as the abscissa and area commence together, we have $C = a^2$. Wherefore

$$A = a^2\left(1 - \cos\frac{x}{a}\right) = a^2 \operatorname{ver}\sin\frac{x}{a}.$$

For the whole area, between $y = 0$ and $y = a$ we have a^2; which is the whole area comprised between two consecutive intersections of the curve with the axis.

2. *The length.* This is expressed by $\int\sqrt{\frac{2a^2 - y^2}{a^2 - y^2}}\,.\,dy$, which is of the same general form as the elliptic arc.

The figure of the cosines is in all respects similar to that of the sines; except that the origin of x is midway between the two consecutive intersections of the curve with that axis.

The whole lengths and areas of the figures of the *tangents, co-tangents, secants, and co-secants,* are infinite, as will be seen on integrating the appropriate equations. For instance, take $y = a\tan\frac{x}{a}$: then

$$\int y dx = a^2\int\tan\frac{x}{a}\,.\,d\left(\frac{x}{a}\right) = a^2\int\frac{\sin\frac{x}{a}\,d\left(\frac{x}{a}\right)}{\cos\frac{x}{a}} = -a^2\int\frac{d\cos\frac{x}{a}}{\cos\frac{x}{a}}$$

$$= -a^2\log_e\cos\frac{x}{a} + C;$$

and, since x and A begin together, we have $C - a^2\log_e\cos 1 = 0$; or $C = 0$. When $\frac{x}{a} = \frac{1}{2}\pi$, then $\cos\frac{x}{a} = 0$, and $\log_e 0 = -$ infinity; and hence we have $A = a^2 \times$ *infinity.*

3. *The area of the figure of the versed sines.* Here we have the equation,

$$y = a\operatorname{vers}\frac{x}{a} = a\left(1 - \cos\frac{x}{a}\right): \text{ and hence}$$

$A = \int y dx = a^2\int\left(1 - \cos\frac{x}{a}\right)d\left(\frac{x}{a}\right) = ax - a^2\sin\frac{x}{a}$, the correction being, as in the figures of tangents, zero.

The area of the whole figure is obviously $2a^2\varpi$, or double the area of the circle.

The area of the figure of the *co-versed sines* is found in the same way: and the length is equal to assignable elliptic.

XI. The Spiral of Archimedes.

This spiral is generated by the uniform angular revolution of the radius-vector, whilst a point moves *from* the pole with a uniform linear velocity. The constant factor a is the radius of the generating circle; or the value of r when $\theta = 2\pi$. The form of its equation is, properly speaking, $r = a\,\frac{\theta}{2\pi}$: but if we take 2π (an entire revolution) as the unit of angular magnitude, it takes the simpler form given at *page* 345, viz. $r = a\theta$.

1. *The length.* This is equally determinable in terms of r or θ. For

$$ds = \sqrt{r^2d\theta^2 + dr^2} = a\sqrt{1 + \theta^2}\,.\,d\theta = \frac{1}{2}\int\sqrt{r^2 + a^2}\,.\,dr.$$

The integrals being taken and corrected, we have the following forms:

$$s = \frac{1}{2}\, a\theta \sqrt{1 + \theta^2} + \frac{1}{2}\, a \log_e \frac{\theta + \sqrt{1 + \theta^2}}{2}, \text{ and}$$

$$s = \frac{r \sqrt{a^2 + r^2}}{2a} + \frac{a}{2} \log_e \frac{r + \sqrt{a^2 + r^2}}{2a}.$$

The latter of these is precisely the length of a parabolic arc, the parameter being $2a$, intercepted between the vertex and a point whose ordinate is equal to r. In fact there is this remarkable relation between the *equable spiral* (a name often given to this one) and the parabola: that if the spiral be "divested of its radiating structure," the curve becomes a parabola. *Leslie's Curve Lines, page* 420.

2. *The area.* Here $\frac{1}{2}\int r^2 d\theta = \frac{1}{2}\, a^2 \int \theta^2 d\theta = \frac{1}{6}\, a^2\theta^3$.

Relative to spiral areas (which are such in *structure*, not merely because they are expressed by *polar equations*) it is important to make one remark.

If we suppose θ to increase indefinitely, the same geometrical space will be repeatedly swept out by succeeding revolutions of the radius-vector, and hence for the entire area between any two convolutions, we must take the integral between the values of θ, which belong to the beginning and the end of the last evolution.

The expressions being taken for $(n-1)^{\text{th}}$, n^{th}, and $(n+1)^{\text{th}}$ revolutions, we have

$$\frac{1}{6}\,\{(n-1) \,.\, 2\varpi\}^3 a^2, \frac{1}{6}\,\{n \,.\, 2\varpi\}^3, \text{ and } \frac{1}{6}\,\{(n+1) \,.\, 2\varpi\}^3.$$

The area, then, lying between the $(n-1)^{\text{th}}$ and n^{th}, and between the n^{th} and and $(n+1)^{\text{th}}$ are respectively.

$$\frac{1}{6}\{(3n^2 - 3n + 1)\,(2\varpi)^3\}a^2, \text{ and } \frac{1}{6}\{(3n^2 + 3n + 1)\,(2\varpi)^3\}a^2.$$

Between these two last expressions, there exists an elegant relation: *viz.*

$$\frac{1}{6}\,\{(3n^2 + 3n + 1)\,(2\varpi)\}a^2 - \frac{1}{6}\,\{3n^2 - 3n + 1)\,(2\varpi)^3\}\,a^2 = n\,(2\varpi)^3 a^2.$$

which is $\frac{n}{6}$ times the area swept out by the first revolution of the radius-vector.

XII. THE HYPERBOLIC OR RECIPROCAL SPIRAL.

1. *The length.* Proceeding as before, we have, since $r = a\theta^{-1}$

$$s = a^2\int\frac{dr}{r\sqrt{r^2 + a^2}} + \int\frac{rdr}{\sqrt{r^2 + a^2}} = a \log_e \frac{\sqrt{r^2 + a^2} - a}{r} + \sqrt{r^2 + a^2} + \text{C}.$$

$$s = a\int\frac{d\theta}{\theta^2\sqrt{1 + \theta^2}} + a\int\frac{d\theta}{\sqrt{1 + \theta^2}} = a\theta^{-1}\sqrt{1 + \theta^2} + a \log_e (\theta + \sqrt{1 + \theta^2}) + c.$$

When r is 0, θ is infinite; and when r is infinite, θ is 0: whence we have no direct method of finding C or c: but we may take either expression between assigned limits.

2. *The area.* Here $\frac{1}{2}\int r^2 d\theta = -\frac{1}{2}\, a \int dr = -\frac{1}{2}\, ar + c$.

Which, from $r = r$ to $r = 0$, gives $\text{A} = \frac{1}{2}\, ar =$ the triangle formed by the radius-vector, tangent and subtangent.

XIII. THE EXPONENTIAL, OR GENERAL LOGARITHMIC SPIRAL.

The general and proper form of the equation of this curve is

$$r = ae^{\mu \cdot \frac{\theta}{2\pi}}$$

or, if we consider 2π the unit of angular magnitude *, it is simply

$$r = ae^{\mu\theta}.$$

The length and area may be found with great facility either by means of the equation above given in $r\theta$, or by means of that in rp. We shall give them, for examples, by both methods.

1. *The length by polar co-ordinates, $r\theta$.* We have in this case

$$dr = \mu a\, e^{\mu\theta}\, d\theta = \mu r d\theta\,; \text{ or } r d\theta = \frac{dr}{\mu}\,; \text{ whence}$$

$$ds = \sqrt{r^2 d\theta^2 + dr^2} = \sqrt{1 + \mu^2}\, r d\theta = \sqrt{1 + \mu^2}\, \frac{dr}{\mu}, \text{ and}$$

$$s = \frac{r}{\mu} \sqrt{1 + \mu^2} = \frac{\sqrt{1 + \mu^2}}{\mu} \,.\, ae^{\mu\theta}.$$

2. *The area by polar co-ordinates.* In this case we get

$$\frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int \frac{r dr}{\mu} = \frac{r^2}{\mu} = \frac{a^2 e^{2\mu\theta}}{\mu}.$$

3. *The area in r and p.* It is shown, generally, (*p.* 509) that

$$p = \frac{r^2 d\theta}{ds} = \frac{r \,.\, \dfrac{dr}{\mu}}{\dfrac{dr}{\mu} \,.\, \sqrt{1 + \mu^2}} = \frac{r}{\sqrt{1 + \mu^2}}.$$

4. *The length by means of r and p.* By (509), we have

$$s = \int \frac{r dr}{\sqrt{r^2 - p^2}} = \int \frac{dr}{\mu} \,.\, \sqrt{1 + \mu^2} = \frac{r}{\mu} \sqrt{1 + \mu^2}$$

6. The tangent makes a *constant angle with the radius vector.*

For $\frac{r}{p} = \sqrt{1 + \mu^2}$ is the cosecant of that angle; and hence if the angle be denoted as at *p.* 396, by ω, we have

$$\omega = \sqrt{1 + \mu^2}, \text{ or } \omega = \mu, \text{ or } \tan \omega = \frac{1}{\mu}\, ,$$

that is, the modulus of the system of logarithms to which the curve is constructed is the trigonometrical tangent of the angle made by the radius-vector and the tangent.

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* The unit of angular magnitude, like that of linear, is *perfectly arbitrary*. Many writers have contended that angular magnitude has "*a natural unit*," namely, the right angle, or four right angles; and reasonings have been founded upon this assumption which are altogether inconclusive. For instance, that of Legendre, on *the theory of parallel lines* in the elements of geometry. That this assumption is *a very convenient one*, we admit; and especially in researches of the present nature. There are other cases in which $\frac{2\pi}{\varpi}$ (that is the angle measured by an arc of the circle which is equal to the radius) is more convenient; especially in researches in *Geodesy*.

Other instances may be easily adduced of inquiries where still different angular units are conveniently employed.

On this account the curve has been likewise called the *logarithmic spiral*, and the *equiangular spiral*.

XIV. The Involute of the Circle.

The equation in r and p is easily found to be, a being the radius of the circle,

$$r^2 = p^2 + a^2.$$

1. *The length.* Applying the formula adapted to the case (*p.* 509, *eq.* 6),

$$ds = \frac{rdr}{a}, \text{ or } s = \frac{r^2}{2a} + \mathrm{C}.$$

To find C, we have to recollect that the origin of s is where $r = a$, and hence at this point,

$$0 = \frac{a^2}{2a} + \mathrm{C}, \text{ or } \mathrm{C} = -\frac{a^2}{2a}; \text{ whence,}$$

$$s = \frac{r^2 - a^2}{2a} = \frac{p^2}{2a}.$$

It is also obvious that p is the radius of curvature of the involute, and hence equal to the arc of the circle which is unwound: whence, if ω be that arc, we have

$$s = \tfrac{1}{2} a\omega^2.$$

2. *The area.* Employing the corresponding formula, *p.* 509, *eq.* 3.

$$\mathrm{A} = \frac{1}{2}\int \frac{pr\,dr}{\sqrt{r^2 - p^2}} = \frac{1}{2a}\int r\sqrt{r^2 - a^2}\,dr = \frac{p^3}{6a} + \mathrm{C}.$$

When $p = 0$, then $\mathrm{A} = 0$, and hence $\mathrm{C} = 0$; and we have simply

$$\mathrm{A} = \frac{p^3}{6a}.$$

XV. The Epicycloïd.

1. *The length.* Let a be the radius of the fixed circle, b that of the rolling one, $c = a + 2b$, and the axis of x the radius drawn to the origin of the curve. Then the equation in r and p becomes

$$p^2 = \frac{c^2(r^2 - a^2)}{c^2 - a^2}, \text{ and } r^2 - p^2 = \frac{a^2(c^2 - r^2)}{c^2 - a^2};$$

whence,

$$s = \int \frac{rdr}{\sqrt{r^2 - p^2}} = -\frac{\sqrt{c^2 - a^2}}{a}\int \frac{rdr}{\sqrt{c^2 - r^2}} = \frac{\sqrt{(c^2 - a^2)(c^2 - r^2)}}{a} + \mathrm{C}.$$

But when $s = 0$, $c = a$, and $\mathrm{C} = 0$.

2. *The area.* By employing the corresponding formula for A, we get

$$\mathrm{A} = \frac{c}{2a}\int \sqrt{\frac{r^2 - a^2}{c^2 - r^2}}\,.\,rdr = \frac{c}{2a}\int \frac{\sqrt{r^2 - a^2}\,.\,rdr}{\sqrt{(c^2 - a^2) - (r^2 - a^2)}}$$

$$= -\frac{c\sqrt{(r^2 - a^2)(c^2 - r^2)}}{4a} + \frac{c(c^2 - a^2)}{4a}\sin^{-1}\sqrt{\frac{r^2 - a^2}{c^2 - a^2}} + \mathrm{C}.$$

When $\mathrm{A} = 0$, then $c = r = a$, and hence $\mathrm{C} = 0$.

Now for an entire revolution of the rolling circle, we have the area of the segment of the fixed circle and of the epicycloïd respectively,

$$ab\varpi, \text{ and } \frac{c\,(c^2 - a^2)\varpi}{4a}, = \frac{b}{a}\,(a^2 + 3ab + 2b^2)\,\varpi.$$

The difference of these, or the proper area of the epicycloïd, is found to be

$$A = \frac{3a + 2b}{a}.\,b^2\varpi.$$

XVI. Evolutes of Curves.

The length of an evolute may be found in the same manner as the length of any other curve, when its equation can be found. It is, however, comparatively seldom that the eliminations required for finding the evolute can be effected; and, therefore, that the general method of rectification can be applied to these curves. Even when such elimination is possible, the equation of the evolute is almost invariably very complex; and it is obvious, that in such cases the expression for ds will become still more complex, and often irresolvable into a function of one of the co-ordinates only, or into a function of any third variable, of which x and y are determinable functions. After all, should one or other of these be possible, the resulting expression may, and most commonly is, one which does not admit of integration in finite terms, or in any of the transcendent functions (the exponential, logarithmic, or trigonometrical) admitted into researches of this kind.

To evade this difficulty, however, the property given at *page* 410 is commonly employed. This scarcely belongs to the integral calculus; but as in this particular case it supersedes the necessity of employing the ordinary method of rectification, the present appears to be the appropriate place for its introduction. It leads to this rule:

The length of a curve between the limits x_1y_1 *and* x_2y_2 *is* $\rho_2 - \rho_1$, *where* ρ_1 *and* ρ_2 *are the radii of curvature at the points* x_1y_1, x_2y_2 *respectively.*

In respect of the *quadrature of evolutes*, there has been no method yet made public by which the general process of integrating ydx or $\frac{1}{2}r^2d\theta$ can be abbreviated or modified from any considerations peculiar to the properties of the evolute; and although the editor is in possession of a method which effects the purpose without much difficulty, want of room prevents his inserting it here *.

(1). *The rectification of the evolute of the common parabola.*

Here, if $y^2 = 4ax$, we have $\rho = \dfrac{2\,(a + x)^{\frac{3}{2}}}{\sqrt{a}} + C.$

To find C put $x = 0$; then the evolute is also 0; and we have $\rho_1 = 2a + C$, where ρ_1 is the radius of curvature at the vertex or origin. Whence

$$s = \rho - \rho_1 = \left\{\frac{2\,(a + x)^{\frac{3}{2}}}{\sqrt{a}} + C\right\} - (2a + C) = \frac{2}{\sqrt{a}}\left\{(a + x)^{\frac{3}{2}} - a^{\frac{3}{2}}\right\}.$$

This curve, which is the *semi-cubical parabola*, is the first whose length was ever found as a finite function of the co-ordinates. It was discovered by William Neill, who was led to it by a remark of Dr. Wallis. See *Wallisii, Opera, tom. i. p.* 551.

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* A few general theorems respecting the *radius of curvature*, *the evolute*, and *the involute*, are given for the student's exercise a little further on, *pp.* 534—5.

(2). *The rectification of the evolute of the ellipse.*

The ellipse referred to its centre and principal axes is $a^2y^2 + b^2x^2 = a^2b^2$; and $\rho = \frac{(a^2 - e^2x^2)^{\frac{3}{2}}}{ab}$. Put ρ_1, ρ_2, for the radii of curvature at the vertices of the major and minor axes: then the corresponding values are $\rho_1 = \frac{b^2}{a}$, $\rho_2 = \frac{a^2}{b}$; and hence, estimated from the cusps, the lengths of any arc corresponding to the point xy of the ellipse, are respectively

$$\frac{a^3 - (a^2 - e^2x^2)^{\frac{3}{2}}}{ab}, \text{ and } \frac{(a^2 - e^2x^2)^{\frac{3}{2}} - b^3}{ab}.$$

The length of the quadrant of the evolute is $\frac{a^2}{b} - \frac{b^2}{a} = \frac{a^3 - b^3}{ab}$.

(3). *The length of an arc of the evolute of the hyperbola* is found in the same manner. For $\rho = \frac{(e^2x^2 - a^2)^{\frac{3}{2}}}{ab}$; and, when $x = a$, we have the radius of curvature at the vertex of the transverse axis, $\rho_1 = \frac{b^2}{a}$. Hence,

$$s = \rho - \rho_1 = \frac{(e^2x^2 - a^2)^{\frac{3}{2}} - b^3}{ab}.$$

But the length of the evolute of the rectangular hyperbola may be presented in another form; viz. by taking the curve in reference to the asymptotes, $xy = a^2$. For then

$$\rho = -\frac{(x^4 + a^4)^{\frac{3}{2}}}{2a^2x^3} + \text{C}.$$

(4). *To rectify the evolute of the cycloïd.* (*See page* 413, 518.)

(5). *To find the length of the evolute of the catenary.*

Here $2y = a\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\}$, $\rho = \frac{y^2}{a}$, disregarding the *sign*, as we may do in these inquiries, since the sign only indicates the *direction* of curvature. Now at the vertex, $x = 0$, and then $y = a$, and $\rho_1 = a$; wherefore,

$$s = \rho - \rho_1 = \frac{y^2}{a} - a = \frac{y^2 - a^2}{a}.$$

(6). *To find the area of the evolute of the parabola.*

When referred to the vertex of the parabola as origin, the parabola itself being $y^2 = 4ax$, the evolute is $27ay^2 = 4(x - 2a)^3$; and transferring the origin to the vertex of the evolute, by writing $x + 2a$ for x, this becomes $27ay^2 = 4x^3$, or

$$y = \frac{2}{3\sqrt{3a}} \,.\, x^{\frac{3}{2}}; \text{ and } y\,dx = \frac{2x^{\frac{3}{2}}dx}{3\sqrt{3a}}, \text{ or integrating,}$$

$$\text{A} = \frac{4x^{\frac{5}{2}}}{15\sqrt{3a}}, \text{ or for both symmetrical parts, } \text{A} = \frac{8x^{\frac{5}{2}}}{15\sqrt{3a}}:$$

the constant being 0, since A and x commence together.

(7). *The area of the evolute of the ellipse.* Here $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (ae)^{\frac{4}{3}}$; or putting $a^{\frac{2}{3}} e^{\frac{4}{3}} = c^2$, and z^3 for x, we have for the area

$$\int y dx = \frac{a}{b} \int (c^2 - x^{\frac{2}{3}})^{\frac{3}{2}} dx = \frac{3a}{b} \int (c^2 - z^2)^{\frac{3}{2}} z^2 dz.$$

This integral has been found at *p.* 480: and we have

$$\int y dx = \frac{a}{16b} \left\{ 3c^6 \sin^{-1} \frac{z}{c} - z\,(8z^4 - 14c^2 z^2 + 3c^4) \sqrt{c^2 - z^2} \right\} + \text{C}.$$

We may ascertain the value between the final limits without restoring the values of z and c in terms of x, a, b; as will appear from the following simple considerations. This, in fact, will be very generally the case with integrations between limits by means of subsidiary substitutions.

When $x = 0$, then $z^3 = 0$, or $z = 0$, and the integral $= 0$; and when $z = c$, or $x^{\frac{2}{3}} = a^{\frac{2}{3}} e^{\frac{4}{3}}$, then the integral becomes $\frac{a^2 e^2 . e^2}{16} . \frac{\varpi}{2}$. Hence,

$$\int_o^c z^2 (c^2 - z^2)^{\frac{3}{2}} dz = \frac{a^2 e^2 . e^2}{16} . \frac{\varpi}{2} = \frac{(a^2 - b^2)^2}{16 a^2} . \frac{\varpi}{2}.$$

The whole area, from the symmetry of the four containing axes, is

$$\frac{3a}{b} . \frac{(a^2 - b^2)^2}{a^2} \frac{\varpi}{8} = \frac{3\,(a^2 - b^2)^2 \varpi}{8ab}.$$

The area might be found, however, by a different method, as follows:

Divide both sides by $(a^2 - b^2)^{\frac{2}{3}}$: then the equation of the curve is changed into

$$\left(\frac{ax}{a^2 - b^2}\right)^{\frac{2}{3}} + \left(\frac{by}{a^2 - b^2}\right)^{\frac{2}{3}} = 1.$$

Now in the original equation of the evolute if we put x and y successively equal to zero, we have

$$x_0 = \frac{a^2 - b^2}{a}, \text{ and } y_0 = \frac{a^2 - b^2}{b}.$$

Denote these by α and β respectively; then the equation of the curve becomes

$$\left(\frac{x}{\alpha}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1.$$

Assume $\frac{x}{\alpha} = \cos^3\theta$, and therefore also $\frac{y}{\beta} = \sin^3\theta$: then,

$$dx = -\,3\alpha \sin\theta \cos^2\theta\, d\theta; \text{ and } y dx = -\,3\alpha\beta \sin^4\theta \cos^2\theta\, d\theta.$$

To integrate this, put $u = \cos\theta$, $dv = \cos\theta \sin^4\theta\, d\theta = \frac{1}{5} d \sin^5\theta$, and hence also $du = -\sin\theta\, d\theta$: wherefore the equation of parts, $\int u dv = uv - \int v du$ becomes

$$\int \sin^4\theta \cos^2\theta\, d\theta = \frac{1}{5} \cos\theta \sin^5\theta + \frac{1}{5} \int \sin^6\theta\, d\theta,$$

$$\int \sin^6\theta\, d\theta = -\left\{\frac{1}{6} \sin^5\theta + \frac{5}{24} \sin^3\theta + \frac{5}{16} \sin\theta\right\} \cos\theta + \frac{5}{16}\theta.$$

Hence we have

$$\text{A} = -\,3\alpha\beta \left\{\left(\frac{1}{6} \sin^5\theta - \frac{1}{24} \sin^3\theta - \frac{1}{16} \sin\theta\right) \cos\theta\right\} - \frac{3}{16}\alpha\beta\theta.$$

When $\theta = 0$, and when $\theta = \pi$, the bracketed part vanishes; and the last term becomes 0 and $\frac{3}{16}\,\alpha\beta\varpi$ in the two cases, respectively: whence,

$$\overset{0}{\underset{\pi}{\mathrm{A}}} = \frac{3}{16}\,\alpha\beta\varpi = \frac{3\,(a^2 - b^2)^2\varpi}{16ab}.$$

But when θ is equal to 0 and π respectively, then x is equal to a and $-a$ respectively: wherefore the limits above employed for θ are those which belong to the extremities of the upper double branch of the curve, and the area just found is the half of the entire surface sought. The entire area, therefore, is, as before,

$$\mathrm{A} = \frac{3\,(a^2 - b^2)^2\varpi}{8ab}.$$

The area of the evolute of the hyperbola may be found by the assumption of $\frac{x}{\alpha} = \sec^3\theta$, $\frac{y}{\beta} = \tan^3\theta$, the curve being $\left(\frac{x}{\alpha}\right)^{\frac{2}{3}} - \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1$, and α, β, determined as before. The differential expression is left for the student's exercise to integrate; viz.

$$\mathrm{A} = \int y\,dx = 3\alpha\beta \int \frac{\sin^4\theta\,d\theta}{\cos^7\theta}.$$

Scholium 1.

The theory of the *involute* is capable of being treated without finding the equation of the curve, in the same manner as is referred to at *page* 530: but for the reason there stated it cannot be introduced into the present volume. The editor hopes, however, to be able to lay it before his readers at some future time, either in a supplementary volume to the present work, or in some periodical work. See also the following page.

Scholium 2.

It will often occur that only the differential equation of the curve, or sometimes the second, third, *etc.* differential equations are either given directly, or implied in the geometrical conditions, from which to determine the curve itself and its properties.

One or two examples of this kind have already occurred in the preceding chapter: as the tractory, the catenary, and the logarithmic curve, viewed as the equitangential curve, its evolute, and the equisubtangential curve.

The general problem implied in this scholium divides itself into two parts: the expression of the defining property in terms of the differential co-efficients of the principal variable, and the subsequent integrations of the resulting equation. Most curves that result from physical inquiries are of this class.

We shall give a single example for the purpose of illustration.

1. *In what curve is the normal in a constant ratio to the radius of curvature?*

Let n express the constant ratio: then the equation expressing the condition is

$$\frac{ds^3}{\mp\,dx\,d^2y} = n.\frac{y\,ds}{dx}, \text{ or } y\,d^2y \pm n\,ds^2 = 0.$$

Assume $\frac{dy}{dx} = y'$; hence $y\,d^2y = y\,dy'\,dx$; and $ds^2 = (1 + y'^2)\,dx^2$. Insert these values, and for dx write its value $\frac{dy}{y'}$; and we get

$$y\,dy' \pm n\,(1 + y'^2)\,\frac{dy}{y'} = 0, \text{ or } \frac{y'dy'}{1 + y'^2} \pm \frac{n\,dy}{y} = 0. \text{ Whence,}$$

$$\log_e (1 + y'^2)^{\frac{1}{2}} \pm \log_e \left(\frac{c}{y}\right)^n = 0, \text{ or } (1 + y'^2)^{\frac{1}{2}} \pm \left(\frac{c}{y}\right)^n = 0.$$

This becomes,

$$1 + y'^2 = \left(\frac{c}{y}\right)^{2n}, \text{ or } dx = \frac{y^n dy}{\sqrt{c^{2n} - y^{2n}}};$$

which is the first differential equation; and represents different classes of curves according to the values of n. We shall examine a few of them.

1. Let $n = 0$; then $dy = 0$, or $y = c_1$; expressing a line parallel to the axis of x at the distance c_1. The radius of curvature is here infinite.

2. Let $n = \frac{1}{2}$: then $x = \int \frac{y^{\frac{1}{2}} dy}{\sqrt{c - y}} = \int \frac{y dy}{\sqrt{cy - y^2}}$

$$= c_2 + \frac{c}{2} \operatorname{versin}^{-1} \frac{2y}{c} - \sqrt{cy - y^2}.$$

3. Let $n = -\frac{1}{2}$: then $x = \int \frac{y^{-\frac{1}{2}} dy}{\sqrt{c^{-1} - y^{-1}}} = \frac{c^{\frac{1}{2}} dy}{2\sqrt{y - c}} = \int \sqrt{cy - c^2} + c_3$;

or $\frac{1}{4}(x - c_3)^2 = cy - c^2$, the equation of a parabola the co-ordinates of whose vertex are c_3, c_1, and parameter is $4c_1$.

4. Let $n = 1$: then $x - c_4 = -\sqrt{c^2 - y^2}$, the equation of a circle.

5. Let $n = -1$: then $dx = \frac{y^{-1} dy}{\sqrt{c^{-2} - y^{-2}}} = \frac{c dy}{\sqrt{y^2 - c^2}}$, which is the differential equation of the *catenary*.

6. Let $n = 2$: then $dx = \frac{y^2 dy}{\sqrt{c^4 - y^4}}$, the differential equation of the *elastic curve*, or the curve into which a perfectly elastic filament forms itself when fixed at one end and pushed out of its rectilinear position at the other.

EXERCISES ON THE LENGTH AND AREAS OF CURVES.

I. *General Theorems.*

1. If a curve be referred to an angle of ordination a, show generally that

$$A = \frac{1}{2} \sin a \int \{x dy - y dx\}.$$

2. Let OM, MP be the co-ordinates of a curve, and OP the chord: then if arc of the curve OP′P divide the area of the triangle OMP always in a given ratio of $\frac{1}{2} m : n$, the curve will be the general parabola, expressed by $y^{m+n} = a^m x^n$.

3. If r be the radius-vector of a point in the curve, and ω the angle formed by the tangent and rectangular ordinate at that point: then the length of the curve traced by the intersection of the tangent with the perpendicular upon it from the origin will be found from

$$s' = \int r d\omega.$$

4. Let AP $= s$ be the arc of a curve unwound to form the arc AP′ $= s'$ of the involute, and ρ be the radius of curvature at P: then,

$$s' = \int \frac{s ds}{\rho}; \text{ and area APP}' = \int \frac{s^2 ds}{2\rho}.$$

5. Let, in the same case, xy be the co-ordinates of P, and $x'y'$ those of P′: then

$$dx' = \frac{s dx}{\rho}; \text{ and } dy' = \frac{s dy}{\rho}.$$

6. Also under the same circumstances we shall have

$$x' = y - \frac{s dy}{ds}; \text{ and } y' = \frac{s dx}{ds} - x.$$

7. When the evolute is rectifiable, the equation of the involute is determinable in finite algebraical terms:—

And when the length of the evolute involves a transcendent, the equation of the involute will involve a similar function.

8. The subtangent and subnormal of a polar curve, in $r_1\theta_1$, intercepted on the polar axis, are respectively

$$\frac{r_1^2 d\theta_1}{d(r_1 \sin\theta_1)}, \text{ and } \frac{r_1 dr_1}{d(r_1 \cos\theta_1)}.$$

9. If the radius of curvature be a given function of the arc, or $\rho = f(s)$: then will the equation of the curve be determinable from the three equations

$$ds^2 = dx^2 + dy^2,\ x = \int ds \sin\int\frac{ds}{f(s)},\ \text{and}\ y = \int ds \cos\int\frac{ds}{f(s)}.$$

Apply the theorem to $f(s) = \text{const.}$, $f(s) = as + b$, and $f(s) = \sqrt{1-s^2}$.

10. If a curve be defined by the equation $\rho = f(\omega)$ where ρ is the radius of curvature, and ω the angle which it makes with a fixed axis; also if ρ' be the radius of curvature of its evolute, and there exist, also, the relation $\rho' = \frac{df(\omega)}{d\omega}$ these curves will be identical: or, which is the same thing, the given curve will be its own evolute, and likewise its own involute.

II. *Particular Curves.*

11. The *involute of the cardioïd* is another cardioïd: find the circle from which this second cardioïd could be generated.

12. The equation of the *syntractory* (generated by a fixed point in the tangent of the tractrix) is

$$x + \sqrt{b^2 - y^2} = a \log_e \frac{b + \sqrt{b^2 - y^2}}{y};$$

when b is the distance of the point from the directrix, and a the same as at *p.* 519.

13. The *sectoral area of the parabola*, the focus being the pole, and vertex the origin, is

$$\overset{\theta}{\underset{0}{\mathrm{A}}} = a^2\left\{\tan\frac{1}{2}\theta + \frac{1}{3}\tan^2\theta\right\}.$$

14. Show that the area of the *curve* $\left(\frac{x}{a}\right)^{\frac{2}{2n+1}} + \left(\frac{y}{b}\right)^{\frac{2}{2n+1}} = 1$, is

$$2\varpi\,(2n+1)\,ab\left\{\frac{1.3...(2n+1)}{2.4...(2n+2)} - \frac{n}{1}\cdot\frac{1.3...(2n+3)}{2.4...(2n+4)} + \frac{n(n-1)}{1.2}\cdot\frac{1.3...(2n+5)}{2.4...(2n+6)} - ...\right\}$$

15. If a be a *given arc of a circle to radius unity*, and $a_1, a_2 \ldots\ldots a_{\text{inf}}$ be the *successive involutes* (the development always commencing at the same point) taken to infinity; show that

$$a + a_1 + a_2 + \ldots\ldots a_{\text{inf}} = e^a - 1.$$

16. If a *catenary* whose parameter is a, and an *equilateral hyperbola* whose semi-axis is also a, have the same vertex and coincident axes, show

(1). That the ordinates of the hyperbola are equal to the corresponding arcs of the catenary.

(2). That the rectangle under the parameter and ordinate of the catenary is double the corresponding hyperbolic sector, the sectoral vertex being the centre of the hyperbola.

17. The difference of the tangent and subtangent is constant, and equal to a: show that the equation of the curve is

$$x = \frac{y^2 - a^2}{4a} + \frac{a}{2} \log_e \frac{a}{y}.$$

18. If the radius of curvature vary as the cube of the normal, the curve is a conic section.

19. Let OA be the axis of one loop of the *lemniscate of Bernoulli*, and draw two radii-vectors OP, OQ, making angles ω_1, ω_2 with OA, such that $\cos \omega_1 \cos \omega_2 = \cos 45^\circ$: then the arcs OP, AQ of the lemniscate are equal to one another; and the line OR which bisects the angle of ordination also bisects the quadrantal arc of the lemniscate.

20. If the rectangle under the radius of curvature and abscissa of a curve be equal to c^2: then that curve is the *elastic curve* (*p.* 534, *case* 6).

21. Deduce the following expressions for the perimeter of a circle, its reciprocal, and the square of its reciprocal.

$$\varpi = 4 \sec \frac{\pi}{4} \sec \frac{\pi}{8} \sec \frac{\pi}{16} \sec \frac{\pi}{32} \ldots \textit{ad inf.}$$

$$\frac{1}{\varpi} = \frac{1}{2} \tan \frac{\pi}{2} + \frac{1}{4} \tan \frac{\pi}{4} + \frac{1}{8} \tan \frac{\pi}{8} + \ldots \textit{ad inf.}$$

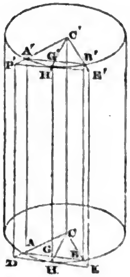
$$\frac{1}{\varpi^2} = \frac{5}{48} - \frac{1}{4} \left\{ \frac{1}{4^2} \tan^2 \frac{\pi}{8} + \frac{1}{4^3} \tan^2 \frac{\pi}{16} + \frac{1}{4^4} \tan^2 \frac{\pi}{32} + \ldots \textit{ad inf.} \right\}$$

CHAPTER X.

SURFACES AND SOLIDS OF REVOLUTION.

PROP. I.

To find the surface and volume of a right cylinder; the altitude and the diameter of the base being given.



1. *The surface.* Let AB be the side of a regular polygon inscribed in the circular base; CC' the axis of the cylinder; CA, CB, CH, radii, CH bisecting the angle formed by the other two; and DE a tangent to the circular base AHE, meeting CA, CB, in D and E. Then DE is the side of a regular polygon of the same number of sides circumscribed to the circular base. Also, if planes be drawn through AB and DE parallel to the axis, they will be the faces of prisms inscribed in and circumscribed about the cylinder.

Let each of the prisms have n faces, and the radius $AC = r$, and height $C'C = h$: then ACH

$= \frac{\pi}{n}$; AB $= 2r \sin \frac{\pi}{n}$; and DE $= 2r \tan \frac{\pi}{2}$. Whence, the surfaces of the inscribed and circumscribed prisms are, respectively,

$$2nrh \sin \frac{\pi}{n} \text{ and } 2nrh \tan \frac{\pi}{n}.$$

But at the limit, the ratio $\frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} = \frac{\frac{\pi}{n}}{\tan \frac{\pi}{n}} = 1$; and hence the cylindrical surface which is intermediate to the two prisms becomes simply,

$$S = 2nrh \,.\, \frac{\pi}{n} = 2\varpi rh.$$

2. *The volume.* In the same manner we find the volumes of the inscribed and circumscribed prisms are

$$nr^2h \sin \frac{\pi}{n} \text{ and } nr^2h \tan \frac{\pi}{n}.$$

Hence, at the limit the volume of the cylinder which is intermediate to these becomes

$$V = nr^2h \frac{\pi}{n} = \pi r^2 h.$$

In very nearly the same way we may find the volume and surface of a right cone whose base and altitude are given; but this will be noticed amongst the examples.

Corollary 1.

If the cylinder be oblique, the same reasoning as to the prisms inscribed and circumscribed applies. Whence, if a be the axis, and α the inclination of the axis to the base, we have $h = a \sin \alpha$, and the volume and surface will be expressed by

$$V = \varpi r^2 h = \varpi r^2 a \sin \alpha,$$
$$S = 2\varpi ra.$$

Corollary 2.

If the bases be of any other form than circular, the proposition is still true. For if polygons be inscribed and circumscribed to the base, and prisms be described upon them whose edges are parallel to the axis; then the surface and volume of the cylinder are ultimately in a ratio of equality with those of each prism. Hence, since the proposition is true with respect to both the prisms, it is true of the cylinder which is intermediate to them.

Then putting A and s for the area and perimeter of the base, we have

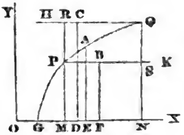
$$V = Aa \sin \alpha, \text{ and } S = as.$$

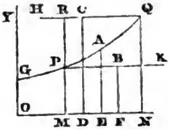
Corollary 3.

If the entire surface of the cylinder be required, the two ends must be added, giving for the value of S,

$$S = as + 2A.$$

PROP. II.





Let PAQ be an arc of a curve, either entirely convex or entirely concave towards the line OX; and let QH, PK drawn from P and Q, be each of them equal to the arc PAQ: then, if this system of lines revolve about OX, so that HQ and PK generate cylindrical surfaces, and PAQ generate a curve surface, the curve surface will have an area greater than the cylindrical surface generated by the line PK nearest to OX and less than that generated by the line QH, which is most remote from OX.

Let C, A, B be such that HC = AP = PB, and draw the ordinates CD, AE, BF. Then the point C generates one of the sections of the cylindrical surface, whose radius is CD; the point A generates a circular section of the surface of revolution whose radius is AE; and the point B generates a circular section of the cylindrical surface whose radius is BF; and these circumferences are, with respect to magnitude, as their radii.

Again, for every point, A, in the curve PAQ, there are corresponding points C and B in the lines HQ, PK; and hence, if circles, whose centres are in OX, move with their planes at right angles to OX, so as to pass co-temporaneously the points C, A, B, equidistant from their respective points of origination, the surfaces generated by them will have the same relation of magnitude (as to greater and less) as their generating circles. Also, those circles are in the order (increasing) of BF, AE, CD. Whence the surface generated by the arc PAQ is intermediate to the two cylinders traced by HQ and PK.

PROP. III.

To find the area of a surface of revolution.

(*See fig. above*).

Let PAQ be a portion of the generating curve, and make the same construction as before. Put

$$OM = x,\ ON = x';\ MP = y,\ NQ = y';$$

and let S, S′ denote the surfaces generated by the segments of the curve GP, GQ about OX; and let s, s' denote the arcs of the same segments.

Then, by the preceding, the curve surface generated by PAQ is of an area intermediate to those of the cylinders generated by PK and QH; or

$$S' - S \text{ is intermediate to } 2(s'-s)\,\varpi y, \text{ and } 2(s'-s)\,\varpi y'.$$

Hence $\dfrac{S'-S}{s'-s}$ is intermediate to $2\varpi y$ and $2\varpi y'$.

But in the limit, the two latter become equal; and hence

$$\left[\frac{S'-S}{s'-s}\right] = \frac{dS}{ds} = 2\varpi y;\ \text{or } dS = 2\varpi y ds;\ \text{whence } S = 2\varpi\int y ds \ldots\ldots\ldots (1).$$

PROP. IV.

To find the volume of a solid of revolution.

(*See fig. p.* 538).

Let V, V′ denote the volumes generated by the segments of the curve estimated from G to P and Q. Then the solid generated by PMNQ is intermediate to those generated by PMNS and RMNQ: that is,

$$V' - V \text{ is intermediate to } (x' - x)\,\varpi y^2 \text{ and } (x' - x)\,\varpi y'^2;$$

$$\text{or, } \frac{V' - V}{x' - x} \text{ is intermediate to } \varpi y^2 \text{ and } \varpi y'^2.$$

Whence at the limit

$$\left[\frac{V' - V}{x' - x}\right] = \frac{dV}{dx} = \varpi y^2; \text{ or } dV = \varpi y^2 dx.$$

Hence

$$V = \varpi \int y^2 dx. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Scholium.

The reasoning employed in the preceding propositions is capable of much more extensive application; but as, generally, the investigations will require the use of the geometry of co-ordinates of three dimensions, we cannot introduce it into this volume.

It may be remarked, however, that if any other regular figure, symmetrically situated in its correspondent parts with respect to OX, be substituted for the circle, the same reasoning applies, requiring only the perimeter or area of the generating figure to be substituted for those of the circle, in the two cases respectively.

We shall presently give an instance amongst others, where a square is substituted for the circle; generating the figure called *a groin*, or that formed by the intersection of two equal right cylinders whose axes intersect at right angles. It is the simplest form of the *groined roof*, so universal in the monastic and ancient architecture.

EXAMPLES.

I. The right cone.

In this case, the surface, as one of revolution, is generated by a right-angled triangle about one of its sides; the other side describing the circular base, and the hypothenuse describing the curved surface of the figure.

1. *The surface.* Let $y = x \tan\beta$ be the equation of the hypothenuse, referred to the axis of revolution as that of x, and the vertex of the cone as origin.

Then $ds = \sec\beta \,.\, dx$, and we have, by insertion in (1),

$$S = 2\varpi \int x \tan\beta \,.\, \sec\beta dx = \varpi \tan\beta \sec\beta\, x^2.$$

no constant being required *.

If we take a for the radius of the base of cone, and h the corresponding height: then $a = h \tan\beta$; and $h \sec\beta$ is the slant side of the cone, or hypothenuse of the generating triangle. Whence

---

* Occasionally, when no constant is required, we shall merely omit mention of it; leaving it for the student to investigate that such constant is really zero.

$$\underset{0}{\overset{h}{\mathrm{S}}} = \varpi \,.\, h \tan\beta \,.\, h \sec\beta = \varpi a \,.\, h \sec\beta = \frac{1}{2}\varpi\,(2a)\,.\,h\sec\beta.$$

That is, the surface of the cone is expressed by half the product of the slant side into the perimeter of the base.

2. *The volume.* Here the general expression (2) becomes,

$$\mathrm{V} = \varpi \int x^2 \tan^2\beta dx = \frac{1}{3}x^3 \,.\, \varpi \tan^2\beta = \frac{1}{3}x \,.\, x^2 \tan^2\beta.$$

$$\underset{0}{\overset{h}{\mathrm{V}}} = \frac{1}{3}h \,.\, \varpi h^2 \tan^2\beta = \frac{1}{3}h \times \text{area of the base.}$$

These results accord with those given in *vol. i. p.* 487.

Scholium.

When a hemisphere, right cone, and right cylinder, stand on the same circular base, and are of the same altitude, they are called the *Archimedian cone, sphere, and cylinder,* from their connexion having been first investigated by that greatest of all the geometers of antiquity.

II. THE SPHERE.

1. *The surface.* Referring the system to the centre of the generating circle as origin, and the axis of x being that of revolution, we have

$$x^2 + y^2 = a^2,\ xdx + ydy = 0,\ \text{and } ds = \frac{adx}{y}:\ \text{whence}$$

$$\mathrm{S} = 2\varpi \int \frac{aydx}{y} = 2\varpi \int adx = 2\varpi ax:\ \text{or again}$$

$$\underset{-a}{\overset{a}{\mathrm{S}}} = 4\varpi a^2.\ \text{The whole surface is therefore } 4\varpi a^2,$$

or four times the area of one of the great circles of the sphere.

From the general formula, also, we deduce at once that the surfaces of portions of the sphere (zones or segments) are to one another as their altitudes. For let $x_1, x_2, x_3, \ldots$ be the distances of any plane sections from that passing through the centre perpendicular to the axis of revolution: then the zones have as surfaces

$$2\varpi a\,(x_2 - x_1),\ 2\varpi a\,(x_3 - x_2),\ 2\varpi a\,(x_4 - x_3),\ etc.$$

and these vary as $x_2 - x_1, x_3 - x_2, x_4 - x_3$, *etc.*; that is, as the distances of their bounding planes *.

Also, these expressions for the surfaces of the zones are equal respectively to the segments of the Archimedian cylinder cut off by the same planes.

2. *The volume.* In this case $\mathrm{V} = \varpi\int y^2 dx = \varpi x\,(a^2 - \frac{1}{3}x^2)$;

$$\text{or } \underset{-a}{\overset{a}{\mathrm{V}}} = \frac{4\varpi}{3}a^3 = \frac{\varpi}{6}(2a)^3.$$

Whence the volume of any segment or zone, as well as of the whole sphere, is determinable; and the rules given in *vol. i. pp.* 491, 492, may be readily deduced from these results.

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* The student will conceive this the better by supposing the ordinary positions of the axes of x and y interchanged.

Also, in the *Archimedian cone and cylinder*, $h = a$; and hence *cone* + *hemisphere* $=\frac{1}{3}\,a\,.\,a^2\varpi + \frac{2}{3}\,a^3\varpi = a^3\varpi = a\,.\,a^2\varpi =$ *cylinder*.

It may further be worth while to remark, that

$$\int\sqrt{a^2 - x^2}\,dx = \frac{1}{2}\,a^2\sin^{-1}\frac{x}{a} + \frac{1}{2}\,x\sqrt{a^2-x^2} = a^2\,.\,\frac{1}{2}\text{ circ. zone }\sin^{-1}\frac{x}{a}$$

$$\int\sqrt{2ax-x^2}\,dx = \frac{1}{2}\,a^2\text{vers}^{-1}\frac{x}{a} - \frac{1}{2}(a-x)\sqrt{2ax-x^2} = a^2.\frac{1}{2}\text{ circ. seg. vers}^{-1}\frac{x}{a}.$$

III. THE SPHEROÏDS, OR ELLIPSOÏDS OF REVOLUTION.

These are called *prolate* or *oblate*, according as the axis of revolution is the transverse, or conjugate diameter.

1. *The surface of the prolate spheroïd.* Refer it to the centre of the generating ellipse as origin: then

$$a^2y^2 + b^2x^2 = a^2b^2 \text{ gives } ds = \frac{\sqrt{a^2 - e^2x^2}}{y}\,.\,dx\text{; and hence}$$

$$dS = 2\varpi\int yds = 2\varpi\int\sqrt{a^2 - e^2x^2}dx$$

$$= \frac{\varpi ab}{e}\left\{\sin^{-1}\frac{ex}{a} + \frac{ex}{a}\left(1 - \frac{e^2x^2}{a^2}\right)^{\frac{1}{2}}\right\}$$

$$S_{-a}^{\;a} = \frac{2\varpi ab}{e}\{\sin^{-1}e + e\sqrt{1 - e^2}\}.$$

2. *The surface of the oblate spheroïd.* Refer it as before, interchange x and y in the equation of the curve, and by a process analogous to the preceding, we have

$$dS = 2\varpi\int yds = 2\varpi\int\sqrt{b^2 + e^2x^2}\,dx\,;$$

and the integration being performed, we have for the whole spheroïd,

$$S_{-b}^{\;b} = 2\varpi b^2\left\{1 + \frac{1 - e^2}{2e}\log_e\frac{1 + e}{1 - e}\right\}.$$

3. *The volume of the prolate spheroïd.* Since $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

$$V = \varpi\int y^2dx = \frac{b^2\varpi}{a^2}\int(a^2 - x^2)\,dx = \frac{b^2\varpi}{a^2}\left\{a^2x - \frac{1}{3}\,x^3\right\}.$$

$$V_{-a}^{\;a} = \frac{4\varpi}{3}\,.\,\frac{b^2}{a^2}\,.\,a^3 = \frac{4\varpi}{3}\,.\,ab^2.$$

4. *The volume of the oblate spheroïd.* Change x and y, and then we have

$$V = \frac{a^2\varpi}{b^2}\{b^2x - \frac{1}{3}x^2\}\text{; and } V_{-a}^{\;a} = \frac{4\varpi}{3}\,.\,a^2b.$$

The relation of these last two to the spheres on a and b are obvious and remarkable. They give

sphere on b : oblate ellipsoïd : : prolate ellipsoïd : sphere on a;
sphere on a × sphere on b = prolate ellipsoïd × oblate ellipsoïd.

IV. THE PARABOLOÏD.

1. *The surface.* Let $4m$ be the parameter: then $y^2 = 4mx$; and

$$ds = \frac{2\sqrt{m}\,.\,\sqrt{m + x}}{y}\,dx\text{, or } S = 2\varpi\int yds = \frac{8\varpi\sqrt{m}}{3}\left\{(x + m)^{\frac{3}{2}} - m^{\frac{3}{2}}\right\}.$$

2. *The volume.* Here $\mathrm{V} = \varpi\int y^2dx = \varpi\int 4mxdx = 2m\varpi x^2$; or

$$\mathrm{V} = 2m\varpi x^2 = \frac{1}{2}x \,.\, 4mx\varpi = \frac{1}{2}x \,.\, y^2\varpi;$$

that is to half the cylinder of the same base and altitude.

V. THE SURFACES GENERATED BY THE CYCLOÏD.

The only two symmetrical axes of revolution are the base and axis: these we shall call the first and second species, respectively.

1. *The surface of the first species.* If a be the diameter of the generating circle, we have, referring to the middle of the base as origin,

$$\frac{dy}{dx} = \sqrt{\frac{a-y}{y}}; \text{ and } \mathrm{S} = 2\varpi\int (ay)^{\frac{1}{2}}\, dx = 2\varpi\sqrt{a}\int \frac{ydy}{\sqrt{a-y}}$$

$$= -\frac{4\varpi\sqrt{a}}{3}(2a+y)\sqrt{a-y}.$$

The whole surface is, therefore, expressed by $\mathrm{S} = \frac{16a^2\varpi}{3}$.

2. *The surface of the second species.* In this case we have

$$\frac{dy}{dx} = \left(\frac{a-x}{x}\right)^{\frac{1}{2}}; \text{ and } \mathrm{S} = 2\varpi\int y \left(\frac{a}{x}\right)^{\frac{1}{2}} dx$$

$$= 4\varpi\sqrt{a}\,\{\sqrt{x}\,.\,y - \int (a-x)^{\frac{1}{2}}dx\}.$$

The entire surface is, hence, equal to $2\varpi a^2\left\{\varpi - \frac{4}{3}\right\}$.

3. *The volume of the first species.* In this case we get

$$\mathrm{V} = \varpi\int \frac{y^3dy}{\sqrt{ay-y^2}} = \frac{5a^3\varpi}{16}\,\text{versin}^{-1}\frac{2y}{a} - \frac{\varpi}{3}\{y^2 + \frac{5ay}{4} + \frac{15}{8}a^2\}\sqrt{ay-y^2},$$

$$\text{and } \mathrm{V}_0^{2a} = \varpi\int_0^{2a}\frac{y^3dy}{\sqrt{ay-y^2}} = \frac{5\varpi^2a^3}{3} \text{ for the whole solid.}$$

4. *The volume of the second species.* Here we get in the same way,

$$\mathrm{V} = \varpi\int y^2dx = \varpi\,\{y^2x - 2\int xy\frac{dy}{dx}\,.\,dx\}.$$

$$= \varpi\left\{y^2x - 2\int y\sqrt{ax-x^2}\,dx\right\}.$$

To find this integral, put $\theta = \text{vers}^{-1}\frac{2x}{a}$: then we have

$$y = \frac{1}{2}a\,(\theta + \sin\theta);\ \sqrt{ax-x^2} = \frac{1}{2}a\sin\theta;$$

$$\text{and } x = \frac{1}{2}a\,(1-\cos\theta), \text{ or } dx = \sin\theta d\theta.$$

Whence the function under the sign of integration becomes

$$\int y\sqrt{ax-x^2}dx = \frac{1}{8}a^3\sin^2\theta\,(\theta + \sin\theta)\,d\theta$$

$$= \frac{1}{8}a^3\theta\sin^2\theta d\theta + \frac{1}{8}a^3\sin^3\theta d\theta$$

$$= \frac{1}{8}a^3\left\{\left(\frac{\theta^2}{4} - \frac{\theta\sin 2\theta}{4} - \frac{\cos 2\theta}{8}\right) - \left(\frac{\sin^2\theta\cos\theta}{3} + \frac{2\cos\theta}{3}\right)\right\};$$

and hence the whole integral between the extreme limits is readily found to be

$$V_0^a = \varpi \left\{3\varpi^2 a^3 - 2a^3\left(\frac{\varpi}{4} + \frac{1}{2}\right)\right\} = \varpi a^3 \left\{\frac{3\varpi^2}{2} - \frac{8}{3}\right\}.$$

EXAMPLES FOR EXERCISE.

1. The volume described by the revolution of the *conchoïd* about the asymptote is

$$V = C - \varpi \left\{ab^2 \sin^{-1}\frac{y}{b} - y^2\sqrt{b^2 - y^2} - \frac{2}{3}(b^2 - y^2)^{\frac{3}{2}}\right\};$$

and the entire volume is

$$V = \varpi\, b^2 \left\{\frac{\varpi a}{2} + \frac{2b}{3}\right\}.$$

2. The entire volume described by the *cissoïd* round its asymptote is $\varpi^2 a$.

3. The *torus* is generated by the revolution of a circle whose radius is a about an axis in the plane of the circle whose distance from the centre of the circle is b: show that the entire surface and volume are

$$S = 4\varpi ab \text{ and } V = 2\varpi a^2 b.$$

4. The whole volume described by the *witch* about its asymptote is

$$V = 4\varpi a^3,$$

5. If the *companion to the cycloïd* revolve about its axis and its base, the entire volumes are respectively

$$V = 3\varpi^2 a^3 \text{ and } V = \varpi(\varpi^2 - 4)\, a^3.$$

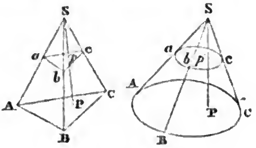
6. The entire surface and volume of the figure generated by the revolution of the cardioïd about its axis are respectively

$$S = \frac{4}{3} a^2 \varpi, \text{ and } V = \frac{8}{3} a^3 \varpi.$$

7. If a parabola whose base is b and height a, revolve about its base so as to form the figure called the *parabolic spindle*, its entire volume will be $\frac{16}{15} a^2 b \varpi$.

PROP. V.

To find the volume of any Pyramid, or of a Cone upon any plane base.



Let S be the vertex, and SP a perpendicular to the base ABC. In SP take $Sp = x$, and through p draw the plane abc parallel to ABC. Then, since the sections are parallel, the figures ABC, abc are similar; and we have (denoting the areas of ABC, abc by A and a, and the perpendicular SP by p).

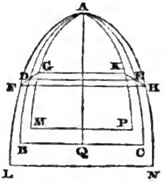
$$a = \frac{Ax^2}{p^2}; \text{ and hence } V = \frac{A}{p^2}\int x^2 dx = \frac{Ax^3}{3p^2}.$$

For the whole volume we have, when $x = p$, the value

$$V_0^p = \frac{1}{3} Ap = \frac{1}{3} \text{ base} \times \text{altitude}.$$

PROP. VI.

To find the volume and surface of the Common Groin, which is thus generated.



Let a square plane FGKH of variable magnitude, moving from the vertex A of a semicircle BAC at right angles to its axis AQ, vary in magnitude so that the middles D, E of the opposite sides FG, HK continually rest upon the semicircle: then the four sides of the square trace out simultaneously the groin-surface.

Let A be the origin, AR $= x$, RD $= y$, and AQ $= a$. Then the area of FGKH $= (2y)^2 = 4y^2$. Hence $dV = 4y^2dx = 4\,(2ax - x^2)\,dx$; and hence

$$V = 4\left\{ax^2 - \frac{1}{3}\,x^3\right\},\ \text{and}\ \overset{a}{\underset{0}{V}} = \frac{8a^3}{3}.$$

Again, since the generating surface, or perimeter of the square, is $8y$,

$$dS = 2yds = 2y\,.\,8dy\,;$$

and hence we have at once

$$S = 8y^2,\ \ \text{and}\ \ \overset{a}{\underset{0}{S}} = 8a^2.$$

The two preceding propositions are specimens of a general principle which is embodied in the following one.

PROP. VII.

Let a plane figure which moves parallel to itself and varies its magnitude (retaining its species, or the constant ratio of its parameters) move so that homologous points may always rest upon a plane curve (or system of lines issuing from a point), and one homologous point in the plane of the moving figure trace a given straight line: then the moving plane figure will generate a solid whose volume and area are expressed by

$$V = \sin\alpha \int A dz \ldots\ldots\ldots (3), \qquad | \qquad S = \int sds' \ldots\ldots\ldots\ldots (4)\,;$$

where α is the angle made by the moving plane with the line of direction of motion, A the area, and S the perimeter of the generating curve at the boundary of the solid; and s' the arc of the curve along which the generatrix moves, and z the distance of that plane along the line from the origin of the motion.

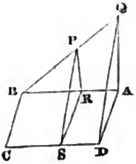
For if instead of the elementary right cylinders which we employed in the reasonings of *propositions iii.* and *iv.* we substitute oblique cylinders, having the adjacent generating curve a for their bases, and for height the common distance between them: the same reasoning respecting the ultimate ratios and their corresponding expressions will apply.

Now in the value of V, the area of the section takes the place of the area of the circle; and in that of S, we have sds' in the place of $2\varpi yds$. Also, since these cylinders are oblique, having α for the angle formed by their axes and bases, the factor $\sin\alpha$ reduces them to the value given above.

It will be obvious that the solid of revolution is a particular case of that here investigated; and it is equally obvious that the principle applies as well to rectilinear generatrices as when they are any specified curves.

EXAMPLES.

1. *Let ABCD be a rectangle, AQ perpendicular to the plane AC at A: then if a variable triangle, PRS, the plane of which is parallel to AQD, move with its angular points upon CD, BA, BQ, it is required to find the volume swept out by the triangle PRS, and the surface generated by PS.*



Put AD $= a$, AB $= b$, AQ $= c$, BR $= x$. Then PR $= \frac{cx}{b}$; and the area A of the generating triangle PRS is $\frac{acx}{2b}$. Whence,

$$\mathrm{V} = \frac{ac}{2b}\int x dx = \frac{acx^2}{4b}; \text{ and } \mathrm{V}_o^b = \frac{abc}{4} = \text{whole volume.}$$

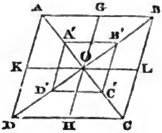
Again, the generating line PS $= \sqrt{\mathrm{PR}^2 + \mathrm{RS}^2} = \frac{\sqrt{c^2x^2 + a^2b^2}}{b}$; and hence

$$\mathrm{S} = \frac{1}{b}\int\sqrt{c^2x^2 + a^2b^2}.dx = \frac{x\sqrt{c^2x^2 + a^2b^2}}{2b} + \frac{a^2b^2}{2b}\int\frac{dx}{\sqrt{c^2x^2 + a^2b^2}}$$

$$= \frac{x\sqrt{c^2x^2 + a^2b^2}}{2b} + \frac{a^2b^2}{2bc}\log_e(cx + \sqrt{c^2x^2 + a^2b^2})$$

$$\text{and } \mathrm{S}_o^b = \frac{b}{2}\left\{\sqrt{a^2 + c^2} + \frac{a^2}{c}\log_e\frac{c + \sqrt{a^2 + c^2}}{a}\right\}.$$

2. *Two equal right cylinders have their axes intersecting under any angle in a point: find the volume which is common to both, and the surface of the figure inclosed.*



Let GH and KL be the axes intersecting in O, and let the plane of the axes cut the cylinders themselves in AB, CD, and AD, BC: also let A′B′C′D′ be any other section of the groined surface made by a plane parallel to ABCD, which will evidently be an equilateral parallelogram similar to ABCD.

Let α be the inclination of the axes, z the distance of the planes ABCD, A′B′C′D′, A the area of A′B′C′D′, and a the common diameter of the cylinders. Then,

$$\mathrm{V} = \int \mathrm{A}dz.$$

Let the vertical from O to the plane ABCD meet the plane A′B′C′D′ in O′; then OO′ $= z$.

Also, AB $=$ BC $= 2a \operatorname{cosec} \alpha$; ABCD $=$ AB.BC $\sin\alpha = 4a^2 \operatorname{cosec}\alpha$; and, plane section A′B′C′D′ $= \frac{\mathrm{A'B'^2}}{\mathrm{AB^2}}$.ABCD.

Wherefore, $\mathrm{A} = \mathrm{ABCD}.\frac{\mathrm{A'B'^2}}{\mathrm{AB^2}} = 4a^2 \operatorname{cosec}\alpha.\frac{a^2 - z^2}{a^2} = 4(a^2 - z^2)\operatorname{cosec}\alpha$,

and we have

$$\mathrm{V} = 4 \operatorname{cosec}\alpha\int(a^2 - z^2)\,dz = 4\,(a^2z - \tfrac{1}{3}z^3)\operatorname{cosec}\alpha.$$

$$\mathrm{V}_{-a}^{a} = \frac{16a^3}{3\sin\alpha} \text{ the volume common to the two cylinders.}$$

Again, the perimeter of the parallelogram A′B′C′D′, , is readily seen to be

$$s = 8 \text{ cosec } \alpha \sqrt{a^2 - z^2}, \text{ and}$$

$$ds' = \frac{adz}{\sqrt{a^2 - z^2}}: \text{ wherefore,}$$

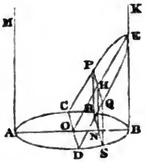
$$S = \int s ds' = 8a \text{ cosec } \alpha \int dz = 8az \text{ cosec } \alpha.$$

This, taken between limits 0, a, and doubled, gives for the entire surface,

$$\overset{a}{\underset{-a}{S}} = 16a^2 \text{ cosec } \alpha.$$

When $\alpha = \frac{1}{2}\pi$, it becomes the *common groin* already discussed, *page* 544.

III. *Let ADBC be the base of a right cylinder, E a point in one of the edges BK, and CED a plane section passing through the centre O of the base: it is required to find the surface and area of the portion CDBE cut off by the plane.*



Let the volume be cut by a plane at right angles to OB, making the section PQSR, which is evidently a rectangle. Put EOB = α, OB = a, and ON = x. Then we have HN = $x \tan \alpha$, and RS = $2\sqrt{a^2 - x^2}$: whence the generating rectangle PQSR (= A) = $2x\sqrt{a^2 - x^2}.\tan \alpha$; and the volume is found from

$$V = 2 \tan \alpha \int \sqrt{a^2 - x^2}\, xdx = \tan \alpha \left\{C - \frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}\right\}.$$

As x and V begin together, when x is 0, V is 0: hence $C = \frac{2}{3}a^3$, and

$$V = \frac{2}{3}\tan \alpha \left\{a^3 - (a^2 - x^2)^{\frac{3}{2}}\right\}, \text{ and } \overset{a}{\underset{0}{V}} = \frac{2}{3}a^3 \tan \alpha.$$

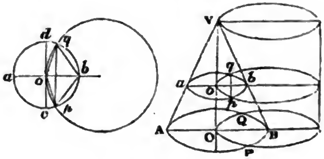
Again, for the surface, the generatrix is the line PQ = EH = $x \tan \alpha = s$; and $ds' = d\text{CQ} = \frac{adx}{\sqrt{a^2 - x^2}}$; whence,

$$S = \int s ds' = \int \frac{ax\, dx \tan \alpha}{\sqrt{a^2 - x^2}} = \{C - a\sqrt{a^2 - x^2}\} \tan \alpha.$$

Now S and x commence together; hence $C = a^2$, and we have

$$S = \left\{a^2 - a\sqrt{a^2 - x^2}\right\} \tan \alpha; \text{ and } \overset{a}{\underset{0}{S}} = a^2 \tan \alpha.$$

IV. *A right cone has its axis coincident with the edge of a right cylinder; and the altitudes of the cone and cylinder and the diameters of their bases are all equal: find the parts into which the surface and volume of the cone are divided by the cylinder.*



Let AB = VO = a and Vo = x; and let the plane *apbq*, parallel to the base, cut the cone and cylinder in the circles *qbp* and *qop* respectively. Then the cone may be conceived to be generated by the variable circle *qbp*; the cylinder by the circle *qop*[*]; and the part that is common to both by the curvilinear figure *oqbp*, or A. Whence we have to find A: which is, evidently,

$$\mathrm{A} = opbq = \text{semicirc. } codb + \text{segm. } poq - \text{trian. } poq - 2 \text{ sector } poc$$

Now $ob = op = \frac{1}{2}x$, $ro = \frac{x^2}{4a}$, and $pr = qr = \frac{x\sqrt{4a^2 - x^2}}{4a}$: whence the expression becomes

$$\mathrm{A} = \frac{\varpi x^2}{8} + \frac{1}{4a^2}\int x^2 dx \sqrt{4a^2 - x^2} - \frac{x^3}{16a^2}\sqrt{4a^2 - x^2} - \frac{x^2}{4}\sin^{-1}\frac{x}{2a},$$

$$= \frac{\varpi x^2}{8} + \frac{2a^2 - x^2}{4}\sin^{-1}\frac{x}{2a} - \frac{x}{8}\sqrt{4a^2 - x^2}; \text{ and hence}$$

$$\mathrm{V} = \int \mathrm{A}dx = \frac{\varpi x^3}{24} + \frac{6a^2x - x^3}{12}\sin^{-1}\frac{x}{2a} + \frac{32a^2 - 5x^2}{72}\sqrt{4a^2 - x^2} - \frac{8a^3}{9}$$

$\overset{a}{\underset{o}{\mathrm{V}}} = \frac{8\varpi + 27\sqrt{3} - 64}{72}a^3$, the part of the cone within the cylinder; and hence the two parts of the volume of the cone are respectively

$$\frac{8\varpi + 27\sqrt{3} - 64}{72}a^3, \text{ and } \frac{64 - 27\sqrt{3} - 2\varpi}{72}a^3.$$

Again, the portion of the surface of the cone within the cylinder is

$$\mathrm{S} = \int s ds' = \int \text{arc } bpq \,.\, d\mathrm{V}b = \int x\frac{\sqrt{5}}{2}dx \,.\, \cos^{-1}\frac{x}{2a}.$$

$$= \frac{\sqrt{5}}{2}\left\{\frac{x^2}{2}\cos^{-1}\frac{x}{2a} + a^2\sin^{-1}\frac{x}{2a} - \frac{x}{4}\sqrt{4a^2 - x^2}\right\}$$

$$\overset{a}{\underset{o}{\mathrm{S}}} = \frac{4\varpi\sqrt{5} - 3\sqrt{15}}{24}a^2.$$

Whence the parts into which the surface is divided are respectively

$$\frac{4\varpi\sqrt{5} - 3\sqrt{15}}{24}a^2, \text{ and } \frac{2\varpi\sqrt{5} + 3\sqrt{15}}{24}a^2.$$

This elegant solution is entirely taken from the *Lady's and Gentleman's Diary* for 1843, *pp.* 61, 62.

EXAMPLES FOR EXERCISE.

1. The axis of a right cylinder passes through the centre of a sphere, their radii being a and a_1: then the volume common to both is

$$\mathrm{V} = \frac{2\varpi}{3}\left\{a_1^3 - (a_1^2 - a^2)^{\frac{3}{2}}\right\}.$$

2. A sphere is cut by a cylinder, the radius of whose base is half that of the sphere, and whose axis bisects the radius of the sphere at right angles: show that if a be the radius of the sphere, the volume of the sphere without the cylinder is one-ninth of the cube of the diameter of the sphere.

3. The surface of the sphere intercepted by the cylinder, in the preceding example is $a^2(\frac{1}{2}\varpi - 1)$: and if the sphere be cut by two such cylinders, the

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[*] For greater distinctness the two circles are exhibited in the margin, as well as in perspective in the principal figure.

surface of the sphere *without* the cylinders is double the square of the diameter of the sphere [*].

4. A paraboloïd of revolution is pierced by a right cylinder, the axis of which passes through the focus and cuts the axis of the paraboloïd at right angles, its radius being the focal distance of the vertex: then the volume common to both is $a^3 \left\{2\varpi + \frac{16}{3}\right\}$, $4a$ being the parameter of the generating parabola.

5. The *cono-cuneus* (or *the shipwrights' circular wedge*) of Wallis is thus generated:

Perpendicular to the plane of a circle, and to each other, two planes are drawn through its centre, and in one of these planes a line parallel to the plane of the circle; then a line moves constantly touching this line and the circle, and always parallel to the other plane, marks out the surface of the figure: it is required by the preceding method to assign its volume and surface [†].

6. A rectangular parallelopiped, whose length and breadth are $2a$, $2b$, and the axis of which passes through the centre of the sphere whose radius is r: show that the portion of the solid common to both is

$$V = 4b\,(r^2 - \tfrac{1}{3}b^2)\sin^{-1}\frac{a}{\sqrt{r^2-b^2}} + 4a\,(r^2 - \tfrac{1}{3}a^2)\sin^{-1}\frac{b}{\sqrt{r^2-a^2}}$$

$$+ \tfrac{8}{3}\sqrt{r^2-a^2-b^2} - \tfrac{8}{3}r^3\sin^{-1}\frac{ab}{\sqrt{(r^2-a^2)(r^2-b^2)}}.$$

---

[*] This is the celebrated enigma of the *oval window*, proposed by Viviani, as a challenge to the cultivators of the differential calculus. Several particulars relative to this problem may be seen in a paper by the editor, in the twelfth volume of the Transactions of the Royal Society of Edinburgh, and in the Gentleman's Diary for 1835. The problem is, however, treated in those places by the method of *spherical co-ordinates*; which subject, should a third volume of this work be printed, will be developed at suitable length, but which want of room did not allow to be introduced into the present one, even in the most contracted form. The form in which Viviani proposed his problem is here given.

Inter venerabilia olim Græciæ monumenta extat adhuc, perpetuo quidem duraturum, Templum augustissimum ichnographia circulari Almæ Geometriæ dicatum, quod testudine intus perfecte hemisphærica operitur: sed in hac fenestrarum quatuor æquales areæ (circum ac supra basin hemisphæræ ipsius dispositarum) tali configuratione, amplitudine, tantaque industria, ac ingenii acumine sunt exstructæ, ut his detractis superstes curva Testudinis superficies, pretioso opere musivo ornata, tetragonismi vere geometrici sit capax. *Acta Eruditorum*, 1692.

[†] Several particulars relative to this surface may be seen in a paper on *Rule Surfaces*, by the editor of this work, in *vol. vi. Leybourn's Mathematical Repository*.

END OF VOL. II.

Gilbert & Rivington, Printers, St. John's Square, London.

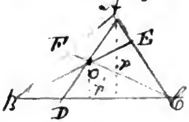
In the "Questions Proposées" in the number for April 1852, I find

Etant donné un point quelconque O dans le plan d'un triangle ABC, on mène les droites AO, BO, CO rencontrant les côtés du triangle en D, E, F. Démontrer la relation

$$\frac{AO}{OD} = \frac{AF}{FB} + \frac{AE}{EC}.$$

This very simple and neat relation I have not before met with (if my memory does not prove recreant) in any collection of geometrical problems. Occured to me years ago. The proof is easy.

$$\frac{AOC}{BOC} = \frac{AF}{FB}, \quad \frac{AOB}{BOC} = \frac{AE}{EC}$$



$$\therefore \frac{AF}{FB} + \frac{AE}{EC} = \frac{AOC + AOB}{BOC}$$

$$= \frac{\Delta - BOC}{BOC} = \frac{AO}{OD}.$$

The above property can be proved from the fig on p 217. by denoting the point O by that of P. Referring to that fig the above property is defined by $\frac{BP}{PE} = \frac{BD}{DC} + \frac{BF}{FA}$

From Δs APE, BPH, we get AE : BH = EP : PB, $\therefore \frac{BP}{PE} = \frac{BH}{AE}$

" Δs CPE, GPB " " BG : EC = BP : PE $\therefore \frac{BP}{PE} = \frac{BG}{EC}$

Whence $\frac{BP^2}{PE^2} = \frac{BH \cdot BG}{AE\,EC}$ From Δs FAC, FBG, we derive BG, and from Δs DCA, DBH we derive BH. Also from AE BF CD = EC . FA BD, we get $\frac{BD}{CD} = \frac{BF}{FA} \cdot \frac{AE}{EC}$. By substituting these values $\frac{BH \cdot BG}{AE \cdot EC}$, and reducing, we get $\frac{BP}{PE} = \frac{BD}{DC} + \frac{BF}{FE}$. Q.E.D

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